

**Math6100**  
**Day 8 Notes**  
**6.1, 6.2 & 6.3, Area**

6.1 Area of Polygonal Regions

Let's first derive formulas for the area of these shapes.

1. Rectangle

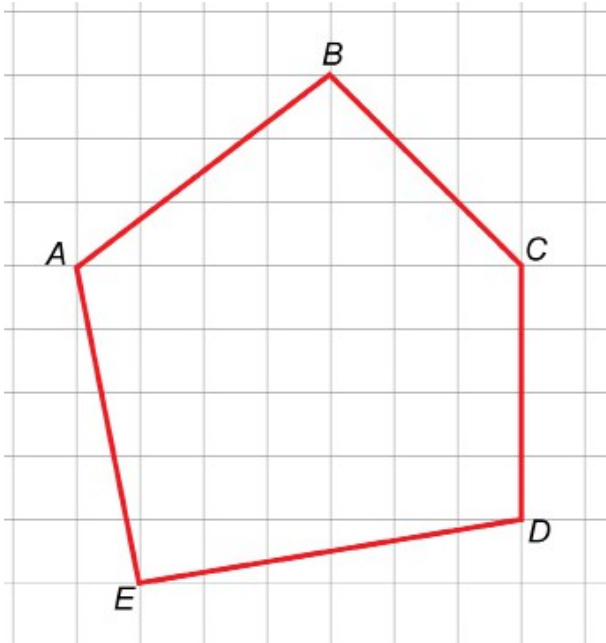
2. Parallelogram

3. Triangle

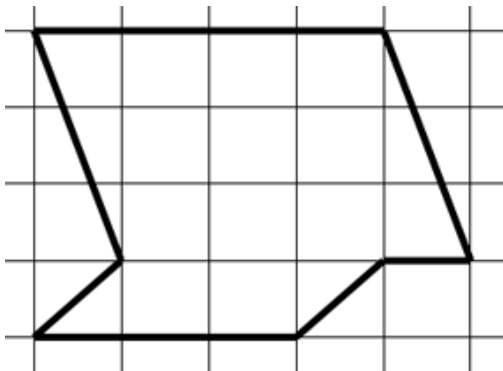
4. Trapezoid

Ex 1: Find the exact area of these polygons. Assume that each grid square has an area of one unit squared.

(a)

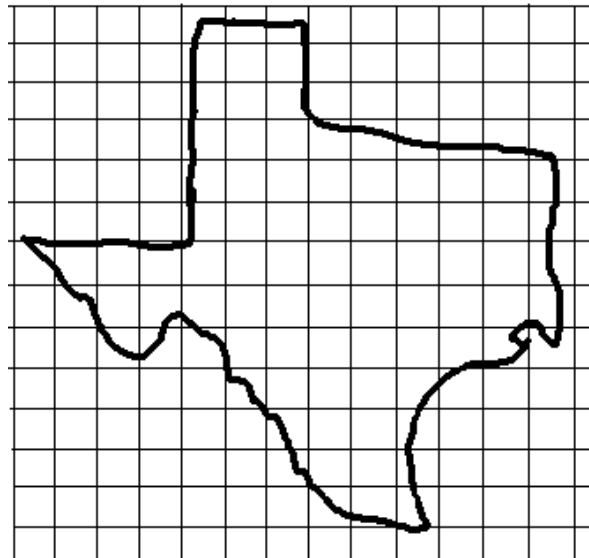


(b)



## 6.2 Approximate Value of Area of Irregular Shapes (non-polygonal)

Ex 1: Use this image, assuming each grid square has an area of one unit squared, to answer the following questions.



(a) Find a reasonable (non-zero) lower bound for the area. Compare with your neighbor.

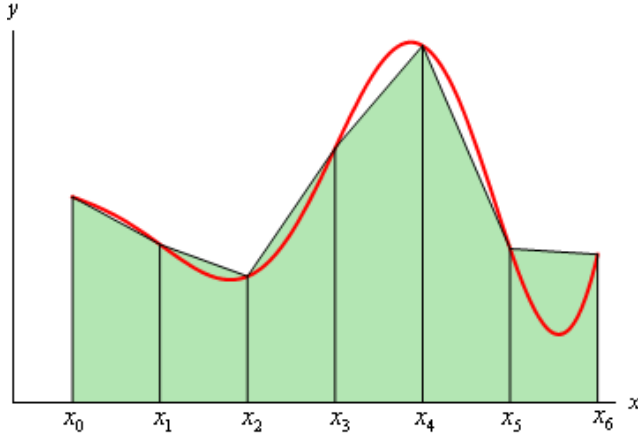
(b) Find a reasonable (finite) upper bound for the area. Compare with your neighbor.

