Sequence

- a predictable pattern
- a set/list of numbers (w/ or w/o detectable pattern)

Notation:

$\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$, $a_n = a(n)$, where $n \in \mathbb{N}$

$a: \{-37, -36, -35, \ldots\} \to \mathbb{R}$

Ex 1

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, . . .

(a) what color is 31?

green (because it's one past a multiple of 3)

(b) 253,679?

blue we know 253,680 is divisible by 3.

$\Rightarrow$ 253,679 is one fewer than some # div. by 3

wuzzittrouble
ipad/tablet/phone
app to practice
modular arithmetic

(c) if $\frac{n}{3}$ has remainder

$$\begin{cases} 0 & \text{red} \\ 1 & \text{green} \\ 2 & \text{blue} \end{cases}$$

if $n \bmod 3 = \begin{cases} 0 & \text{red} \\ 1 & \text{green} \\ 2 & \text{blue} \end{cases}$

Ex 2 black bead pattern: 1, 2, 3, 4, . . . 7

white bead pattern: 2, 4, 8, 16, 32, . . .

behind cloud: $\underbrace{5+6}_{\text{black}} \underbrace{+16+32+64-7}_{\text{white}}$

$= 4 + 80 + 32 = 116$

1, 2, 2, 4, 3, 8, 4, 16,
5, 32, 6, 64, 7

Ex 3:

	col 1	col 2	col 3	col 4	col 5
		2	3	4	5
	9	8	7	6	
		10	11	12	13
...	17	16	15	14	

(a) col 4 (b) col 2

(c) col 3 (d) col 5

$n \bmod 8 = \begin{cases} 0 & \text{col } 2 \\ 1 & \text{col } 1 \\ 2 & \text{col } 2 \\ 3 & \text{col } 3 \\ 4 & \text{col } 4 \\ 5 & \text{col } 5 \\ 6 & \text{col } 4 \\ 7 & \text{col } 3 \end{cases}$

Ex 4 1 2 3 4 5 6 7 8 9 10 11 12

13 14 15 16 17 18 19 20 21 22

23 24 25 26

perfect squares are only #s w/ odd # of factors

25: 1, 5, 25

26: 1, 2, 13, 26

36: 1, 2, 3, 4, 6, 9, 12, 18, 36

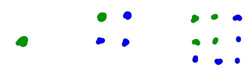
Ex5

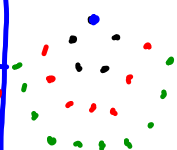
n	t_n	s_n	p_n
1	1	1	1
2	3	4	5
3	6	9	12
4	10	16	22
5	15	25	35
6	21	36	51
n	$\frac{n(n+1)}{2}$	n^2	$\frac{(n-1)n}{2}+n^2 = \frac{3}{2}n^2-\frac{n}{2}$

$n=2,3,4,\dots$

$s_n = s_{n-1} + 2n-1,\ s_1=1$

$2(n-1)+1$





$p_1=1$

$p_n = p_{n-1} + 3n-2$

$(n+n+n-2)$

$t_n = t_{n-1}+n,\ t_1=1$ (recursive)

Prove: $t_n = t_{n-1}+n,\ t_1=1$

$\longrightarrow t_n = \frac{n(n+1)}{2}$ $n=1,2,3,\dots$

Pf (By Induction)

① check $n=1$. $t_1=1$ $\frac{1(1+1)}{2}=1$ ✓

② Assume $t_n=\frac{n(n+1)}{2}$ for some $n\geq 1$.

③ Check $n+1$ case.

$t_{n+1} = t_n + n+1$

$= \frac{n(n+1)}{2} + (n+1)$

$= (n+1)\left(\frac{n}{2}+1\right) = (n+1)\left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2}$ ✗

$p_n = an^2 + bn + c$

(n, p_n) $(1,1)$ $(2,5)$ $(3,12)$

as an exercise:

Prove $p_n = p_{n-1} + 3n - 2$, $p_1 = 1$

$\Rightarrow p_n = \frac{n(n-1)}{2} + n^2 = \frac{3}{2}n^2 - \frac{n}{2}$, $n = 1, 2, 3 \ldots$

(by induction)

arithmetic seq.:

- has common difference (d)
- recursive formula:

  $a_1 = k, \quad a_n = a_{n-1} + d$
- iterative formula:

  $a_n = a_1 + (n-1)d$

geometric seq.

