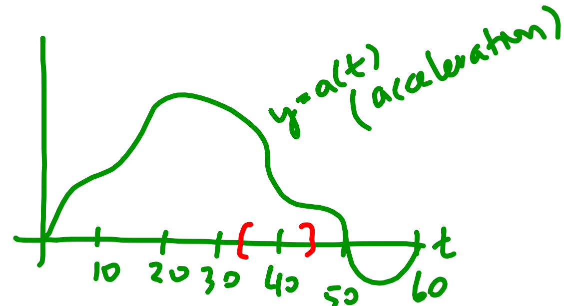
4.2 (pg 175)

(b) $f(x) = \dfrac{x^6 - 3x^5 + 2x - 7}{-4x^9 + 5x^6 - 3x + 2}$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^6}{-4x^9} = \lim_{x \to \pm\infty} \frac{1}{-4x^3} = 0$$

19)



$a(20) > a(40)$

(a) $v(t)$ on $[35, 45]$ is increasing

(b) $v(t)$ concavity on $[15, 20]$

$a(t) \uparrow \Rightarrow v(t)$ concave up

(c) $v(t)$ concave down on $[30, 40]$

(d) $v(40) < v(20)$

when $a(t) = 0$,

$v'(t) = 0$

(slope of $v(t) = y$ curve is zero)

when $a(t) > 0 \Leftrightarrow v'(t) > 0$

slope of $y = v'(t)$ curve is positive

$v'(t) = y$ is increasing

on $[0, 50]$, $a(t) > 0 \Leftrightarrow v'(t) > 0$

$\Rightarrow v(t)$ increasing on $[0, 50]$

$\Rightarrow v(20) < v(40)$

on $[55, 59]$, $a(t) < 0$

$\Leftrightarrow v'(t) < 0$

$\Rightarrow v(t)$ dec.

$\Rightarrow v(55) > v(59)$

4.3 (cont) (pg 3)

(d) $f(x) = 2^x$

$$f'(x) = \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h\to 0} \frac{2^x(2^h - 1)}{h} = 2^x \lim_{h\to 0}\left(\frac{2^h-1}{h}\right)$$

$$= \cancel{2^x \lim_{h\to 0}\left(\frac{2^h}{h} - \frac{1}{h}\right)}$$ doesn't help

$$\ln(f'(x)) = \ln\left(2^x \lim_{h\to 0}\left(\frac{2^h-1}{h}\right)\right)$$

$$= \ln(2^x) + \ln\left(\lim_{h\to 0}\left(\frac{2^h-1}{h}\right)\right)$$

$$= \cancel{x(\ln 2) + \lim_{h\to 0}\left(\ln\left(\frac{2^h-1}{h}\right)\right) = x(\ln 2) + \lim_{h\to 0}\left(\ln(2^h-1) - \ln h\right)}$$

not helpful

$$f'(x) = 2^x \lim_{h\to 0}\left(\frac{2^h-1}{h}\right) = 2^x \lim_{h\to 0}\left(\frac{e^{\ln 2^h} - 1}{h}\right)$$

$$= 2^x \lim_{h\to 0}\left(\frac{e^{h\ln 2} - 1}{h}\right)$$ not helpful

$\ln 2 \simeq 0.693147$

h	$\frac{2^h-1}{h}$
0.001	0.6933874
0.0001	0.693171204
−0.001	0.692907
−0.0001	0.693123

$$\lim_{h\to 0} \frac{2^h-1}{h} \simeq 0.693$$

$$\Rightarrow f'(x) \simeq 2^x(0.693)$$

(e) $g(x) = \left(\frac{3}{5}\right)^x$

$$g'(x) = \lim_{h\to 0} \frac{\left(\frac{3}{5}\right)^{x+h} - \left(\frac{3}{5}\right)^x}{h}$$

$$= \left(\frac{3}{5}\right)^x \left(\lim_{h\to 0} \frac{\left(\frac{3}{5}\right)^h - 1}{h}\right)$$

$$\approx \left(\frac{3}{5}\right)^x (-0.5108) \approx \left(\frac{3}{5}\right)^x \ln\left(\frac{3}{5}\right)$$

h	$\frac{\left(\frac{3}{5}\right)^h - 1}{h}$
0.001	-0.510695
0.0001	-0.5108
-0.0001	-0.51083