(c) Alice says that 168 square units is her estimate for the area. That's clearly an over-estimate (why?). What would you say is the approximate magnitude of her error?

(d) If you estimate your area by taking the average (arithmetic mean) of the upper and lower bounds, what is an approximate magnitude of that error?

How can we estimate the area “under” a curve (i.e. between the curve and the x-axis)?

### 1. Trapezoid Method



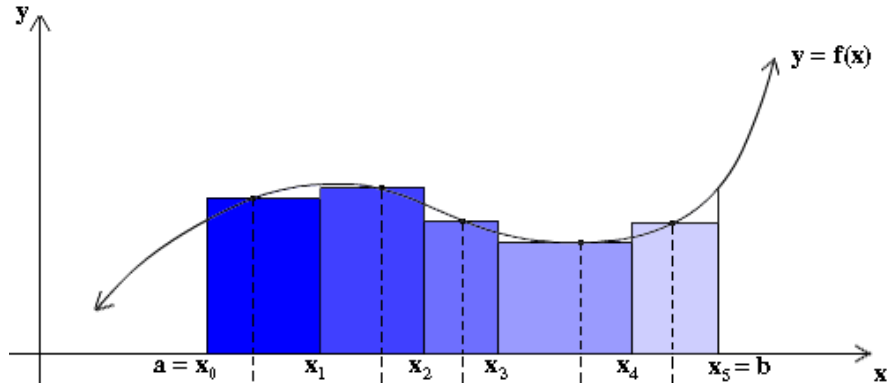
Use this image of a generic graph of  $y = f(x)$  to help create the trapezoid method for estimating area between a generic curve  $y = f(x)$  and the x-axis on the interval  $[a, b]$  (assume  $a < b$ ). Assume  $x_0, x_1, x_2, \dots$  are all arbitrarily chosen.

Now, let's be a bit more efficient and choose x-values that are uniformly distributed on the interval  $[a, b]$ . Assume we'll estimate the area with  $n$  trapezoids.

Then  $\Delta x = \frac{b-a}{n}$ . What is the area formula now for that total estimated area?

## 2. Rectangular Method

Use  $n$  rectangles now, in the three designated ways, to estimate the area between the curve  $y=f(x)$  and the  $x$ -axis on the interval  $[a, b]$ . Again assume the  $x$ -values are uniformly distributed on that interval, which still gives us  $\Delta x = \frac{b-a}{n}$ .



Develop the estimated area formula using rectangles whose heights are given by :

(a) the left endpoints.

(b) the right endpoints.

(c) the midpoints.

What happens with all these formulas if the curve is below the x-axis? (Still assume  $a < b$ .)

Ex 1: Use three different methods to approximate  $\int_0^2 x^3 dx$  .

### 6.3 Exact Value of Area of Irregular Shapes

What happens in all the area estimate formulas we derived in section 6.2 if we let  $n$  go to infinity?

Ex 1: Use the definition of the definite integral (one of the sums we derived as  $n$  goes to infinity) to find the area under the curve  $f(x) = x^2 + 2$  on the interval  $[-1, 3]$ .

## Mystery Notes

(For this lecture, you'll have to suspend some mathematical ideas that you already think you know, so that we can explore this as if for the first time!)

$$D_x\left(\frac{x^3}{3}\right)=x^2$$

$$D_x\left(\frac{x^2}{2}\right)=x$$

$$D_x\left(\frac{x}{1}\right)=1$$

$$D_x(??)=x^{-1}$$

$$D_x\left(\frac{-1}{x}\right)=x^{-2}$$

$$D_x\left(\frac{-1}{2x^2}\right)=x^{-3}$$

What function goes in the place of the ??? This is our driving question. In other words, we are on the hunt for the function whose derivative is  $\frac{1}{x}$ .

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Remember what the First Fundamental Theorem of Calculus states?

If  $f$  is continuous on an open interval  $I$  containing  $a$  and  $x$ , then  $F(x)=\int_a^x f(t)dt$  is a function such that  $F'(x)=f(x)$ .

So, we're guaranteed that  $F(x)=\int_1^x \frac{1}{t} dt$  exists since  $f(t)=\frac{1}{t}$  is a continuous function on any interval that doesn't include zero!

