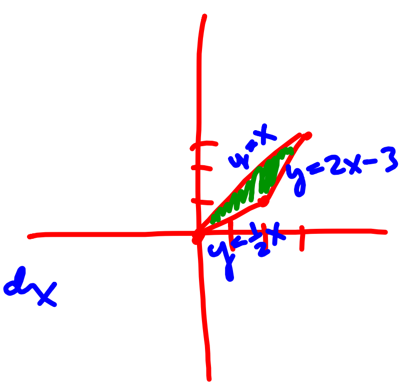
6.4 #2) $A = \pi r^2$, $A_1 = 3r^2$, $A_2 = \frac{c^2}{12} = \frac{(2\pi r)^2}{12} = \frac{\pi^2 r^2}{3}$

(b) $A_1 - A = 3r^2 - \pi r^2 = (3-\pi)r^2$

(c) $\frac{(3-\pi)\cancel{r^2}}{\pi \cancel{r^2}} = \frac{3-\pi}{\pi}$

6.5 #9) A(0,0) B(2,1) C(3,3)

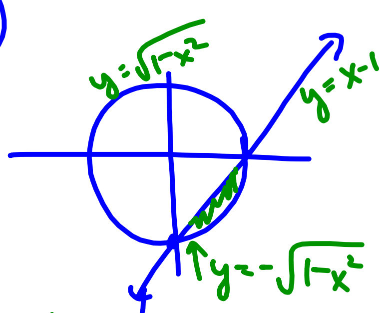



$A = \int_0^2 \left(x - \frac{1}{2}x\right) dx + \int_2^3 (x - 2x + 3)\, dx$

$=$

6.5

11)




$x^2+y^2=1$

$y=-\sqrt{1-x^2}$

$$A=\int_0^1 x-1-(-\sqrt{1-x^2})\,dx$$

$$=\int_0^1 (x-1)\,dx+\int_0^1\sqrt{1-x^2}\,dx$$

7.1

2) $y=\frac{1}{x}$ $x\in[1,3]$ $\Delta x=\frac{2}{n}$ $x_i=1+\frac{2i}{n}$

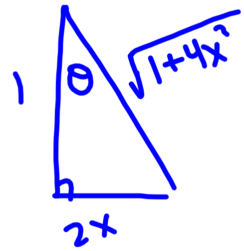
$$L\approx\sum_{i=1}^{n}\Delta s_i\approx\sum_{i=1}^{n}\sqrt{\Delta x_i^2+\Delta y_i^2}=\sum_{i=1}^{n}\sqrt{\left(\frac{2}{n}\right)^2+\left(\frac{1}{1+\frac{2i}{n}}-\frac{1}{1+\frac{2(i-1)}{n}}\right)^2}$$

$$L\approx\sum_{i=0}^{n-1}\sqrt{\left(\frac{2}{n}\right)^2+\left(\frac{1}{1+\frac{2(i+1)}{n}}-\frac{1}{1+\frac{2i}{n}}\right)^2}$$

7.1

#5) $f(x) = 1 - x^2$ $\quad x \in [0,1]$ $\quad f'(x) = -2x$

$$L = \int_0^1 \sqrt{1+4x^2}\, dx$$




$\frac{2x}{1} = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$\sqrt{1+4x^2} = \sec\theta$

$$= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta$$

$u = \sec\theta \qquad v = \tan\theta$

$du = \sec\theta\tan\theta\, d\theta \qquad dv = \sec^2\theta\, d\theta$

$$= \frac{1}{2}\left(\sec\theta\tan\theta\Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} \sec\theta\tan^2\theta\, d\theta\right)$$

$$\frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec(\tan^{-1}(2))(2) - \int_0^{\tan^{-1}(2)} (\sec^3\theta - \sec\theta)\, d\theta\right)$$

$$\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta = \sec(\tan^{-1}(2)) + \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec\theta\, d\theta$$

$$= \sec(\tan^{-1}(2)) + \frac{1}{2}\left(\ln|\sec\theta + \tan\theta|\right)\Big|_0^{\tan^{-1}(2)}$$