$\Rightarrow$ we suspect $D_x(b^x) = b^x \ln b \quad b > 0,\ b \neq 1$

$b \neq 1,\ b > 0$

(f) <u>claim</u> if $f(x) = b^x$, then $f'(x) = kb^x$ w/

$k = \lim_{h \to 0} \frac{b^h - 1}{h}$

<u>Pf</u> $f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = b^x \left(\lim_{h \to 0} \frac{b^h - 1}{h} \right)$

If $k = \left(\lim_{h \to 0} \frac{b^h - 1}{h} \right)$ exists and is finite,

then $f'(x) = kb^x$ #

(suspect $k = \ln b$)

$a \neq 1,\ a > 0$

$D_x(e^x) = e^x \ln e = e^x$ $\qquad D_x(a^x) = a^x \ln a$

Ex1 (a) $f(x) = \dfrac{xe^{-2x}}{e^x+5}$

$$f'(x) = \frac{(e^x+5)(e^{-2x} + xe^{-2x}(-2)) - (xe^{-2x})(e^x)}{(e^x+5)^2}$$

(b) $f(x) = 4^{2x^2-1}$

$f'(x) = 4^{2x^2-1}(\ln 4)(4x)$

(c) $f(x) = (2^x - x^2)^x = e^{\ln(2^x-x^2)^x} = e^{x \ln(2^x - x^2)}$

1 2

$D_x(x^n) = nx^{n-1}$ fixed n

$D_x(n^x) = n^x \ln n$, $n > 0$

$D_x(\ln x) = \frac{1}{x}$ $(x > 0)$

(need logarithmic differentiation)

$$f'(x) = e^{x\ln(2^x-x^2)}\left[\ln(2^x-x^2) + \frac{x(2^x\ln 2 - 2x)}{2^x - x^2}\right]$$

5.1 Antiderivatives

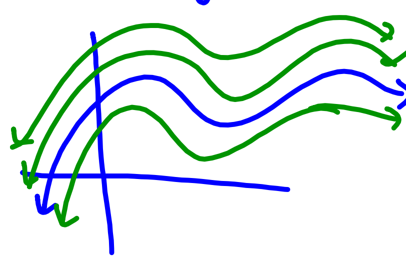
$F(x)$ is antiderivative of $f(x)$ if $F'(x) = f(x)$

1. $G(x) = F(x) + c$ $\quad c \in \mathbb{R}$ (arbitrary constant)

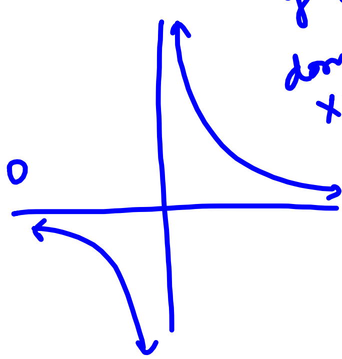
   $G'(x) = F'(x) = f(x)$

2. $\Rightarrow$ $F(x)$ is not the unique antiderivative.

3. $y = F(x)$



4) are all fns integrable? No

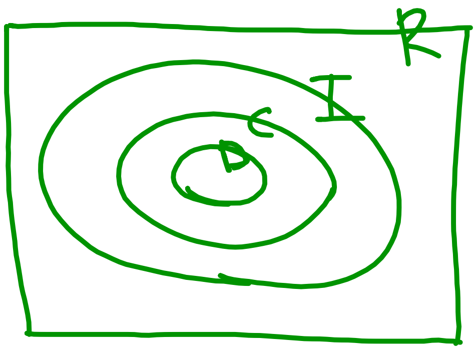


ex $\quad D_x(\ln x) = \frac{1}{x} \quad \forall x \neq 0$

$\Rightarrow F(x) = \ln x, \; f(x) = \frac{1}{x}$

but this statement true only for $x > 0$
(not for $x \neq 0$)

Ex1



$R = \{ f : f \text{ is } \mathbb{R}\text{-valued on } E \subseteq \mathbb{R} \}$

① diff. $\Rightarrow$ cont.

