2.1 #5 (pg 71)

$h(t) = -16t^2 + 1000$

(a) $h(t) - h(1) = h(t+1)$

check: $\boxed{h(t+1) - h(t) = -h(1)}$ false

$h(t+1) - h(t) = -16(t+1)^2 + \cancel{1000} - (-16t^2 + \cancel{1000})$

$= -32t + -16$

$$\frac{h(t+1)-h(t)}{t+1-t} = \frac{h(t+1)-h(t)}{1} = h(t+1) - h(t)$$

#55 (pg 77) Claim: there are no fns $f: \mathbb{R} \to \mathbb{R}$ $g: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) + g(y) = xy$ $\forall\, x, y \in \mathbb{R}$.

Pf (by contradiction)

let $x_0 = 0$. cases ① $f(x_0) = 0$ or ② $f(x_0) \neq 0$

case ①: if $f(x_0) = 0$, then $f(x_0) + g(y) = x_0 y$

$g(y) = 0$ $\forall y \in \mathbb{R}$ $\Rightarrow g(y)$ doesn't span $\mathbb{R}$

$\Rightarrow$ case 1 can't be true.

Case ②: $f(0) \neq 0$

$\Rightarrow f(0) + g(y) = 0$ must mean $g(y) = -f(0)$

by $g(y) = -f(0)$ also Does not span $\mathbb{R}$

$\Rightarrow$ this case cannot be true #

2.1 (pg 77)

#57) $f(x) = 2x-3$, $h(x) = x+6$ $\quad x \in \mathbb{R}$

$f \circ g_1 = h \qquad g_2 \circ f = h$ $\quad$ for some $g_1 : \mathbb{R} \to \mathbb{R}$

$g_2 : \mathbb{R} \to \mathbb{R}$

$g_1(x) = \dfrac{x+9}{2}$

$g_2(f(x)) = x+6$

$g_2(2x-3) = x+6$

$g_2(x) = \frac{1}{2}x + c$

$g_2(2x-3) = \frac{1}{2}(2x-3) + c = x - \frac{3}{2} + c$

want $-\frac{3}{2} + c = 6$

$c = \frac{15}{2}$

$g_2(x) = ax + c$

$g_2(2x-3) = \underbrace{a(2x-3)+c = x+6}$ (true $\forall x \in \mathbb{R}$)

$2ax - 3a + c = x + 6$

equate like coefficients:

$2a = 1 \qquad -3a + c = 6$

$a = \frac{1}{2}$

pg 102 #29)b)

$$x^n + x^{n-1} + \dots + x + 1 = \frac{x^{n+1}-1}{x-1}$$

$n \in \mathbb{N}$

$$\sum_{j=0}^{n} x^j = \frac{(1-x^{n+1})}{1-x} = \frac{x^{n+1}-1}{x-1}$$

Finite Sum of Geom. Series

$$\sum_{n=0}^{p} ar^n = \frac{a(1-r^{p+1})}{1-r} = \sum_{n=1}^{p+1} ar^{n-1}$$

$$\lim_{x\to 1}\frac{x^{n+1}-1}{x-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(\text{some } x^n \text{ polynomial})}{\cancel{(x-1)}} = \lim_{x\to 1}(x^n + x^{n-1} + \dots + 1)$$

$$= 1(n+1) = n+1$$

some x^n poly. $= (x^n + x^{n-1} + \dots + 1)$

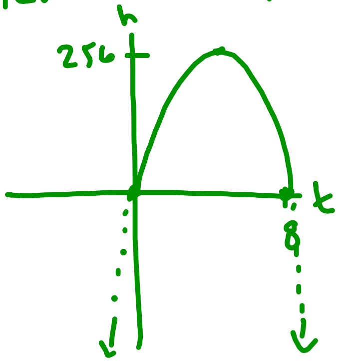
$$(x-1)(x^n + x^{n-1} + \dots + 1) = (x^{n+1} + \cancel{x^n} + \dots + \cancel{x^2} + \cancel{x}) - (\cancel{x^n} + \cancel{x^{n-1}} + \dots + \cancel{x} + 1)$$

$$= x^{n+1} - 1$$

3.3

$h(t) = -16t^2 + 128t = -16t(t-8)$

(a)