$$= \sec(\tan^{-1}(2)) + \frac{1}{2}\left(\ln|\sec(\tan^{-1}(2)) + 2|\right) - 0$$

$$= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$$

#6) $f(x) = x^3 - x \qquad x \in [-5, 2]$

$f'(x) = 3x^2 - 1$

$$L = \int_{-5}^{2} \sqrt{1 + (3x^2 - 1)^2}\, dx$$

#7) $f(x) = 2x^2 + x \qquad x \in [-3, 3]$

$f'(x) = 4x + 1$

$$L = \int_{-3}^{3} \sqrt{1 + 16x^2 + 8x + 1}\, dx = \int_{-3}^{3} \sqrt{2(8x^2 + 4x + 1)}\, dx$$

$$= \sqrt{2} \int_{-3}^{3} \sqrt{8(x^2 + \tfrac{1}{2}x + \tfrac{1}{16}) + 1 - \tfrac{1}{2}}\, dx$$

$$= \int_{-3}^{3} \sqrt{16(x + \tfrac{1}{4})^2 + 1}\, dx$$

$= xe^x(2+x)$

8) $f(x) = x^2 e^x \quad x \in [-1,2] \quad f'(x) = 2xe^x + x^2 e^x$

$$L = \int_{-1}^{2} \sqrt{1 + (2xe^x + x^2 e^x)^2}\, dx = \int_{-1}^{2} \sqrt{1 + x^2 e^{2x}(4 + 4x + x^2)}\, dx$$

11) $L = \int_0^2 \sqrt{1+x^2}\, dx \qquad x = f'(x) \Rightarrow \frac{x^2}{2} + c = f(x)$

14) $L = \int_1^4 x^{-2}\sqrt{x^4+1}\, dx = \int_1^4 \sqrt{1 + x^{-4}}\, dx$

$$= \int_1^4 \sqrt{1 + (x^{-2})^2}\, dx$$

$x^{-2} = (x^{-4})^{1/2}$

$f'(x) = x^{-2} \Rightarrow f(x) = \frac{-1}{x} + c$

$x^{-2}\sqrt{x^4+1} = \sqrt{x^{-4}}\sqrt{x^4+1}$

$= \sqrt{x^{-4}(x^4+1)}$

$= \sqrt{1 + x^{-4}}$

16) $f(x) = \frac{25}{2}\left(e^{x/25} + e^{-x/25}\right)$

$f'(x) = \frac{25}{2}\left(\frac{1}{25}e^{x/25} - \frac{1}{25}e^{-x/25}\right)$

$25/2\,(e^{x/25} + e^{-x/25})$

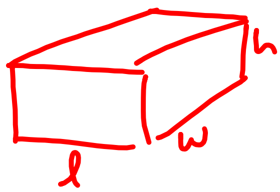
$\neq 25/(2\,(e^{x/25} + e^{-x/25}))$

$25/2(x) = 25/2x$

$\frac{25(x)}{2}$

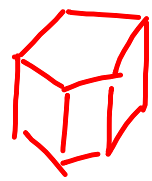
# 7.3 Volume

rect. prism




$V = lwh$

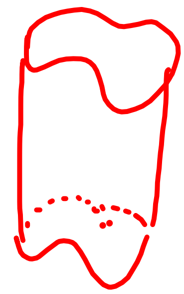
any prism:



$V = Ah$

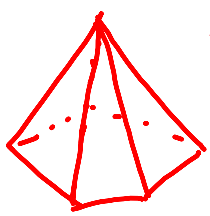
$A$ = area of base

any cylinder:

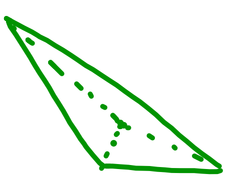


$V = Ah$

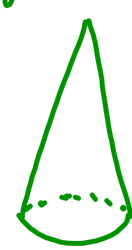
any pyramid.