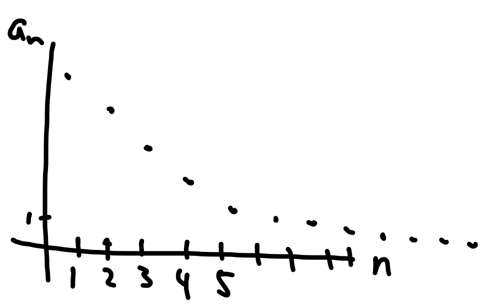
- has common ratio (r)
- recursive formula:

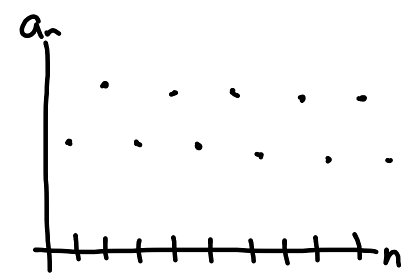
  $a_1 = k, \quad a_n = r\, a_{n-1}$
- iterative formula:

  $a_n = k r^{n-1}$

Convergent sequence:

$\{a_n\}$ is convergent when $\lim\limits_{n\to\infty} a_n = L$, $L < \infty$.

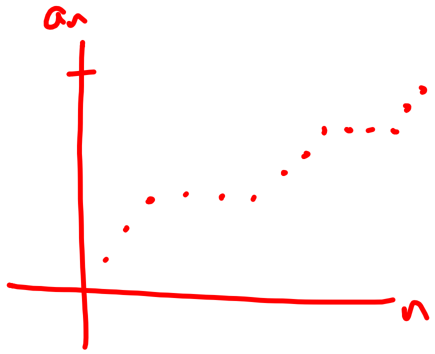




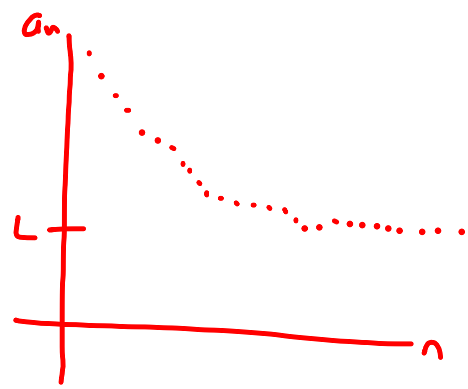
divergent sequence:

$\{a_n\}$ diverges if $\lim\limits_{n\to\infty} a_n = \pm\infty$

OR $\lim\limits_{n\to\infty} a_n$ DNE.



$\{a_n\}$ nondecreasing



Ex 6 (a) $x_n = \frac{1}{n^{1/2}}$ $\{x_n\}$ convergent or divergent?

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$$

(b) $y_n = \frac{3n}{n+100}$

$$\lim_{n\to\infty} \frac{3n}{n+100} = \lim_{n\to\infty} \frac{3n}{n} = 3$$

$$\lim_{n\to\infty} \left(\frac{3n}{n+100}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) = \lim_{n\to\infty} \frac{3}{1+\frac{100}{n}} = \frac{3}{1+\lim_{n\to\infty}\frac{100}{n}} = \frac{3}{1+0} = 3$$

(c) $d_n = 1+(-1)^n$

$\lim_{n\to\infty} (1+(-1)^n)$ DNE

$\Rightarrow \{d_n\}$ diverges

(d) $b_n = \frac{1+(-1)^n}{n}$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{1+(-1)^n}{n} \le \lim_{n\to\infty} \frac{2}{n}$$

$$0 \le \lim_{n\to\infty} \frac{1+(-1)^n}{n} \le 0$$

Ex7

(a) $\lim\limits_{n\to\infty} \frac{(-5)^{n-2}}{3^n}$

$= \lim\limits_{n\to\infty} \frac{(-1)^{n-2}\,5^{n-2}}{3^n}$

$= \lim\limits_{n\to\infty} \frac{(-1)^{n-2}}{5^2}\left(\frac{5}{3}\right)^n$

$= \frac{1}{5^2} \lim\limits_{n\to\infty} (-1)^n \cdot \left(\frac{5}{3}\right)^n$