vertex at $t=4$

$(4, 256)$

domain: $t \in [0, 8]$

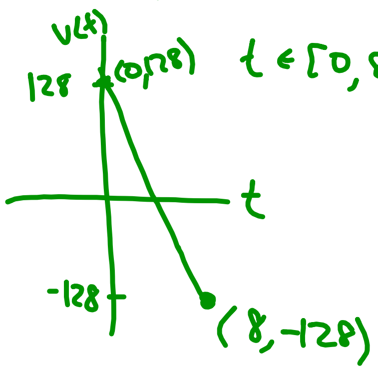
h in ft

t in sec

(b) $v(t) = h'(t) = -32t + 128$

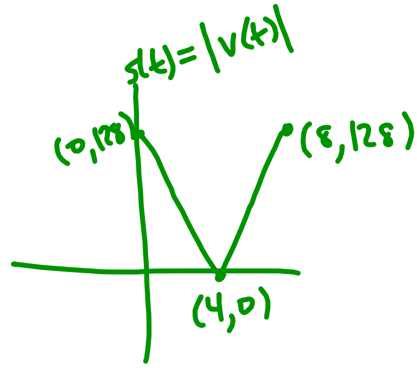
$\frac{dh}{dt}$

(c)



$t \in [0, 8]$

(d)



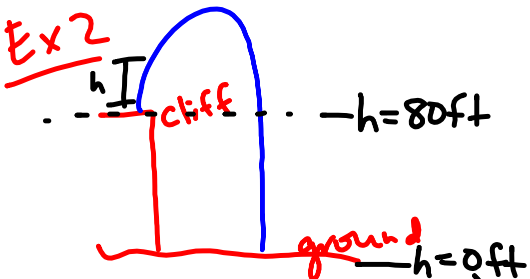
(e) $t = 4$ sec

(f) 256 ft

(g) 8 sec

(h) $a(t) = \frac{dv}{dt} = -32\ ft/sec^2$

Ex2



$h(t) = -16t^2 + 64t + 80$

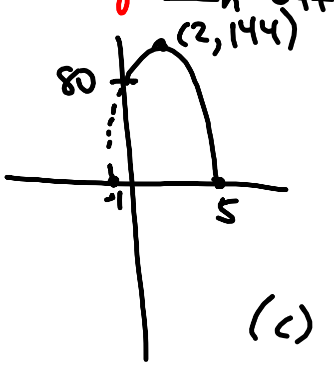
h in ft t in sec

(at $t=0$, we threw rock)

$h(t) = -16(t^2 - 4t - 5)$

$= -16(t-5)(t+1)$

(a)



(b) $v(t) = \frac{dh}{dt} = -32t + 64$

(c) 64 ft/sec = initial velocity

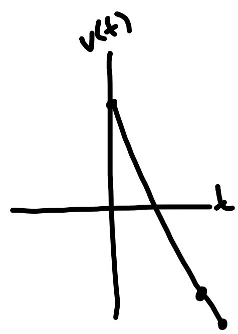
(d) $h=0$ at $t=5$ sec

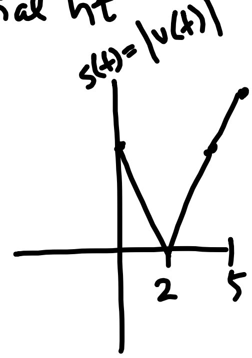
$v(5) = -32(5) + 64 = -96$ ft/sec

(e) 80 ft is initial ht

(f) speed increasing $(2,5)$

speed decreasing $(0,2)$





(g) -32 ft/sec^2

3.4 & 3.5

assume f differentiable

1. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Claim: ① $D_x(f(x)+g(x)) = D_x(f(x)) + D_x(g(x))$

② $D_x(cf(x)) = c\,D_x(f(x))$

Pf

① $D_x(f(x)+g(x)) = \lim\limits_{h \to 0} \dfrac{f(x+h)+g(x+h) - (f(x)+g(x))}{h}$

$$= \lim_{h\to 0}\left[\left(\frac{f(x+h)-f(x)}{h}\right) + \left(\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] + \left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]$$

$$= D_x(f(x)) + D_x(g(x)).$$

② $D_x(cf(x)) = \lim\limits_{h\to 0}\dfrac{cf(x+h)-cf(x)}{h} = c\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= c\,D_x(f(x))$$

2.

n	$f(x)=x^n$	$f'(x)$
0	1	0
1	x	1
2	x^2	$2x$
3	x^3	$3x^2$
4	x^4	$4x^3$
5	x^5	$5x^4$

$\underline{n=0}$

$f(x)=1$

$$f'(x)=\lim_{h\to 0}\frac{1-1}{h}=\lim_{h\to 0}0=0$$

$\underline{n=1}$ $f(x)=x$

$$f'(x)=\lim_{h\to 0}\frac{x+h-x}{h}=1$$

$\underline{n=2}$

$f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}=\lim_{h\to 0}(2x+h)=2x$$