② cont $\Rightarrow$ int.

① $D \subset C$

② $C \subset I$

5. $D_x(x^n) = nx^{n-1}$

$$\int nx^{n-1}\,dx = n\frac{x^{n-1+1}}{n} + c = x^n + c$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

6. ① $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

② $\int kf(x)\,dx = k\int f(x)\,dx \quad \forall k \in \mathbb{R}$ (fixed)

Ex 2 (a) $\int x^{3/2}\,dx$

$= \frac{2}{5}x^{5/2} + c$

(b) $\int \frac{5}{x^3}\,dx = \int 5x^{-3}\,dx$

$= -\frac{5}{2}x^{-2} + c$

(c) $\int \left(8x^2 + \frac{2x^{-2/3}}{\cancel{x^{2/3}}}\right)dx$

$= \frac{8x^3}{3} + 6x^{1/3} + c$

Ex 3 $\int e^x\,dx = e^x + c$

$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c \longrightarrow$ check $D_x\left(\frac{e^{kx}}{k} + c\right)$

$= \frac{e^{kx}(k)}{k} = e^{kx}$ ✓

$\int e^{kx+b}\,dx = e^b \int e^{kx}\,dx$

$= \frac{e^b e^{kx}}{k} + c = \frac{e^{kx+b}}{k} + c$

Ex 4 $F(x) = \begin{cases} 1 - \frac{1}{x} & x < 0 \\ 3 - \frac{1}{x} & x > 0 \end{cases}$ Is $F(x)$ antider. for $f(x) = \frac{1}{x^2}$?

$F'(x) = \begin{cases} \frac{1}{x^2} & x < 0 \\ \frac{1}{x^2} & x > 0 \end{cases} \equiv \frac{1}{x^2},\ x \neq 0$

(b) $F(x) - F\left(\frac{-1}{x}\right) = \begin{cases} 1 - \frac{1}{x} - \left(1 - \frac{1}{-1/x}\right) & x < 0 \\ 3 - \frac{1}{x} - \left(3 - \frac{1}{-1/x}\right) & x > 0 \end{cases}$

$= \begin{cases} \frac{-1}{x} - x & x < 0 \\ \frac{-1}{x} - x & x > 0 \end{cases} \equiv -x - \frac{1}{x},\ x \neq 0$

(b) Is $F(x) = \frac{-1}{x}$, $x \neq 0$ antider. for $f(x) = \frac{1}{x^2}$.

Ex 5: (a) $\int \sqrt{x}\,(x-2)^2\,dx =$

$$= \int \left(x^{5/2} - 4x^{3/2} + 4x^{1/2}\right) dx =$$

$$= \int x^{5/2}dx - 4\int x^{3/2}dx + 4\int x^{1/2}dx =$$

$$= \frac{2}{7}x^{7/2} - \frac{8}{5}x^{5/2} + \frac{8}{3}x^{3/2} + C$$

(b) $\left(\int \sqrt{x}\,dx\right)\left(\int (x-2)^2\,dx\right)$

$$= \left(\int x^{1/2}\,dx\right)\left(\int (x^2 - 4x + 4)\,dx\right)$$

$$= \left(\frac{2}{3}x^{3/2} + c\right)\left(\frac{x^3}{3} - 2x^2 + 4x + K\right)$$

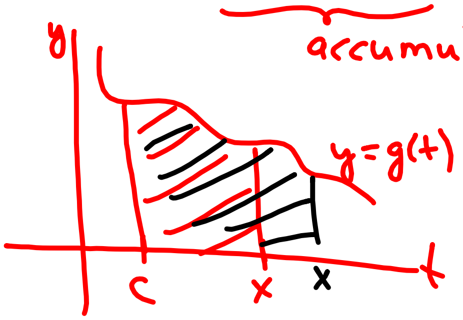
(b) Is there a "product rule" for integration?

no

5.2 Definite Integrals

1. $G(x) = \underbrace{\int_c^x g(t)\,dt}_{\text{accumulation fn}}$, then $G'(x) = g(x)$ $\Leftrightarrow$ $D_x\left(\int_c^x g(t)\,dt\right) = g(x)$

$\int_c^x g(t)\,dt = G(x) - G(c)$



2. $\int_a^b f(x)\,dx = F(b) - F(a)$

Ex5 (practice using First Fund. Thm of Calc.

