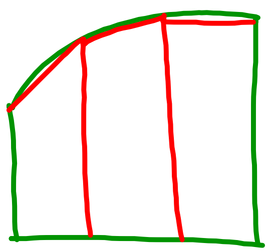
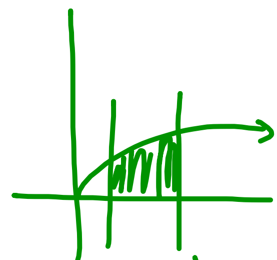
6.2

#2) $y=\sqrt{x}$, x-axis, $x=1$, $x=4$





$$A=\frac{1}{2}\left(f(a)+f(b)+2\sum_{i=1}^{n-1}f(x_i)\right)\Delta x$$

$$=\frac{1}{2}\left(f(1)+f(4)+2\sum_{i=1}^{n-1}\sqrt{1+\frac{3i}{n}}\right)\frac{3}{n}$$

$$=\frac{3}{2n}\left(3+2\sum_{i=1}^{n-1}\sqrt{1+\frac{3i}{n}}\right)$$

$\Delta x=\frac{4-1}{n}=\frac{3}{n}$

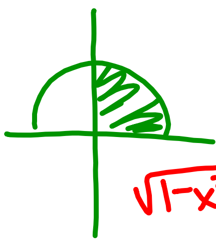
$x_i=1+\frac{3i}{n}$

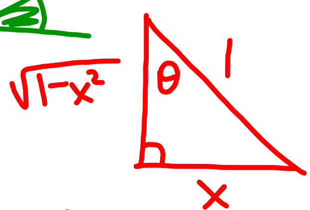
6.3 #13)

$f(x) = \sqrt{1-x^2}$ $\quad a=0,\ b=1$

$A = \int_0^1 \sqrt{1-x^2}\,dx$

$\cancel{u = 1-x^2}$ doesn't work
$\cancel{du = -2x\,dx}$
$\cancel{-\tfrac{1}{2}du = x\,dx}$





$\frac{x}{1} = \sin\theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\sqrt{1-x^2} = \cos\theta$

$x=0,\ \sin\theta = 0$
$\theta = 0$

$x=1,\ \sin\theta = 1$
$\theta = \pi/2$

$A = \int_0^{\pi/2} \cos\theta\cos\theta\,d\theta$

$= \int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta$

$= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$

$= \left(\frac{1}{2}\theta + \frac{1}{2}\left(\frac{1}{2}\right)\sin(2\theta)\right)\Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}\left(\cancel{\sin\pi - \sin 0}\right)^{0}$

$= \frac{\pi}{4}$

6.3

19) $f(x) = \dfrac{x}{x^2+1}$ $a = -\frac{1}{2}$, $b = \frac{3}{2}$

$$\int_{-1/2}^{3/2} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{5/4}^{13/4} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{5/4}^{13/4}$$

$u = x^2+1$

$du = 2x\,dx$

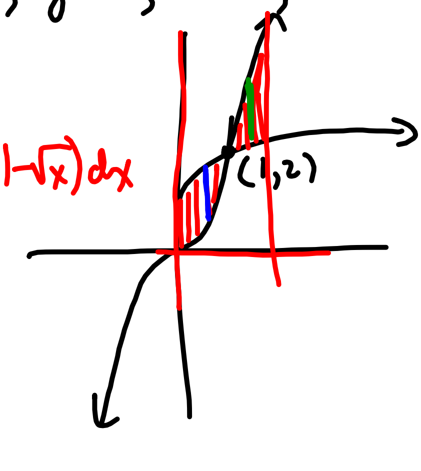
$\frac{1}{2}\,du = x\,dx$

$$= \frac{1}{2}\left(\ln\left(\frac{13}{4}\right) - \ln\left(\frac{5}{4}\right)\right)$$

