Math6100
Day 7 Notes
5.2 & 5.3, Definite Integrals and Techniques of Integration

5.2 Definite Integrals

Fundamental Theorem(s) of Calculus
Let $a, b, c \in \mathbb{R}$ (i.e. they are constants) and $g(x)$ is an integrable function.

1. If $G(x)=\int_c^x g(t)\,dt$, then $G'(x)=g(x)$.
2. If $F(x)$ is the antiderivative of $f(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Ex 1: Compute these integrals.

(a) $\int_1^3 (2x^2+1)\,dx$

(b) $\int_0^2 e^x\,dx$

Note:

(a) $\int_a^b f(t)\,dt = \int_a^b f(w)\,dw = \int_a^b f(x)\,dx$ i.e. variable of the definite integral is a "dummy variable." The result of a definite integral will be a number (assuming a and b are constants).

(b) Geometrically, how can we interpret $\int_a^b f(t)\,dt$? (Is it a "signed" or "unsigned" value?)

Ex 2: Let $f(x) = x^3$ and $g(x) = \int_0^x f(t)\,dt$.

(a) $g(x) = ?$

(b) $g'(x) = ?$

(c) $g(0) = ?$

Properties of Definite Integrals

1. $\int_a^a f(x)\,dx =$

2. $\int_a^b f(x)\,dx =$

   (we can change the order of integration limits)

3. $\int_a^c f(x)\,dx =$

   (we can split up a definite integral into the sum of two integrals)

4. $\int_a^b k\, f(x)\,dx$ ($\forall k \in \mathbb{R}$, constant) $=$

   (we can factor out a constant coefficient of a definite integral)

5. $\int_a^b (f(x) \pm g(x))\,dx =$

   (definite integrals distribute through addition/subtraction)

Question: Which two of the above conditions tell that definite integrals are linear operators?

Ex 3: Compute this definite integral.

$$\int_{-4}^{3} |x|\,dx$$

Ex 4: $\int_{0}^{2} f(x)\,dx$ = ? if $f(x)=\begin{cases} x^2-3\text{x} & \text{if } x\geq 1 \\ e^x & \text{if } x<1 \end{cases}$

Ex 5: Find $G'(x)$ for these functions.

(a) $G(x)=\int_{1}^{x} \frac{t^2}{t^5+1}\,dt$

(b) $G(x)=\int_{1}^{x^3-x^2} \frac{t^2}{t^5+1}\,dt$

(c) $G(x)=\int_{x}^{x^3-x^2} \frac{t^2}{t^5+1}\,dt$

(d) $G(x)=\int_{1}^{x} \frac{x^3 t^2}{t^5+1}\,dt$

5.3 Techniques of Integration

We'll go over three different techniques of integration, which will include a couple of the "project" techniques from this book, because they are valuable techniques and worth mentioning.

1. **u-substitution**
2. **integration by parts**
3. **integration using partial fraction decomposition (PFD)**

Ex 1: Evaluate these integrals using u-substitution.
(a) $\int (3x^2+1)e^{x^3+x+2}\,dx$

(b) $\int 5x^2 \sqrt[3]{x^3+89}\,dx$

(c) $\int e^{2x+e^{2x}}\,dx$

(d) $\int \cos x \, \cos(\sin x)\,dx$

Common ("classic") cases for integration by parts, when the integrand function is:

1. polynomial * exponential function.
2. polynomial * sine/cosine function.
3. exponential * sine/cosine function.
4. any function we don't know how to integrate, but we know how to differentiate.

Why does this integration technique work?

Let's look at the product rule for differentiation.

$$D_x(u(x)v(x))=u'(x)v(x)+v'(x)u(x)$$

Or written more shortly: $D_x(uv)=u'v+v'u$

$$\Leftrightarrow v'u=D_x(uv)-u'v$$

$$\Rightarrow \int uv'\,dx=\int (D_x(uv)-u'v)\,dx$$

But remember that notationally $v'=\dfrac{dv}{dx}$, $u'=\dfrac{du}{dx}$, $D_x(uv)=\dfrac{d(uv)}{dx}$.

So we have $\displaystyle\int u\left(\frac{dv}{dx}\right)dx=\int \frac{d(uv)}{dx}\,dx-\int\left(\frac{du}{dx}\right)v\,dx$

And this leads to the integration by parts formula:

Note: $u\,dv$ must account for everything in the entire integral (including the dx).
This is like doing a double u-substitution.

Ex 2: Evaluate these integrals, using integration by parts.

(a) $\displaystyle\int (x^2+1)\cos x\,dx$

(b) $\int e^x \sin x \, dx$

(c) $\int x^3 e^x \, dx$

(d) $\int \ln x \, dx$

Ex 3: Evaluate these integrals, using Partial Fraction Decomposition (PFD).

(a) $\int \frac{6x^3 - 6x^2 - 3x - 2}{x^2(x+1)(x+2)} dx$

(b) $\int \frac{2x^4+4x^3-12x^2+5x-4}{x^3-2x^2+x}\,dx$

Ex 4: Use any techniques you find reasonable to evaluate these integrals.

(a) $\int \frac{(x-2)^2}{\sqrt{x}}\,dx$

(b) $\int \frac{x^2}{\sqrt{x-2}}\,dx$

(c) $\int_{-2}^{0} \frac{3x^5}{\sqrt[3]{(x^3+1)^4}}\,dx$

(d) $\int_{1}^{5} \frac{\sqrt{2x-1}}{8x-3}\,dx$