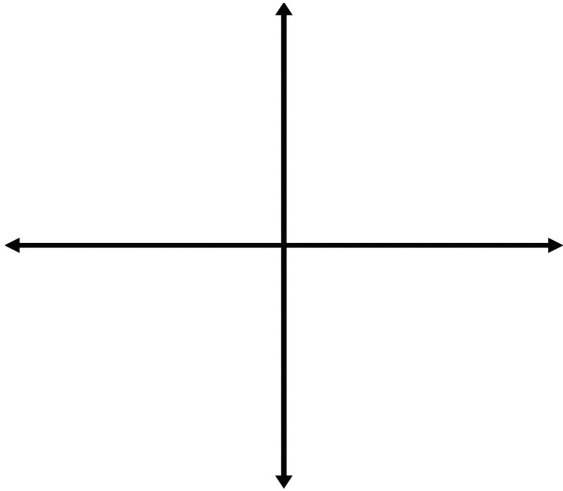


For now, we need to create our own symbol that represents this function that we know exists for all  $x > 0$ .

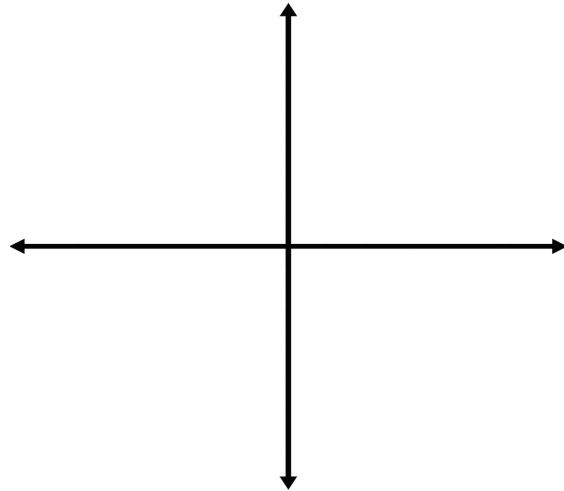
$$= \int_1^x \frac{1}{t} dt, \forall x > 0$$

Let's draw what this means.

$$0 < x < 1$$



$$x > 1$$



Our new function is well defined for  $x > 0$ , but not defined for  $x < 0$  (nor for  $x = 0$ ) because we can't integrate over the interval containing  $x = 0$ .

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Claim:

Proof:

$\implies$

This fills in our gap, namely \_\_\_\_\_.

Properties of our new operator:

$$a, b \in \mathbb{R}^+, r \in \mathbb{Q}$$

(i)

(ii)

(iii)

(iv)

Proof:

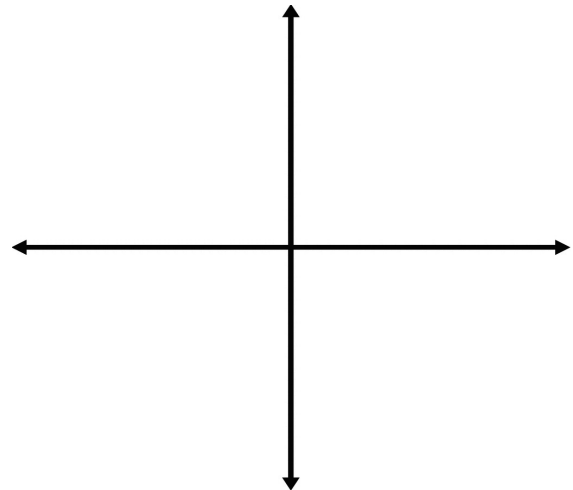
(i)

(ii)

(iii)

(iv)

Let's investigate the shape of our new function.



We can see that an inverse exists!

Let's define the inverse by some new function (that also needs new notation/symbol) as:

Definition: Let  $e \in \mathbb{R}^+$  be a number such that \_\_\_\_\_.

Definition: For  $a > 0$ , \_\_\_\_\_.

Let  $f(x) = a^x$ ,  $a \in \mathbb{R}^+$ ,  $a \neq 1$ . Then we have the \_\_\_\_\_ function that we learned about in algebra.

Definition: For  $a \in \mathbb{R}^+$ ,  $a \neq 1$ , \_\_\_\_\_

$\Leftrightarrow$  \_\_\_\_\_.

Now we can finally prove  $D_x(e^x) = e^x$  (without resorting to a table of values).

Start with  $D_x(\ln x) = \frac{1}{x}$  and  $x = \ln y \Leftrightarrow e^x = y$  and use implicit differentiation.

Our definition of  $e$  is that it's the constant such that  $\ln(e) = 1$ .

Let's look at  $e$  two different ways, in addition to our definition.

(1) Prove that  $e = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}}$  (or equivalently  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ ) .

(2) Prove that  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$  .