$$= \frac{1}{2}\ln\left(\frac{13}{5}\right) = \ln\sqrt{\frac{13}{5}}$$

6.3

#23) $f(x) = 2x^3$, $g(x) = 1+\sqrt{x}$, $y=0$, $x=0$, $x=2$



$$A = \int_0^1 (1+\sqrt{x}-2x^3)\,dx + \int_1^2 (2x^3-1-\sqrt{x})\,dx$$

$$= \left(x + \tfrac{2}{3}x^{3/2} + \frac{2x^4}{4}\right)\Big|_0^1$$

$$+ \left(\frac{x^4}{2} - x - \tfrac{2}{3}x^{3/2}\right)\Big|_1^2$$

$$= \left(\cancel{1} + \tfrac{2}{3} + \cancel{\tfrac{1}{2}}\right) + \left(8 - \cancel{2} - \frac{4\sqrt{2}}{3}\right) - \left(\cancel{\tfrac{1}{2}} - \cancel{1} - \tfrac{2}{3}\right) = \frac{4-4\sqrt{2}}{3} + 8$$

$$= \frac{28-4\sqrt{2}}{3}$$

$\int k e^x \, dx$ k is constant

$= k \int e^x \, dx = k e^x + c$

(b) $\int (3x-1) e^x \, dx = \int 3x e^x \, dx - \int e^x \, dx$

$u = 3x \quad v = e^x$

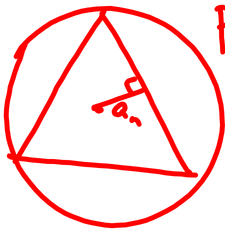
$du = 3\,dx \quad dv = e^x \, dx$

$= 3xe^x - 3\int e^x \, dx - e^x + c = 3xe^x - 3e^x - e^x + C$

$= 3xe^x - 4e^x + c$

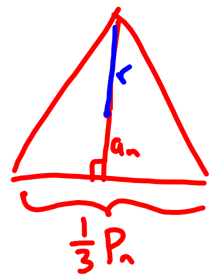
6.4

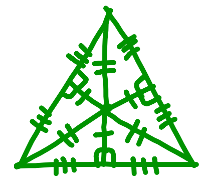
2. Method of Exhaustion



P_n, a_n given

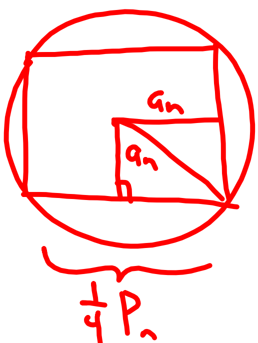
$A_3 = \frac{1}{2}\left(\frac{1}{3}P_n\right)(r + a_n)$



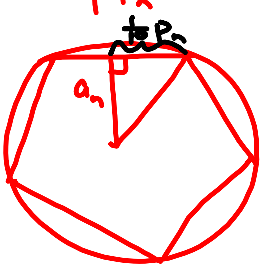


$A_{small} = \frac{1}{2}a_n\left(\frac{1}{6}P_n\right)$

$A_3 = 6\left(\frac{1}{2}a_n\left(\frac{1}{6}P_n\right)\right) = \frac{1}{2}a_nP_n$



$A_4 = 8\left(\frac{1}{2}a_n\left(\frac{1}{8}P_n\right)\right) = \frac{1}{2}a_nP_n$



$A_5 = 10\left(\frac{1}{2}a_n\left(\frac{1}{10}P_n\right)\right) = \frac{1}{2}a_nP_n$

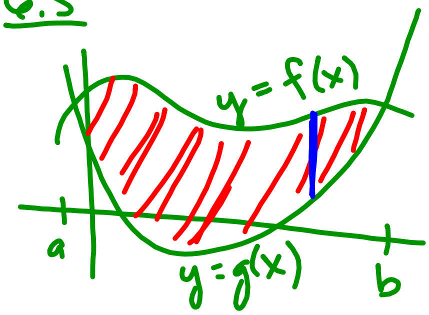
$\lim_{n\to\infty} a_n = r$

$\lim_{n\to\infty} P_n = 2\pi r \Rightarrow \lim_{n\to\infty} A_n =$

$\lim_{n\to\infty} \frac{1}{2}(a_nP_n)$

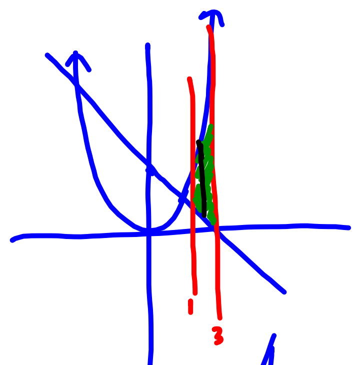
$= \frac{1}{2}(r)(2\pi r) = \pi r^2$

6.5



$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Ex 1 (a) $y = x^2$, $y = 1 - x$, $x = 1$, $x = 3$



intersection:

