4.3 Exponential Change

Rational exponents are well-defined. Let $b>0, \; m,n \in \mathbb{N}, \; b \in \mathbb{R}$.
Then

(a) $b^m = \underbrace{b \cdot b \cdot \ldots \cdot b}_{m \text{ times}}$

(b) $b^{-m} = \dfrac{1}{b^m} = \dfrac{1}{\underbrace{b \cdot b \cdot \ldots \cdot b}_{m \text{ times}}}$

(c) $b^{\frac{1}{n}} = \sqrt[n]{b}$ = a positive number whose nth power is b

(d) $b^{\frac{m}{n}} = \sqrt[n]{b^m}$

(e) $b^{-\frac{m}{n}} = \dfrac{1}{b^{\frac{m}{n}}} = \dfrac{1}{\sqrt[n]{b^m}}$

What about irrational exponents?

Fix an irrational x. Let's choose $x=\pi$. Its decimal representation is something like
$\pi \approx 3.1415926535897\ldots$. We can construct sequences $\{y_n\}$ and $\{z_n\}$ of rational
numbers:

$y_0=3$, $y_1=3.1$, $y_2=3.14$, $y_3=3.141$, $y_4=3.1415$, $y_5=3.14159\ldots$ and
$z_0=4$, $z_1=3.2$, $z_2=3.15$, $z_3=3.142$, $z_4=3.1416$, $z_5=3.14160\ldots$

1. What can you say about $\lim\limits_{n\to\infty} y_n$ and $\lim\limits_{n\to\infty} z_n$?

2. According to the definition of rational exponents, the number 2^{y_n} is well-defined for each
   n-value. What is $\lim\limits_{n\to\infty} 2^{y_n}$? So is the sequence $\{2^{y_n}\}$ convergent or divergent? And,
   what about $\{2^{z_n}\}$?

3. What does this suggest about a definition for 2^π ?

What if we extend this argument to a general case? Let x be an arbitrary irrational number. Then, there exist two sequences $\{y_n\}$ and $\{z_n\}$ of rational numbers with the following properties:

(a) y_n and z_n are rational numbers for each n,

(b) $y_1 \le y_2 \le y_3 \le \ldots$ and $z_1 \ge z_2 \ge z_3 \ge \ldots$,

(c) $y_n \le x \le z_n$ for each n,

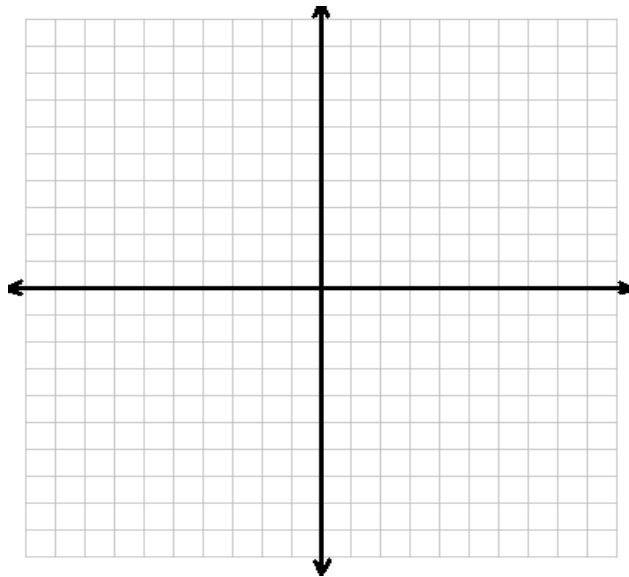(d) $\lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = x$.

Then, how can we define 2^x ?

How can we define b^x for $b > 0$, an arbitrary real number, and for x, an arbitrary irrational number?

The function $f(x) = b^x$, for $b > 0$, is now well-defined on the set of real numbers and its called the exponential function.
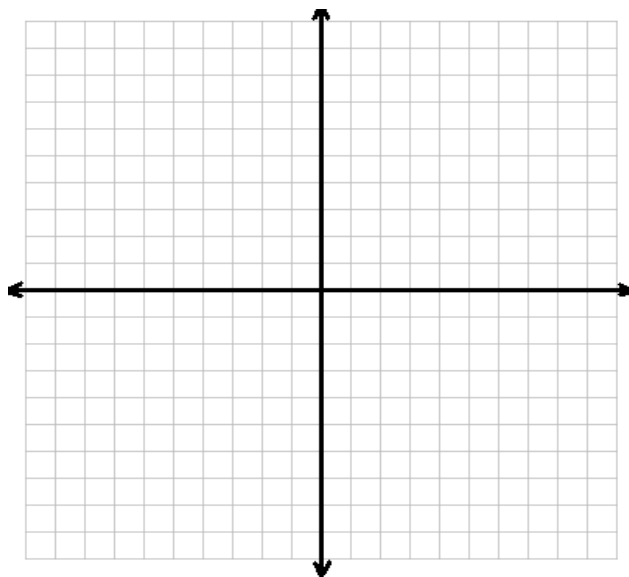
What is the derivative of an exponential function?