(a) $G(x) = \int_1^x \frac{t^2}{t^5+1}\,dt$

$G'(x) = \frac{x^2}{x^5+1}$

(b) $G(x) = \int_1^{x^3-x^2} \frac{t^2}{t^5+1}\,dt$

$G'(x) = \left(\frac{(x^3-x^2)^2}{(x^3-x^2)^5+1}\right)(3x^2-2x)$

$D_x\left((x^3-x^2)^{3/5}\right)$

$= \frac{3}{5}(x^3-x^2)^{-2/5}(3x^2-2x)$

$u = x^3 - x^2$

$G(u) = \int_1^u \frac{t^2}{t^5+1}\,dt$

$D_x(G(u)) = G'(u)\frac{du}{dx}$

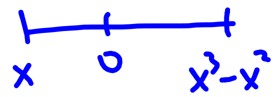
$= G'(x^3-x^2)\left(\frac{d(x^3-x^2)}{dx}\right)$

ex $$G(x) = \int_2^{\sqrt{x+1}} \sin(t^3)\,dt$$

$$D_x\left((x+1)^{1/2}\right) = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}}$$

$$G'(x) = \sin\left(\left(\sqrt{x+1}\right)^3\right)\left(\frac{1}{2\sqrt{x+1}}\right)$$

(c) $$G(x) = \int_x^{x^3-x^2} \frac{t^2}{t^5+1}\,dt$$



$$= -\int_0^x \frac{t^2}{t^5+1}\,dt + \int_0^{x^3-x^2} \frac{t^2}{t^5+1}\,dt$$

$$G'(x) = \frac{-x^2}{x^5+1} + \frac{(x^3-x^2)^2}{(x^3-x^2)^5+1}(3x^2-2x)$$

(d) $$G(x) = \int_1^x \frac{x^3t^2}{t^5+1}\,dt$$ (assume $x \perp\!\!\!\perp t$)

independent

$$= \underbrace{x^3}_{①} \underbrace{\int_1^x \frac{t^2}{t^5+1}\,dt}_{②}$$

$$G'(x) = 3x^2\int_1^x \frac{t^2}{t^5+1}\,dt + x^3\left(\frac{x^2}{x^5+1}\right)$$

Ex 1

(a) $\int_1^3 (2x^2+1)\,dx$

$= \left(\frac{2x^3}{3}+x\right)\Big|_1^3$

$= \left(\frac{2}{\cancel{3}}\left(\overset{9}{\cancel{3^3}}\right)+3\right) - \left(\frac{2\cdot 1^3}{3}+1\right)$

$= (18+3) - \frac{2}{3} - 1 = 20 - \frac{2}{3}$

$= 19\frac{1}{3}$ or $\frac{58}{3}$

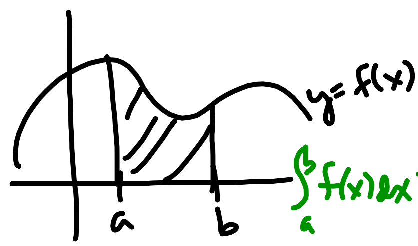
(b) $\int_0^2 e^x\,dx$

$= e^x\Big|_0^2 = e^2 - e^0$

$= e^2 - 1$

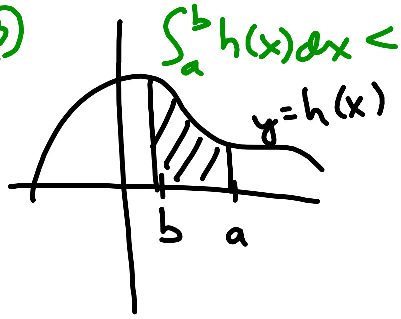
Note: $\int_a^b f(t)\,dt = \int_a^b f(y)\,dy$