$x^2 = 1 - x$

$x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{1+4}}{2}$

$x = \frac{-1 \pm \sqrt{5}}{2}$

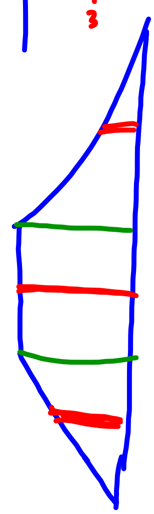
$$A = \int_1^3 \left(x^2 - (1 - x) \right) dx$$

$$= \left(\frac{x^3}{3} - x + \frac{x^2}{2} \right) \Big|_1^3$$

$$= \left(9 - 3 + \frac{9}{2} \right) - \left(\frac{1}{3} - 1 + \frac{1}{2} \right)$$

$$= 6 + \frac{8}{2} + 1 - \frac{1}{3} = 11 - \frac{1}{3}$$

$$= 10\tfrac{2}{3} \text{ or } \frac{32}{3}$$



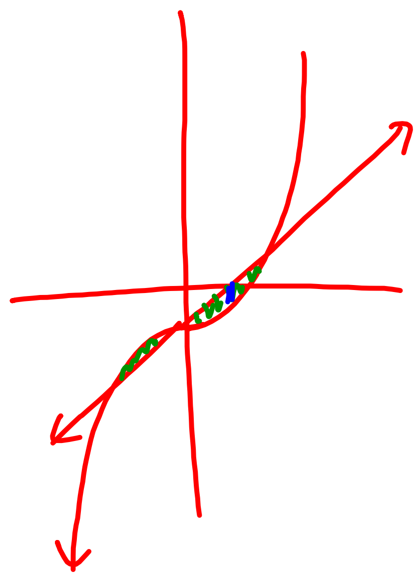
(b) $y = x^3 - 1$, $y = x - 1$

int. pts:

$x^3 - 1 = x - 1$

$x = 1$, $x^3 = x$ $x^2 = 1$

$x = 0$ $x = -1$

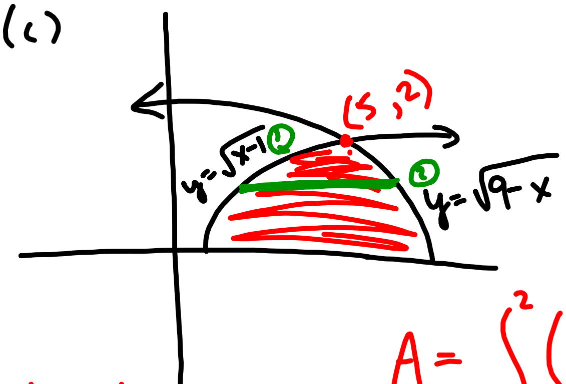


$A = \int_0^1 2(x - \cancel{1} - x^3 + \cancel{1})\,dx$

$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$

(c)



① $y^2 = x-1$
$x = y^2+1$

② $y^2 = 9-x$
$x = 9-y^2$

int. pts:

$y^2+1 = 9-y^2$

$2y^2 = 8$

$y^2 = 4 \Rightarrow y=2$ is pt we want

$$A = \int_0^2 \left(9-y^2-(y^2+1)\right)dy$$

$$= \int_0^2 (8-2y^2)\,dy = \left(8y - \frac{2y^3}{3}\right)\Big|_0^2$$

$$= \left(16 - \frac{16}{3}\right) - 0 = \frac{32}{3}$$

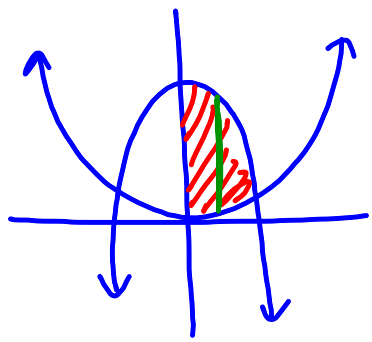
(d) $y = \frac{1}{3}x^2$, $y = 4 - x^2$

$$A = 2\int_0^{\sqrt{3}} \left(4 - x^2 - \frac{1}{3}x^2\right) dx$$

$$= 2\int_0^{\sqrt{3}} \left(4 - \frac{4}{3}x^2\right) dx$$

$$= 2\left(4x - \frac{4}{9}x^3\right)\Big|_0^{\sqrt{3}} = 2\left(4\sqrt{3} - \frac{4}{9}(3\sqrt{3})\right) - 0$$

$$= 2\left(4\sqrt{3} - \frac{4}{3}\sqrt{3}\right) = \frac{16}{3}\sqrt{3}$$



int pts:

$\frac{1}{3}x^2 = 4 - x^2$

$\frac{4}{3}x^2 = 4$

$x^2 = 3$

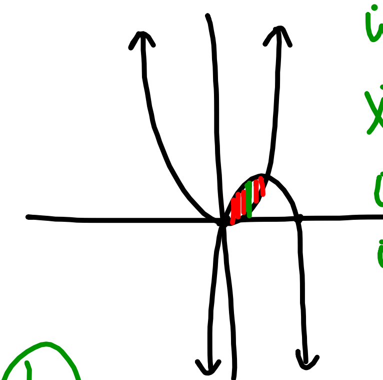
$x = \pm\sqrt{3}$

(e) $y = x^2$, $y = 2x - x^2$

$$A = \int_0^1 (2x - x^2 - x^2)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

$$= \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = \left(1 - \frac{2}{3}\right) - 0 = \frac{1}{3}$$



int pts:

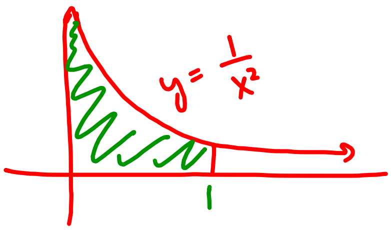
$x^2 = 2x - x^2$

$0 = 2x - 2x^2$

$0 = x - x^2$

$x = 0, 1$

Improper Integrals



$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{x^2}\,dx$$

$$= \lim_{b\to 0^+}\left(\frac{-1}{x}\Big|_b^1\right)$$

$$= \lim_{b\to 0^+}\left(-1 - \frac{-1}{b}\right) \quad \text{diverges}$$

Ex1 (a) $\int_0^1 \frac{1}{x^p}\,dx$ p = constant

1. $p=1$
2. $p>1$
3. $0<p<1$
4. $p=0$
5. $p<0$

① $p=1$: $\lim_{b\to 0^+}\int_b^1 \frac{1}{x}\,dx = \lim_{b\to 0^+} \ln|x|\Big|_b^1$

$$= \ln 1 - \lim_{b\to 0^+}\ln b$$

diverges

② $p>1$: $\lim_{b\to 0^+}\int_b^1 \frac{1}{x^p}\,dx = \lim_{b\to 0^+}\left(\frac{x^{-p+1}}{-p+1}\right)\Big|_b^1$

$$= \frac{1}{-p+1} - \lim_{b\to 0^+}\left(\frac{1}{-p+1}\right)\left(\frac{1}{b^{p-1}}\right)$$

diverges

③ $0<p<1$:

$$\frac{1}{1-p} - \lim_{b\to 0^+}\left(\frac{1}{-p+1}\right)\left(b^{-p+1}\right) = \frac{1}{1-p} - 0 = \frac{1}{1-p}$$

④ $p=0$: $\frac{1}{1-p} - \lim_{b\to 0^+}\left(\frac{1}{1-p}\right)\left(b^1\right) = \frac{1}{1-p} = 1$

⑤ $p<0$: $\frac{1}{1-p} - \lim_{b\to 0^+}\left(\frac{1}{1-p}\right)\left(b^{-p+1}\right) = \frac{1}{1-p}$

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{diverges}, & \text{if } p \ge 1 \\ \frac{1}{1-p}, & \text{if } p<1 \end{cases}$$

(b) $\int_1^{\infty} \frac{1}{x^p}\,dx$

① $p=1$
② $p>1$
③ $p<1$

① $p=1$: $\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln x \Big|_1^a$

$= \lim_{a\to\infty} \ln a - \ln 1$ diverges

② $p>1$: $\lim_{a\to\infty}\int_1^a \frac{1}{x^p}\,dx = \lim_{a\to\infty}\left(\frac{x^{-p+1}}{-p+1}\right)\Big|_1^a$