Let $f(x) = 2^x$ be defined for all real numbers. Graph the function.

(a) Fill in the first derivative sign line for this function's graph.

(b) Graph the first derivative function, based on the sign line you just found.

(c) Compare the two graphs (of the function and its derivative). What do you notice?

(d) Compute the derivative using the definition of the derivative.
(*Note: You may have to use a numerical argument to finish this. And, then, can we do the limit with calculus?)

3

(e) Following the same process, what is the derivative of $g(x)=\left(\dfrac{3}{5}\right)^{x}$ then?

(f) Prove the following statement:

If $f(x)=b^{x}$, then $f'(x)=kb^{x}$ where $k=\lim\limits_{h\to 0}\dfrac{b^{h}-1}{h}$.

(g) What is the value of k?

4

Finish filling in this derivative rule:
$$D_x(e^x)=$$

$$D_x(a^x)= \qquad\qquad \forall\, a>0, a\neq 1$$

Ex 1: Find the derivative for each of these functions.

(a) $f(x)=\dfrac{x\,e^{-2x}}{e^x+5}$

(b) $f(x)=4^{2x^2-1}$

(c) $f(x)=(2^x-x^2)^x$

5.1 Indefinite Integrals (Antiderivatives)

Definition: Antiderivative
A function $F(x)$ is antiderivative of the function $f(x)$ if $F'(x)=f(x)\ \forall\, x\in$ domain(f) .

1. If $G(x)=F(x)+c$, $c\in\mathbb{R}$, then what's $G'(x)$?

2. What does this mean about the uniqueness of $F(x)$?

5

3. If we give you the graph of $y = F(x)$, then can you graph all the possible $y = G(x)$ for the $G(x)$ defined as in #1 above? What does the c-value tell you graphically?

4. Are all functions integrable, i.e. does the antiderivative exist for all functions?

Ex 1: Draw a Venn diagram for these sets.

R = { $f : f$ is $\mathbb{R}$ - valued function on some subset $E \subseteq \mathbb{R}$ }
D = { $f : f$ is $\mathbb{R}$ - valued, differentiable function on some subset $E \subseteq \mathbb{R}$ }
C = { $f : f$ is $\mathbb{R}$ - valued, continuous function on some subset $E \subseteq \mathbb{R}$ }
I = { $f : f$ is $\mathbb{R}$ - valued, integrable function on some subset $E \subseteq \mathbb{R}$ }

5. How can we "undo" the power rule for differentiation?

$$D_x(x^n)=n\,x^{n-1} \;\Rightarrow\; \int n\,x^{n-1}\,dx=x^n+C \;, \;\forall\, n\neq 0$$

State the typical power rule for integration:

$$\int x^n\,dx \;=$$

6. Is the antiderivative a linear operator? What does this mean?

Ex 2: Find the antiderivative/indefinite integral.

(a) $\int x^{3/2}\,dx$

(b) $\int \dfrac{5}{x^3}\,dx$

(c) $\int \left(8x^2+\dfrac{2}{x^{2/3}}\right)dx$

Ex 3: What is $\int e^x dx$? What about $\int e^{kx} dx$ for some nonzero constant k? What about $\int e^{kx+b} dx$?

Ex 4: Is $F(x)=\begin{cases} 1-\dfrac{1}{x} & \text{if } x<0 \\ 3-\dfrac{1}{x} & \text{if } x>0 \end{cases}$ an antiderivative for $f(x)=\dfrac{1}{x^2}$?

(a) Describe $F(x)-F\left(\dfrac{-1}{x}\right)$.

(b) Is $F(x)=\dfrac{-1}{x}$ $\forall x\neq 0$ also an antiderivative for $f(x)=\dfrac{1}{x^2}$?

8

Ex 5: Evaluate these integrals.

(a) $\int \sqrt{x}(x-2)^2 dx$

(b) $\left(\int \sqrt{x}\,dx\right)\left(\int (x-2)^2 dx\right)$

(c) Is there a "product rule" for integration?