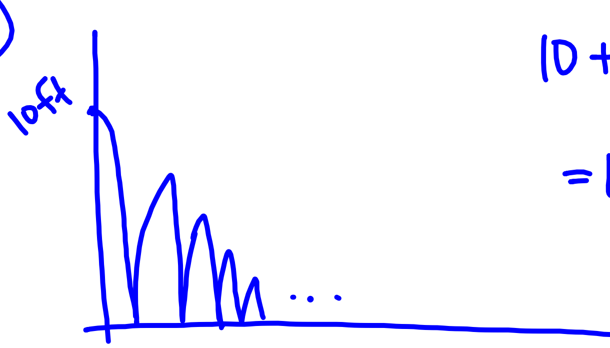
1.2 #21)



$$10 + 2\left(\tfrac{3}{4}(10) + \left(\tfrac{3}{4}\right)^2(10) + \left(\tfrac{3}{4}\right)^3(10) + \dots\right)$$

$$= 10 + 2(10)\sum_{n=1}^{\infty}\left(\tfrac{3}{4}\right)^n$$

$$= 10 + 20\left(\frac{3/4}{1 - 3/4}\right)$$

$$= 10 + 20(3) = 70 \text{ ft.}$$

Inverse Fns

- if we think of fn going "forward", then inverse fn goes "backward"
- graphically $y = f(x)$ and $y = f^{-1}(x)$ are reflections across line $y = x$.

ex

	$f(x)$	$f^{-1}(x)$
domain	$(-\infty, 0]$	$[1, 5]$
range	$[1, 5]$	$(-\infty, 0]$

i.e.

range of f = domain of f^{-1}

and

domain of f = range of f^{-1}

Ex8 $\cancel{f: A \to B}$ bad notation

$B \neq \text{Image}(f)$, $A = \text{domain}(f)$

$f(x) = y \quad x \in A, \ y \in B$

$I = \text{range}(f) \subset B$

$f: A \to I$

If f^{-1} exists, then

$f^{-1}: I \to A$

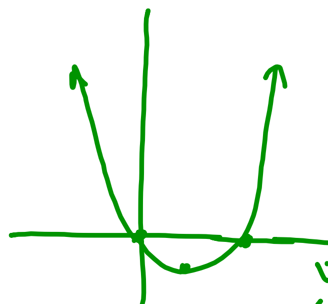
(not $f^{-1}: B \to A$)

Ex9 1-1 means $\forall x \ \exists! f(x)$

Ex 11:

(a) $f(x) = x^2 - x$, $\cancel{f: \mathbb{R} \to \mathbb{R}}$

$f: \mathbb{R} \to [\frac{-1}{4}, \infty)$



$f^{-1}(x)$ DNE

(doesn't pass HLT)

(b) $f(x) = x^2 - x$, $f: (-\infty, \frac{1}{2}] \to [\frac{-1}{4}, \infty)$ $f^{-1}(x)$ exists

$y = x^2 - x \qquad \frac{1}{4} + y = x^2 - x + \frac{1}{4}$

$y = x(x-1) \qquad \frac{1}{4} + y = (x - \frac{1}{2})^2$

$-\sqrt{\frac{1}{4} + y} = x - \frac{1}{2} \quad \longleftarrow$

$\frac{1}{2} - \sqrt{\frac{1}{4} + y} = x$

$f^{-1}(x) = \frac{1}{2} - \sqrt{\frac{1}{4} + x}$ domain: $[\frac{-1}{4}, \infty)$ ✓

range: $(-\infty, \frac{1}{2}]$

(c) $f(x) = x^2 - x \qquad f: [\frac{1}{2}, \infty) \to [-\frac{1}{4}, \infty)$

$f^{-1}(x)$ does exist

$f^{-1}(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}$

2.2 Limits of Fns

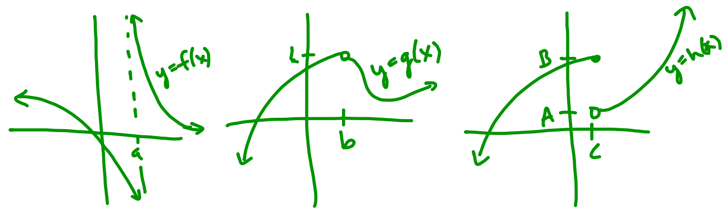
Ex 1: $f(x) = \dfrac{x^2(x-1)(x+3)}{x^2(x-1)(x+1)}$ (a) domain(f) $= \{x \mid x \in \mathbb{R}, x \neq 0, \pm 1\}$