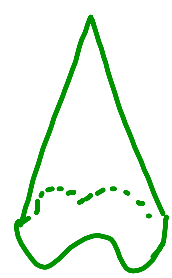
$V = \frac{1}{3}Ah$



right circular cone:



$V = \frac{1}{3}Ah$



Sphere:

$V = \frac{4}{3}\pi r^3$

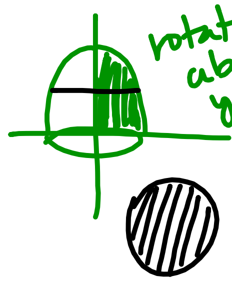


front view

$$V=\int_a^b \pi\left(r_o^2-r_i^2\right)dx$$
(or dy)

Washer

| | disk/washer | shell |
| --- | --- | --- |
| rotate about horiz. axis | dx | dy |
| rotate about vert. axis | dy | dx |

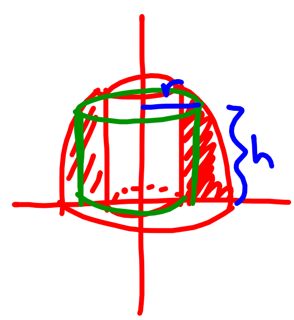



disk

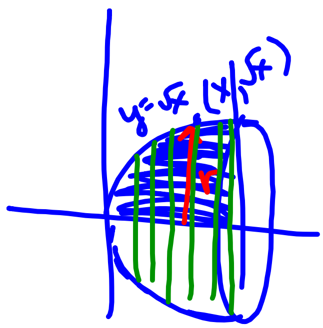
$$V=\pi\int_c^d r^2\,dy$$
(or dx)

shell

$$V=2\pi\int_a^b rh\;dx$$
(or dy)



Ex 1 (a) $y=\sqrt{x}$, x-axis, $x=9$, about x-axis



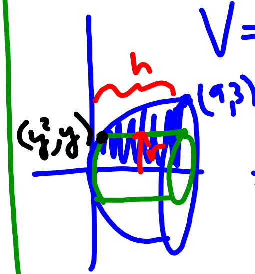

disk

$$V=\pi\int_0^9 r^2\,dx$$

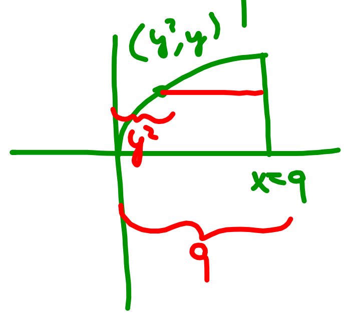
$$=\pi\int_0^9(\sqrt{x})^2\,dx$$

shell $y=\sqrt{x}\Rightarrow x=y^2$

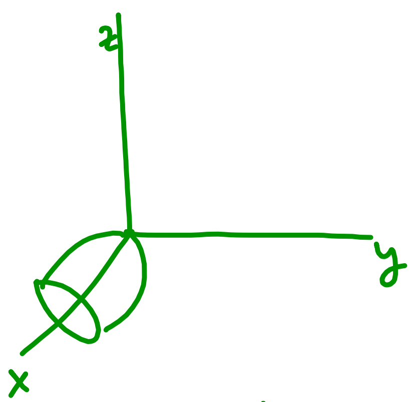
$$V=2\pi\int_0^3 rh\,dy$$




$$=2\pi\int_0^3 y(9-y^2)\,dy$$




Use Multivariable Calculus:

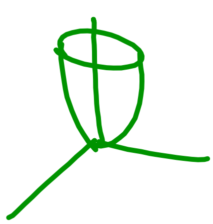



$x=0$ to $x=9$

$y^2+z^2=x$ (paraboloid)