$\underline{n=5}$

$f(x)=x^5$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^5-x^5}{h}=\lim_{h\to 0}\frac{\cancel{x^5}+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-\cancel{x^5}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(5x^4+10x^3h+10x^2h^2+5xh^3+h^4)}{\cancel{h}}$$

$$=5x^4$$

$n\in\mathbb{W}$

$$D_x(x^n)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\frac{\sum_{p=0}^{n}\left[\binom{n}{p}x^{n-p}h^p\right]-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^n}+\sum_{p=1}^{n}\left[\binom{n}{p}x^{n-p}h^p\right]-\cancel{x^n}}{h}=\lim_{h\to 0}\frac{\cancel{h}\sum_{p=1}^{n}\binom{n}{p}x^{n-p}h^{p-1}}{\cancel{h}}$$

$$=\lim_{h\to 0}\left[\binom{n}{1}x^{n-1}+\cancel{\sum_{p=2}^{n}\binom{n}{p}x^{n-p}h^{p-1}}\right]=nx^{n-1}$$

3. (a) $D_x(x^2\cdot x^3) = D_x(x^2)\cdot D_x(x^3)$? false

$D_x(x^5) = 5x^4$ $\qquad$ $(2x)(3x^2) = 6x^3$

(b) claim $\dfrac{f(x+h)\,g(x+h)-f(x)g(x)}{h} = \dfrac{f(x+h)-f(x)}{h}\,g(x+h) + f(x)\left(\dfrac{g(x+h)-g(x)}{h}\right)$

Pf

$$\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h) + f(x)\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \frac{g(x+h)f(x+h)-\cancel{f(x)g(x+h)}+\cancel{f(x)g(x+h)}-f(x)g(x)}{h}$$

$$= \frac{f(x+h)\,g(x+h)-f(x)g(x)}{h} \quad \#$$

(c) claim $D_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Pf

$$D_x(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\,g(x+h) + f(x)\,\frac{g(x+h)-g(x)}{h}\right]$$

$$= \left[\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\right]\left(\lim_{h\to 0} g(x+h)\right) + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)\,g(x) + f(x)\,g'(x) \quad \#$$

4. $f(x) = \frac{1}{x} \quad \forall x \in \mathbb{R},\ x \neq 0$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{\cancel{x} - (\cancel{x}+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2} = -x^{-2}$$

$$D_x(x^{-1}) = -1x^{-2}$$

5. $g(x)=\frac{1}{f(x)}$ $\forall x \in$ domain of f s.t. $f(x)\neq 0$.

(a) Claim:
$$\frac{g(x+h)-g(x)}{h}=\frac{-1}{f(x+h)f(x)}\cdot\frac{f(x+h)-f(x)}{h}$$

Pf.
$$\frac{g(x+h)-g(x)}{h}=\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}=\frac{\frac{f(x)-f(x+h)}{f(x)f(x+h)}}{h}$$
$$=\frac{f(x)-f(x+h)}{hf(x)f(x+h)}=\frac{-1}{f(x+h)f(x)}\cdot\frac{f(x+h)-f(x)}{h} \quad \#$$

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\left(\frac{-1}{f(x+h)f(x)}\cdot\frac{f(x+h)-f(x)}{h}\right)$$
$$=\underbrace{\lim_{h\to 0}\left(\frac{-1}{f(x+h)f(x)}\right)}_{\frac{-1}{(f(x))^2}}\cdot\underbrace{\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)}_{f'(x)}=\frac{-1}{(f(x))^2}f'(x)$$

$$D_x\left(\frac{1}{f(x)}\right)=\frac{-1}{(f(x))^2}f'(x)$$

(b) $g(x)\cdot f(x)=1$

$$D_x\left(g(x)\cdot f(x)\right)=D_x(1)$$
$$g'(x)f(x)+g(x)f'(x)=0$$
$$g'(x)=\frac{-g(x)f'(x)}{f(x)}=-\left(\frac{1}{f(x)}\right)\frac{f'(x)}{f(x)}=\frac{-1}{(f(x))^2}f'(x)$$