$= \lim_{a\to\infty} \frac{a^{-p+1}}{-p+1} - \frac{1}{-p+1} = \lim_{a\to\infty}\left(\frac{1}{-p+1}\right)\left(\frac{1}{a^{p-1}}\right) - \frac{1}{1-p}$

$= 0 - \frac{1}{1-p} = \frac{1}{p-1}$

③ $p<1$: $\lim_{a\to\infty} \frac{a^{-p+1}}{-p+1} - \frac{1}{-p+1}$ diverges

$$\int_1^{\infty}\frac{1}{x^p}\,dx \begin{cases} \text{diverges}, & \text{if } p\le 1 \\ \frac{1}{p-1}, & \text{if } p>1 \end{cases}$$

(c) $\displaystyle\int_{-2}^{0} \frac{3x^5}{\sqrt[3]{(x^3+1)^4}}\,dx = \int_{-2}^{0} \frac{x^3\,(3x^2)\,dx}{(x^3+1)^{4/3}} = \int_{-7}^{1} \frac{u-1}{u^{4/3}}\,du$

$u = x^3+1 \Rightarrow x^3 = u-1$

$du = 3x^2\,dx$

$x=-2,\; u=-8+1=-7$

$x=-1,\; u=-1+1=0$

$x=0,\; u=1$

$$= \lim_{a\to 0^-} \int_{-7}^{a} \left(u^{-1/3} - u^{-4/3}\right) du + \lim_{b\to 0^+} \int_{b}^{1} \left(u^{-1/3} - u^{-4/3}\right) du$$

diverges

(d) $\int_{-\infty}^{0} 3(2x-1)^{-4/3}\,dx = \lim_{b\to-\infty} \int_b^0 \frac{3}{(2x-1)^{4/3}}\,dx$

$= 3\lim_{b\to-\infty}\int_b^0 (2x-1)^{-4/3}\,dx = 3\lim_{b\to-\infty} \left.\frac{-3(2x-1)^{-1/3}}{2}\right|_b^0 = \frac{-9}{2}\left(-1 - \lim_{b\to-\infty}\cancel{\frac{1}{\sqrt[3]{2b-1}}}\right)$ $\to 0$

$= \frac{9}{2}$

$u = 2x-1$ $\quad x\to-\infty, u\to-\infty$

$du = 2\,dx$ $\quad x=0, \; u=-1$

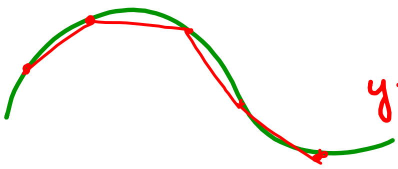
$\frac{1}{2}\,du = dx$

$\lim_{b\to-\infty}\int_b^{-1} \frac{3}{2} u^{-4/3}\,du$

$= \frac{3}{2}\left(\frac{1}{\frac{4}{3}-1}\right) = \frac{3}{\frac{8}{3}-2}\cdot\frac{3}{3}$

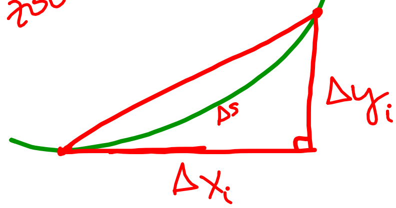
$= \frac{9}{8-6} = \frac{9}{2}$

7.1

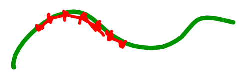


$y=f(x)$

zoomed in: $S=$ arc length



$$\Delta s \approx \sqrt{\Delta x_i^2 + \Delta y_i^2}$$



$$\Delta x_i = x_{i+1} - x_i$$

$$\Delta y_i = y_{i+1} - y_i = f(x_{i+1}) - f(x_i)$$

$$f'(x_{i+1}) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{\Delta y_i}{\Delta x_i} \Rightarrow \Delta y_i \approx f'(x_{i+1})\Delta x_i$$

$$\Rightarrow \Delta s \approx \sqrt{(\Delta x_i)^2 + \left(f'(x_{i+1})\Delta x_i\right)^2}$$

$$= \sqrt{\Delta x_i^2\left[1 + \left(f'(x_{i+1})\right)^2\right]} = \sqrt{1 + \left(f'(x_{i+1})\right)^2}\ \Delta x_i$$

$$\Rightarrow ds = \sqrt{1 + (f'(x))^2}\ dx$$

$$L = \int_a^b ds = \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\ dx}$$