(b) $f(x) = \dfrac{(x+3)}{x+1},\ x \neq 0, 1$ yes

(c) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x+3}{x+1} = \frac{3}{1}$ $\qquad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x+3}{x+1} = \frac{4}{2} = 2$

$\lim_{x \to 0^+} f(x) = 3$ $\qquad \lim_{x \to 1^+} f(x) = 2$

ex



$\left.\begin{array}{l} \lim_{x \to a^-} f(x) = -\infty \\ \lim_{x \to a^+} f(x) = \infty \end{array}\right\}$ VA

$\lim_{x \to a} f(x)$ DNE

$\left.\begin{array}{l} \lim_{x \to b^-} g(x) = L \\ \lim_{x \to b^+} g(x) = L \\ \lim_{x \to b} g(x) = L \end{array}\right\}$ hole

$g(b) \neq L$

$\left.\begin{array}{l} \lim_{x \to c^-} h(x) = B \\ \lim_{x \to c^+} h(x) = A \\ \lim_{x \to c} h(x) \text{ DNE} \end{array}\right\}$ jump

(d) $f(x) = \dfrac{x^2(x-1)(x+3)}{x^2(x-1)(x+1)}$

$x=0$ and at $x=1$ there is a hole.

what about at $x=-1$? $\lim_{x \to -1^-} \dfrac{\cancel{x^2}\cancel{(x-1)}(x+3)}{\cancel{x^2}\cancel{(x-1)}(x+1)} = \lim_{x \to -1^-} \dfrac{x+3}{x+1}$

$\left(\frac{\text{nonzero}}{0} \text{ case}\right)$ $\qquad = -\infty \qquad \cancel{= \frac{-1+3}{0} = \infty}$

$\lim_{x \to -1^+} \dfrac{x+3}{x+1} = \infty$

ex $f(x) = \dfrac{x+3}{(x+1)^2}$

$\left.\begin{array}{l} \lim_{x \to -1^-} \frac{x+3}{(x+1)^2} = \infty \\ \lim_{x \to -1^+} \frac{x+3}{(x+1)^2} = \infty \end{array}\right\} \lim_{x \to -1} f(x) = \infty$

Ex2 $f(x) = \dfrac{(x+2)^2(x-3)}{(x-4)(x+2)}$

(a) domain $\{x \mid x \neq 4, -2\}$

(b) $f(x) = \dfrac{(x+2)(x-3)}{x-4}$, $x \neq -2$

(c) $x=-2$ $\lim\limits_{x\to -2} f(x) = 0$ hole at $x=-2$

(d)

$x=4$ VA $\lim\limits_{x\to 4^+} \dfrac{(x+2)(x-3)}{x-4} = \infty$

$\lim\limits_{x\to 4^-} f(x) = -\infty$

ex $f(x) = \dfrac{(x+2)(x-3)}{(x-4)(x+2)^2} = \dfrac{x-3}{(x-4)(x+2)}$

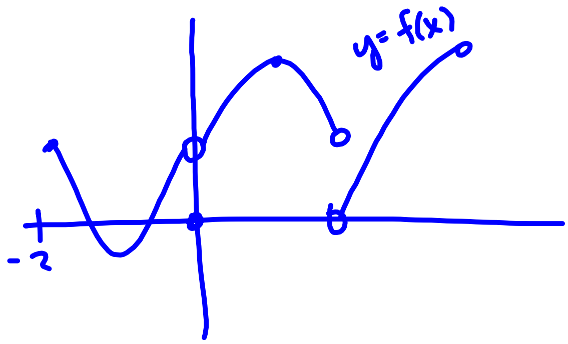
VA at $x=4$ and at $x=-2$

Ex 3

$$\lim_{x \to 3} \frac{x-3}{x^2-9} = \lim_{x \to 3} \frac{1}{x+3} = \frac{1}{6}$$

x doesn't actually = 3.