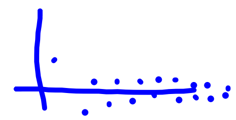
DNE

$\lim\limits_{n\to\infty} \left(\frac{5}{3}\right)^n = \infty$

(b) $\lim\limits_{n\to\infty} \frac{(-5)^{n-2}}{6^n}$

$= \frac{1}{5^2} \lim\limits_{n\to\infty} (-1)^n \left(\frac{5}{6}\right)^n$

$= 0$



(c) $\lim\limits_{n\to\infty} (5-2n)$

$= -\infty$

(d) $\lim\limits_{n\to\infty} \frac{\sin n}{3^n}$

$= \cancel{\frac{\lim\limits_{n\to\infty} \sin n}{\lim\limits_{n\to\infty} 3^n}}$

$\lim\limits_{n\to\infty} \frac{-1}{3^n} \le \lim\limits_{n\to\infty} \frac{\sin n}{3^n} \le \lim\limits_{n\to\infty} \frac{1}{3^n}$

$0 \le \lim\limits_{n\to\infty} \frac{\sin n}{3^n} \le 0$

Sum of an arithmetic seq.

$a_n = a + (n-1)d$

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} [a + (i-1)d]$$

$$= [a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-2)d) + (a+(n-1)d)]$$

$$= \left(\frac{a + (a+(n-1)d)}{2}\right) n = \frac{(a+a+nd-d)n}{2} = \frac{(2a+d(n-1))n}{2}$$

Claim: $S_n = \sum_{i=1}^{n} [a + (i-1)d] = an + \frac{n(n-1)d}{2}$

Pf (by induction)

① $n=1$ case, $a(1) + \frac{1(0)}{2}d = a$, $S_1 = a + 0d = a$ ✓

② Assume $S_n = an + \frac{n(n-1)d}{2}$ for some $p-1$, s.t. $p-1 \geq 1$.

③ check $n=p$ case.

$$S_p = \sum_{i=1}^{p} [a + (i-1)d] = \underbrace{\sum_{i=1}^{p-1} [a + (i-1)d]}_{S_{p-1}} + \underbrace{(a + (p-1)d)}_{a_p}$$

$$= a(p-1) + \frac{(p-1)(p-2)d}{2} + a + (p-1)d$$

$$= ap - \cancel{a} + \cancel{a} + (p-1)d\left[\frac{p-2}{2} + 1\right]$$

$$= ap + (p-1)d\left(\frac{p}{2} - \cancel{1} + \cancel{1}\right)$$

$$= ap + \frac{(p-1)dp}{2}$$

\#

Sum of a geometric seq.

$a_n = a(r^{n-1})$

$S_p = \sum_{n=1}^{p} a(r^{n-1})$

$S_p = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^{p-2}} + \cancel{ar^{p-1}}$

$- rS_p = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \dots + \cancel{ar^{p-1}} + ar^p$

$S_p - rS_p = a - ar^p$

$S_p(1-r) = a(1-r^p)$

$S_p = \dfrac{a(1-r^p)}{1-r}$

Ex8 (a) 300

$24+23+\dots+1$

$=\frac{(24+1)24}{2}$

$=300$

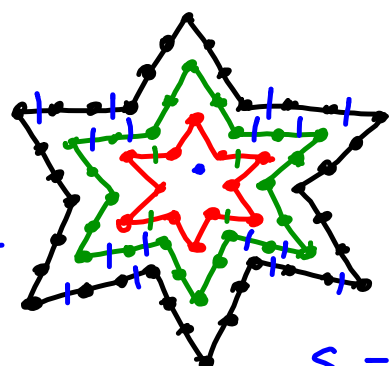
$\binom{25}{2}=\frac{25!}{23!\,2!}=\frac{25\cdot 24}{2}=300$

(b) $27+26+25=78$

Ex 9

n	S_n
1	1
2	13
3	37
4	73
⋮	
n	an^2+bn+c $=6n^2-6n+1$

12, 24, 36; 12, 12



$S_1=1$

$S_n=S_{n-1}+4(2n-1)+4(n-2)$

$S_n=S_{n-1}+12n-12$

$S_n=S_{n-1}+6(2n-3)+6$

$S_n=S_{n-1}+12n-12$

$S_n=an^2+bn+c$ (n,S_n) $(1,1)$ $(2,13)$ $(3,37)$

$a=6$

$b=-6$

$c=1$