Side Note

① If we have $x = g(y)$, then

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\ dy$$

$x = g(t)$
$y = t$

② If we have $x = h(t)$, $y = \ell(t)$, then

$$L = \int_p^q \sqrt{(h'(t))^2 + (\ell'(t))^2}\ dt$$

most general

Ex1 $y=x^2$ $(0,0)$ to $(2,4)$ $f'(x)=2x$

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx=\int_0^2\sqrt{1+(2x)^2}\,dx$$

Ex2 $y=10\left(e^{\frac{x}{20}}+e^{\frac{-x}{20}}\right)$ $y'=10\left(\frac{1}{20}e^{x/20}-\frac{1}{20}e^{-x/20}\right)$

$$=\frac{1}{2}e^{x/20}-\frac{1}{2}e^{-x/20}$$

$$L=\int_0^{40}\sqrt{1+\frac{1}{4}\left(e^{x/20}-e^{-x/20}\right)^2}\,dx$$

$$=\int_0^{40}\sqrt{1+\frac{1}{4}\left(e^{x/10}-2+e^{-x/10}\right)}\,dx$$

$$=\int_0^{40}\sqrt{1+\left[\frac{1}{4}e^{x/10}-\frac{1}{2}+\frac{1}{4}e^{-x/10}\right]}\,dx=\int_0^{40}\sqrt{\frac{1}{4}e^{x/10}+\frac{1}{2}+\frac{1}{4}e^{-x/10}}\,dx$$

$$=\int_0^{40}\sqrt{\left(\frac{1}{2}e^{x/20}+\frac{1}{2}e^{-x/20}\right)^2}\,dx=\frac{1}{2}\int_0^{40}\left(e^{x/20}+e^{-x/20}\right)dx$$

$$=\frac{1}{2}\left(20e^{x/20}-20e^{-x/20}\right)\Big|_0^{40}=10\left[\left(e^2-e^{-2}\right)-\left(e^0-e^0\right)\right]$$

$$=10\left(e^2-\frac{1}{e^2}\right)$$

Ex 3 $y = \frac{1}{3}(x^2+2)^{3/2}$ $x \in [0,2]$ $y' = \frac{1}{2}(x^2+2)^{1/2}(2x)$

$= x\sqrt{x^2+2}$

$$L = \int_0^2 \sqrt{1 + x^2(x^2+2)}\, dx$$

$$= \int_0^2 \sqrt{x^4+2x^2+1}\, dx = \int_0^2 \sqrt{(x^2+1)^2}\, dx = \int_0^2 (x^2+1)\, dx$$

$$= \left(\frac{x^3}{3}+x\right)\Big|_0^2 = \frac{8}{3}+2 = \frac{14}{3}$$

Ex 4 $x = r\cos\theta$ r fixed $\theta \in [0, 2\pi]$

$y = r\sin\theta$ $x' = -r\sin\theta$, $y' = r\cos\theta$

$$L = \int_c^d \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta = r\theta\Big|_0^{2\pi} = r(2\pi - 0) = 2\pi r$$

Ex5

$x = \frac{1}{4}t^4 + 5 \quad y = \frac{1}{6}t^6 - 1 \quad t \in [0, 2]$

$x'(t) = t^3 \quad y'(t) = t^5$

$$L = \int_0^2 \sqrt{(t^3)^2 + (t^5)^2}\, dt = \int_0^2 \sqrt{t^6 + t^{10}}\, dt = \int_0^2 t^3 \sqrt{1 + t^4}\, dt$$

$u = 1 + t^4$

$du = 4t^3\, dt$

$\frac{1}{4}\, du = t^3\, dt$

$$= \frac{1}{4}\int_1^{17} \sqrt{u}\, du = \frac{1}{4}\left(\frac{2}{3} u^{3/2}\Big|_1^{17}\right)$$

$$= \frac{1}{6}\left(17^{3/2} - 1\right)$$