Ex 4



$f(0) = 0$

$\lim_{x \to 0} f(x) = 1$

$\lim_{x \to 2^-} f(x) = 1$

$f(2)$ undefined

$\lim_{x \to 2^+} f(x) = -1$

$\lim_{x \to 2} f(x)$ DNE

2.3 Continuity

$f(x)$ cont. at $x=c$ iff $\lim_{x\to c} f(x) = f(c)$.

① $\lim_{x\to c} f(x)$ exists (& is finite)

② $f(c)$ exists

③ they are the same

Ex1 (a) $f(x) = \begin{cases} 2x+3 & x<0 \\ x^2+3 & 0 \le x < 2 \\ 4 & x \ge 2 \end{cases}$

no (x=0 is fine)

at x=2 jump

(b) $y=\sqrt{x}$

yes (right-cont at x=0)

(c) $f(x) = |x+5| - 1$

yes

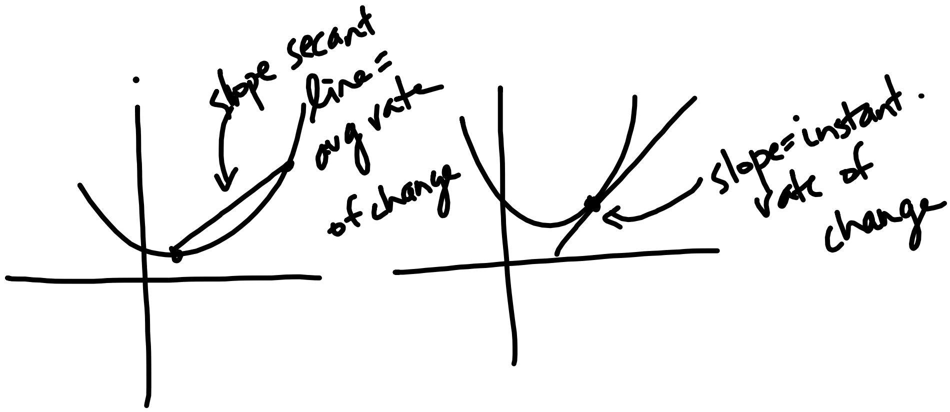
(d) $f(x) = \dfrac{(x-1)^2(x+2)}{(x-1)^2}$

at x=1 hole

(e) $f(x) = \dfrac{(x-1)^2(x+2)}{(x-1)^3}$

at x=1, VA

3.1 & 3.2



$$\text{avg slope} = \frac{f(b)-f(a)}{b-a}$$

Ex 1 $\quad d(t) = 60t$

$[1,4] \quad \frac{60(4)-60(1)}{4-1} = \frac{60(3)}{3} = 60$

$[6,7] \quad \frac{60(7)-60(6)}{7-6} = 60$

$[0,25] \quad \frac{60(25)-60(0)}{25-0} = 60$

(b) avg speed = 60 $\qquad$ (assume $b \neq a$)

$$\frac{f(b)-f(a)}{b-a} = \frac{60b-60a}{b-a} = \frac{60\cancel{(b-a)}}{\cancel{(b-a)}} = 60$$

Ex 2

(a) $g(x)=mx+c$ m, c constants

$$\frac{g(b)-g(a)}{b-a} = \frac{mb+\cancel{c}-(ma+\cancel{c})}{b-a} = \frac{m\cancel{(b-a)}}{\cancel{b-a}} = m \quad \left(\begin{matrix}\text{assume}\\ b\neq a\end{matrix}\right)$$