$$V=\int_e^f\int_c^d\int_a^b dV$$

$\Leftrightarrow$

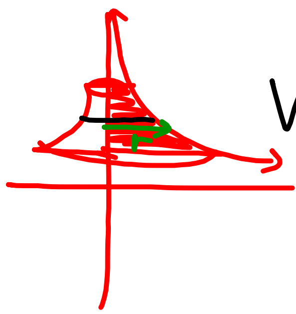


$x^2+y^2=z \rightarrow z=r^2$

$$V=\int_0^{2\pi}\int_0^3\int_{r^2}^9 dz\,(r\,dr\,d\theta)$$

Ex2

(a) $x = \frac{2}{y}$, $y=2$, $y=6$, $x=0$, about y-axis

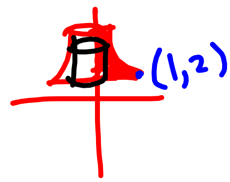


disk

$$V = \pi \int_2^6 r^2 \, dy$$

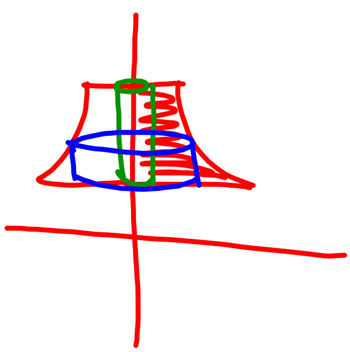
$$= \pi \int_2^6 \left(\frac{2}{y}\right)^2 dy$$

shell

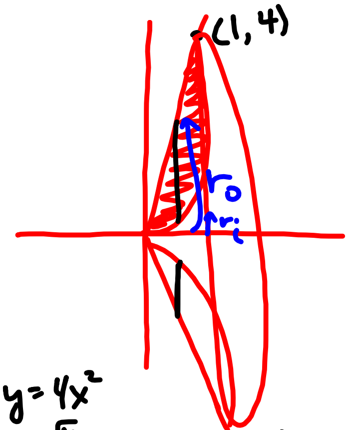



$$V = 2\pi \int_a^b rh \, dx$$

$$V = 2\pi \int_0^{1/3} x(6-2)\,dx + 2\pi \int_{1/3}^{1} x\left(\frac{2}{x} - 2\right) dx$$



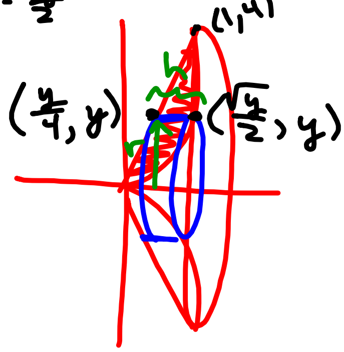
(b) $y = 4x$ , $y = 4x^2$ about x-axis




washer: $V = \pi \int_0^1 (r_o^2 - r_i^2)\, dx$

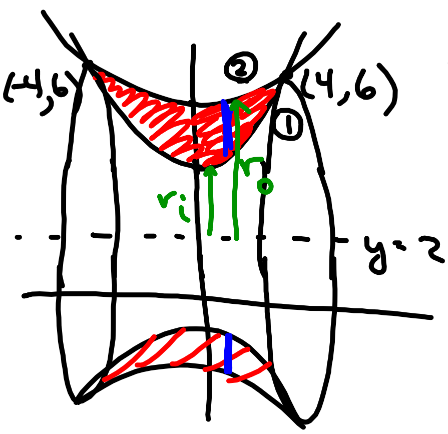
$= \pi \int_0^1 \left( (4x)^2 - (4x^2)^2 \right) dx$

$y = 4x^2$
$x = \frac{\sqrt{y}}{2}$




shell $V = 2\pi \int_0^4 r h \, dy = 2\pi \int_0^4 y \left( \frac{\sqrt{y}}{2} - \frac{y}{4} \right) dy$

(c) ① $y = \frac{3}{16}x^2 + 3$, ② $y = \frac{1}{16}x^2 + 5$, about $y = 2$

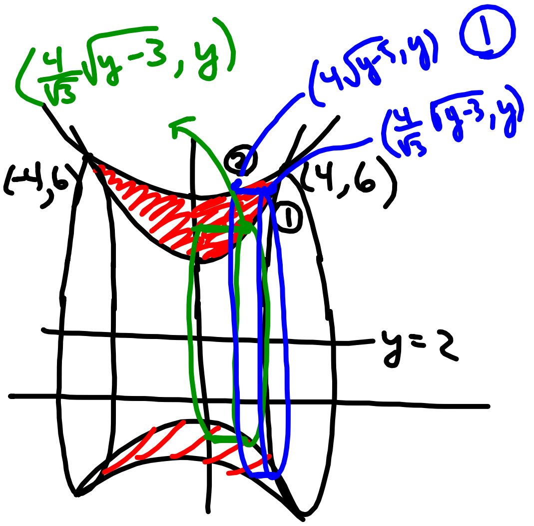



int pts

$$\frac{3}{16}x^2 + 3 = \frac{1}{16}x^2 + 5$$
$$\frac{1}{8}x^2 = 2$$
$$x^2 = 16$$
$$x = \pm 4$$