①

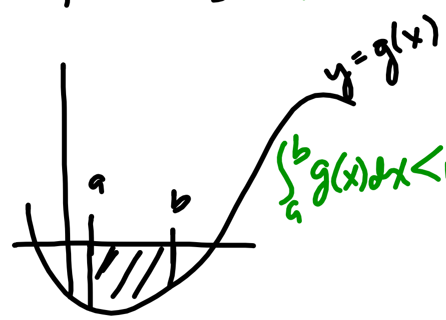


$\int_a^b f(x)\,dx > 0$

③ $\int_a^b h(x)\,dx < 0$



②



$\int_a^b g(x)\,dx < 0$

Ex2 $f(x) = x^3$ $g(x) = \int_0^x f(t)\,dt$

(a) $g(x) = \int_0^x t^3\,dt = \frac{t^4}{4}\Big|_0^x = \frac{x^4}{4}$

(b) $g'(x) = x^3 = f(x)$

(c) $g(0) = \int_0^0 f(t)\,dt = 0$

1) $\int_a^a f(x)\,dx = 0$ 2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

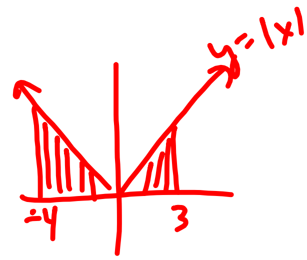
3) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

4) $\int_a^b k\,f(x)\,dx =$
$= k\int_a^b f(x)\,dx$

5) $\int_a^b \left(f(x) + g(x)\right)dx$
$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Ex3

$$\int_{-4}^{3} |x|\,dx = \int_{-4}^{0} -x\,dx + \int_{0}^{3} x\,dx$$



$$= -\frac{x^2}{2}\Big|_{-4}^{0} + \frac{x^2}{2}\Big|_{0}^{3}$$

$$= -0 - \frac{-(-4)^2}{2} + \frac{3^2}{2} - 0 = 8 + \frac{9}{2} = \frac{25}{2}$$

Ex4

$$\int_0^2 f(x)\,dx \qquad f(x) = \begin{cases} x^2 - 3x & x \geq 1 \\ e^x & x < 1 \end{cases}$$

$$= \int_0^1 e^x\,dx + \int_1^2 (x^2 - 3x)\,dx$$

$$= e^x\Big|_0^1 + \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_1^2$$

$$= e - 1 + \left(\frac{8}{3} - \frac{12}{2}\right) - \left(\frac{1}{3} - \frac{3}{2}\right)$$

$$= e - 1 + \frac{7}{3} - \frac{9}{2}$$

5.3 Techniques of Integration

Ex1 (a) $\int (3x^2+1)\, e^{x^3+x+2}\, dx$

$u = x^3+x+2$

$du = (3x^2+1)\,dx$

$= \int e^u\, du = e^u + C$

$= e^{x^3+x+2} + C$

(b) $\int 5x^2 (x^3+89)^{1/3}\, dx$

$u = x^3+89$

$du = 3x^2\, dx$

$\frac{1}{3}\, du = x^2\, dx$

$= \int 5\left(\frac{1}{3}\right) u^{1/3}\, du$

$= \frac{5}{3}\left(u^{4/3}\left(\frac{3}{4}\right)\right) + C$

$= \frac{5}{4}(x^3+89)^{4/3} + C$

(c) $\int e^{2x+e^{2x}}\, dx = \int e^{2x} e^{e^{2x}}\, dx$

$u = e^{2x}$

$du = 2e^{2x}\, dx$

$\frac{1}{2}\, du = e^{2x}\, dx$

$= \int e^u \left(\frac{1}{2}\right) du = \frac{e^u}{2} + C$

$= \frac{1}{2} e^{e^{2x}} + C$

check: $D_x\left(\frac{1}{2} e^{e^{2x}} + C\right)$

$= \frac{1}{2} e^{e^{2x}} (e^{2x})(2) = e^{e^{2x}} e^{2x} = e^{2x+e^{2x}}$ ✓

(d) $\int \cos x \cos(\sin x)\,dx = \int \cos u\,du$

$u = \sin x$

$du = \cos x\,dx$

$= \sin u + C$

$= \sin(\sin x) + C$