(b) $\frac{g(b)-g(a)}{b-a}=m$

$g(b)-g(a)=m(b-a)$

$g(b)-g(a)=mb-ma+c-c$

$g(b)-g(a)=(mb+c)-(ma+c)$

$g(b)=g(a)+mb-ma$

$$\frac{g(b)-g(x)}{b-x}=m$$

$g(b)-g(x)=m(b-x)$

$g(b)-mb=g(x)-mx$

$mx+g(b)-mb=g(x)$

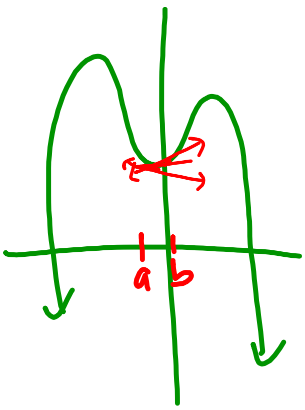
let $c=g(b)-mb$

$g(x)=mx+c$

(c) (a) proved $\Rightarrow$

(b) " $\Leftarrow$

so yes

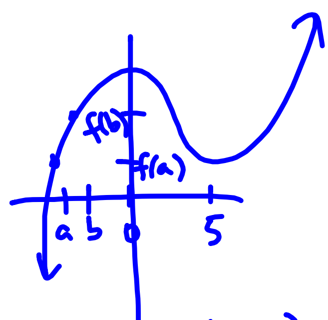


$$\lim_{b\to a} \frac{f(b)-f(a)}{b-a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Defn Increasing/Decreasing fn

$f(x)$ is increasing (decreasing) on (a,b) if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ $\forall$ x_1, x_2 in (a,b).

[$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$]

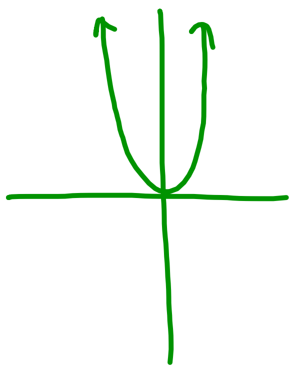


inc. on $(-\infty,0) \cup (5,\infty)$

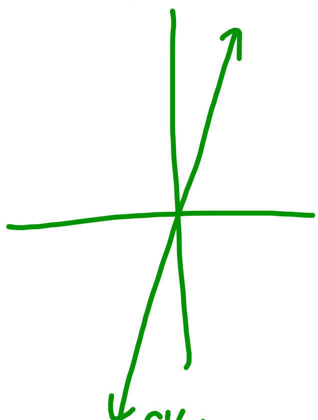
dec. on $(0,5)$

<u>Ex 3</u>

$f(x) = x^2$



$f'(x) = 2x$



(a) $f(x)$ inc. on $(0, \infty)$

dec on $(-\infty, 0)$

$f'(x)$ inc. on $(-\infty, \infty)$

on $(-\infty, 0)$ $f'(x) < 0$; on $(0, \infty)$ $f'(x) > 0$

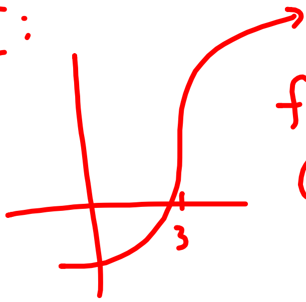
(b) $f'(x_0) = 0 \Rightarrow x_0 = 0$; at x_0 there's a pt of zero-slope

(c) slope on $f(x) = y$ graph correlates w/ y-values on $y = f'(x)$ graph

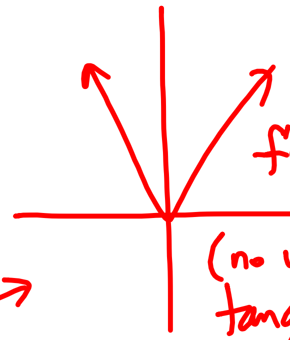
Ex4 slope of zero defines horizontal line

can remove: differentiable

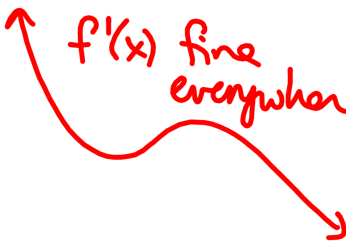
Ex5:



$f'(3)$ undefined (vert slope)

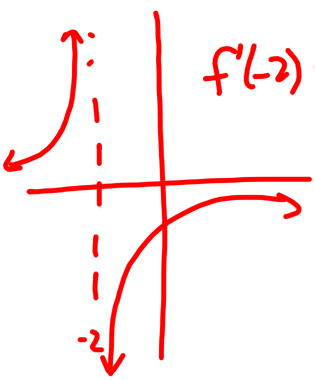


$f'(0)$ undefined (no unique tangent line)



$f'(x)$ fine everywhere

$f'(-2)$ and $f'(3)$ undefined (no unique tangent)



$f'(-2)$ undefined because $f(x)$ is discont. at $x=-2$

In general, if $f'(c)$ is undefined, then one of these is true.

① $f(x)$ is discontinuous at $x=c$

or ② no unique tangent line (it's too "pointy")

or ③ tangent line is vertical.

True or False?

True Diff $\Rightarrow$ Cont

others are false

Ex 6

(a) $f(x) = |x|$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{c \to x} \frac{f(c) - f(x)}{c - x}$$

$$f'(x) = \lim_{h \to 0} \left(\frac{|x+h| - |x|}{h} \right) \left(\frac{|x+h| + |x|}{|x+h| + |x|} \right) = \lim_{h \to 0} \frac{|x+h|^2 - |x|^2}{h(|x+h| + |x|)}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h(|x+h| + |x|)} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}(|x+h| + |x|)}$$

$$= \lim_{h \to 0} \frac{2x+h}{|x+h| + |x|} = \frac{2x}{2|x|} = \frac{x}{|x|} = \frac{|x|}{x}$$

(b) $f(x)=2x^2-1$

$$f'(x)=\lim_{h\to 0}\frac{2(x+h)^2-1-(2x^2-1)}{h}=\lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2\cancel{-1}\cancel{-2x^2}\cancel{+1}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(4x+2h)}{\cancel{h}}=4x+2(0)=4x$$

(c) $f(x)=\sqrt{x+5}$

$$f'(x)=\lim_{h\to 0}\left(\frac{\sqrt{x+h+5}-\sqrt{x+5}}{h}\right)\left(\frac{\sqrt{x+h+5}+\sqrt{x+5}}{\sqrt{x+h+5}+\sqrt{x+5}}\right)$$

$$=\lim_{h\to 0}\frac{\cancel{x}+h+\cancel{5}-(\cancel{x}+\cancel{5})}{h(\sqrt{x+h+5}+\sqrt{x+5})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+5}+\sqrt{x+5})}$$

$$=\frac{1}{\sqrt{x+5}+\sqrt{x+5}}=\frac{1}{2\sqrt{x+5}}$$

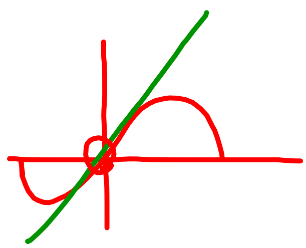
(d) $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos(h) + \sin(h)\cos x - \sin x}{h}$$

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos(h)-1}{h}\right) + \cos x\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \sin x\left[\underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0}\right] + \cos x\left[\underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1}\right] = \cos x$$

$$\lim_{\heartsuit\to 0} \frac{\sin \heartsuit}{\heartsuit} = 1$$



$$\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)\left(\frac{\cos(h)+1}{\cos(h)+1}\right) = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(1+\cos(h))}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(1+\cos(h))}$$

$$= \lim_{h\to 0}\underbrace{\left(\frac{\sin h}{h}\right)}_{1}\underbrace{\left(\frac{-\sin h}{1+\cos h}\right)}_{0} = 1\cdot 0 = 0$$