washer

$$V = 2\pi \int_0^4 \left(r_o^2 - r_i^2\right) dx$$
$$= 2\pi \int_0^4 \left[\left(\frac{1}{16}x^2 + 5 - 2\right)^2 - \left(\frac{3}{16}x^2 + 3 - 2\right)^2\right] dx$$




① $\sqrt{\frac{(y-3)16}{3}} = x$

$$x = \frac{4\sqrt{y-3}}{\sqrt{3}}$$

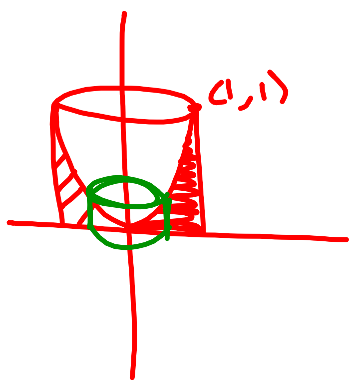
② $x = \sqrt{(y-5)16}$

$$x = 4\sqrt{y-5}$$

$$V = 2\left[V_1 + V_2\right]$$
$$V_1 = 2\pi \int_3^5 rh\, dy$$
$$V_2 = 2\pi \int_5^6 rh\, dy$$

$$V_1 = 2\pi \int_3^5 (y-2)\left(\frac{4}{\sqrt{3}}\sqrt{y-3}\right) dy$$
$$V_2 = 2\pi \int_5^6 (y-2)\left(\frac{4}{\sqrt{3}}\sqrt{y-3} - 4\sqrt{y-5}\right) dy$$

(d) $y=x^3$, $x=1$, $y=0$, about y-axis

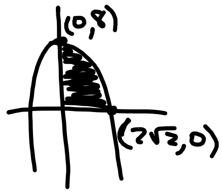


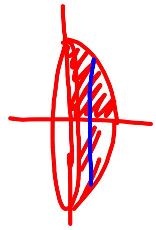

shell:

$$V = 2\pi\int_0^1 rh\,dx$$

$$= 2\pi\int_0^1 x(x^3)\,dx$$

(e) $y = 8 - x^2$, $x=0$, $y=0$ (in QI)

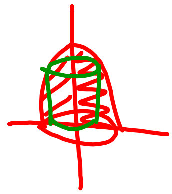



1) 

about x-axis

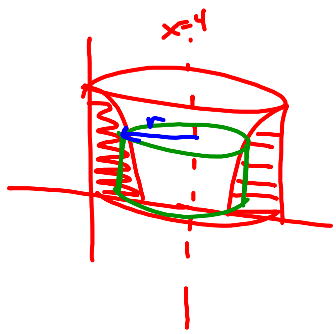
disk $V = \pi \int_0^{2\sqrt{2}} r^2 \, dx$

$$V = \pi \int_0^{2\sqrt{2}} (8-x^2)^2 \, dx$$

2) 

shell: $V = 2\pi \int_0^{2\sqrt{2}} rh \, dx$

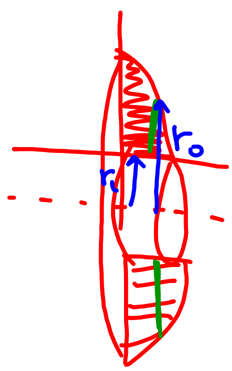
$$= 2\pi \int_0^{2\sqrt{2}} x(8-x^2)\, dx$$