6. $D_x(x^n) = ?\ \forall n \in \mathbb{Z}$

We know $D_x(x^n) = nx^{n-1}\ \forall\ n \in \mathbb{W}$

and $D_x\left(\frac{1}{f(x)}\right) = \frac{-1}{(f(x))^2} f'(x)$

let $n = -p$ where $p \in \{1, 2, 3, \ldots\}$, then $n \in \mathbb{Z}^-$

Then $x^n = x^{-p} = \frac{1}{x^p}$

$$D_x(x^n) = D_x\left(\frac{1}{x^p}\right) = \frac{-1}{(x^p)^2}\left(D_x(x^p)\right) = \frac{-1}{x^{2p}}\left(px^{p-1}\right)$$

$$= -p\,x^{p-1-2p} = -p\,x^{-1-p} = nx^{n-1}$$

$\Rightarrow D_x(x^n) = nx^{n-1}\ \forall x \in \mathbb{Z}.$

7. $D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x)\cdot\frac{1}{g(x)}\right)$

remember:
$D_x\left(\frac{1}{f(x)}\right) = \frac{-1}{(f(x))^2} f'(x)$

$= f'(x)\cdot\frac{1}{g(x)} + f(x)\, D_x\left(\frac{1}{g(x)}\right)$

$= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-1}{(g(x))^2}\, g'(x)$

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

8. $g(x) = \sqrt{f(x)}$

claim

$$\frac{g(x+h)-g(x)}{h} = \frac{f(x+h)-f(x)}{h} \cdot \frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}}$$

Pf

$$\frac{f(x+h)-f(x)}{h} \cdot \frac{1}{\sqrt{f(x+h)}+\sqrt{f(x)}}$$

$$= \frac{\cancel{f(x+h)-f(x)}}{h} \cdot \frac{1}{\cancel{\left(\sqrt{f(x+h)}+\sqrt{f(x)}\right)\left(\sqrt{f(x+h)}-\sqrt{f(x)}\right)}} \cdot \left(\sqrt{f(x+h)}-\sqrt{f(x)}\right)$$

$$= \frac{\sqrt{f(x+h)}-\sqrt{f(x)}}{h} = \frac{g(x+h)-g(x)}{h}$$

$D_x(g(x)) =$

$$D_x\left(\sqrt{f(x)}\right) = \lim_{h\to 0} \frac{\sqrt{f(x+h)}-\sqrt{f(x)}}{h}$$

$$= \lim_{h\to 0}\left(\underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)} \cdot \underbrace{\frac{1}{\sqrt{f(x+h)}+\sqrt{f(x)}}}_{\frac{1}{2\sqrt{f(x)}}}\right)$$

$$D_x\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$$

9. Find $D_x\left((f\circ g)(x)\right)$.

claim

$$\frac{f(g(x+h))-f(g(x))}{h} = \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}$$

Pf (trivial).

$$D_x\left((f\circ g)(x)\right) = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}$$

$$= \left(\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right)\cdot\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

let $t = g(x+h)-g(x)$ as $h\to 0$, $t\to 0$

$$= \left(\lim_{t\to 0}\frac{f(t+g(x))-f(g(x))}{t}\right)\left(g'(x)\right) = f'(g(x))\,g'(x)$$

$$D_x\left(f(g(x))\right) = f'(g(x))\,g'(x) \quad \text{(Chain Rule)}$$

10. $f(x) = x^{a/b} \qquad a, b \in \mathbb{Z}, \; b \neq 0.$

$$(f(x))^b = x^a$$

$$D_x\left((f(x))^b\right) = D_x(x^a)$$

$$b(f(x))^{b-1}(f'(x)) = ax^{a-1}$$

$$b\left(x^{\frac{a}{b}}\right)^{b-1} f'(x) = ax^{a-1}$$

$$f'(x) = \frac{ax^{a-1}}{bx^{\frac{a(b-1)}{b}}} = \frac{a}{b} x^{a-1-\frac{a(b-1)}{b}}$$

$$= \frac{a}{b} x^{\cancel{a}-1-\cancel{a}+\frac{a}{b}}$$

$$= \frac{a}{b} x^{\frac{a}{b}-1}$$

$$D_x\left(x^{\frac{a}{b}}\right) = \frac{a}{b} x^{\frac{a}{b}-1}$$

$$\Rightarrow D_x(x^n) = nx^{n-1}, \; n \in \mathbb{Q}$$

Ex 1

(a) $f(x) = \sqrt[3]{x^2 + 20x - \frac{1}{x^2} + \pi^3} = \left(x^2 + 20x - x^{-2} + \pi^3\right)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}\left(x^2 + 20x - x^{-2} + \pi^3\right)^{-2/3}\left(2x + 20 + 2x^{-3}\right)$