3) 


shell
$V = 2\pi \int_0^{2\sqrt{2}} rh \, dx$

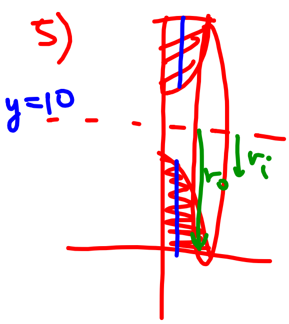
$$= 2\pi \int_0^{2\sqrt{2}} (4-x)(8-x^2)\, dx$$

4) about $y=-2$




washer
$V = \pi \int_0^{2\sqrt{2}} (r_o^2 - r_i^2)\, dx$

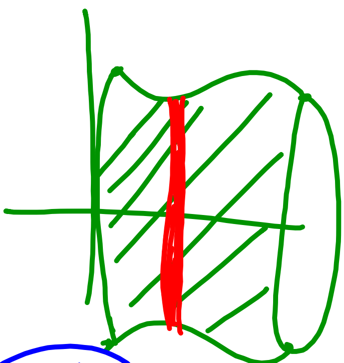
$$= \pi \int_0^{2\sqrt{2}} \left( (8-x^2+2)^2 - 2^2 \right) dx$$

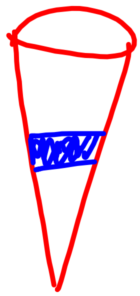
5) 


washer $V = \pi \int_0^{2\sqrt{2}} (r_o^2 - r_i^2)\, dx$

$$= \pi \int_0^{2\sqrt{2}} \left( 10^2 - \left(10 - (8-x^2)\right)^2 \right) dx$$

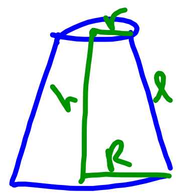
# 7.2 Surface Area





frustrum

$$SA \approx 2\pi \underbrace{\left(\frac{r+R}{2}\right)}_{r \text{ in the integral}} \underbrace{\ell}_{ds}$$




approx SA $= \sum_{i=1}^{n} 2\pi r_i \ell_i$

$$SA = 2\pi \int_a^b r \, ds$$

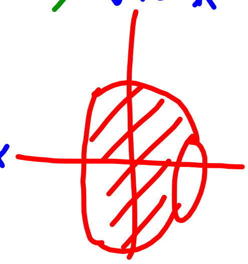
about x-axis

$$SA = 2\pi \int_a^b r \sqrt{1+(f'(x))^2}\, dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1+(f'(x))^2}\, dx$$

Ex 1

$y = \sqrt{16-x^2}$ on $[-1,3]$ $\quad y' = \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{16-x^2}}$

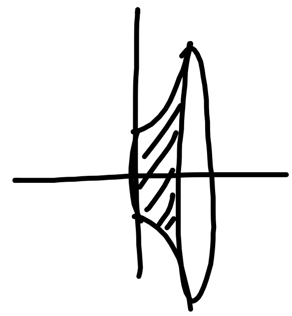


$$SA = 2\pi \int_{-1}^{3} \sqrt{16-x^2}\sqrt{1+\frac{x^2}{16-x^2}}\,dx$$

$$= 2\pi \int_{-1}^{3} \sqrt{16-\cancel{x^2}+\cancel{x^2}}\,dx = 2\pi(4)\left(x \Big|_{-1}^{3}\right)$$

$$= 8\pi\left(3-(-1)\right) = 32\pi$$

Ex 2 $\quad y=e^x \quad$ on $[0,2]$



$$SA = 2\pi\int_0^2 e^x\sqrt{1+e^{2x}}\,dx$$

$u=1+e^{2x} \Rightarrow u-1=e^{2x} \Rightarrow \sqrt{u-1}=e^x$

$du=2e^{2x}dx$

$\frac{2}{u-1}du=dx$

$$SA=2\pi\int_2^{e^4+1}\frac{\sqrt{u-1}\,\sqrt{u}\;2}{u-1}\,du$$

$$=4\pi\int_2^{e^4+1}\sqrt{\frac{u}{u-1}}\,du$$ getting nowhere

$$SA=\int_0^2 2\pi\, e^x\sqrt{1+e^{2x}}\,dx = 2\pi\int_1^{e^2}\sqrt{1+u^2}\,du$$

$u=e^x$

$du=e^x dx$