5.3 (pg 207)

18) $\displaystyle\int \frac{2x-1}{x^2(x-1)^2}\,dx$

$$\frac{2x-1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$2x-1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

$x=0$: $-1 = B$

$x=1$: $1 = D$

$x=-1$: $-3 = -4A - 4 - 2C + 1$

$4A = -2C$

$C = -2A$

$x=2$: $3 = 2A + -1 + 4(-2A) + 4$

$0 = -6A \Rightarrow A = 0 \Rightarrow C = 0$

$$\int \frac{2x-1}{x^2(x-1)^2}dx = \int\left(\frac{-1}{x^2} + \frac{1}{(x-1)^2}\right)dx = \int\left(-x^{-2} + (x-1)^{-2}\right)dx$$

$$= -\frac{x^{-1}}{-1} + \frac{(x-1)^{-1}}{-1} + C$$

$$= \frac{1}{x} - \frac{1}{x-1} + C$$

$m, b \in \mathbb{R}$ (constants), $m \neq 0$

$\displaystyle\int (mx+b)^n\,dx$

$u = mx+b$

$du = m\,dx$

$\frac{1}{m}du = dx$

$= \displaystyle\int \frac{1}{m}u^n\,du$

$= \dfrac{1}{m}\dfrac{u^{n+1}}{n+1} + C$

$= \dfrac{1}{m}\dfrac{(mx+b)^{n+1}}{(n+1)} + C$

$\displaystyle\int e^{mx+b}\,dx$

$= \frac{1}{m}\displaystyle\int e^u\,du$

$= \frac{1}{m}e^u + C$

$= \frac{1}{m}e^{mx+b} + C$

$\displaystyle\int \sin(mx+b)\,dx$

$= \frac{1}{m}\displaystyle\int \sin u\,du$

$= -\frac{1}{m}\cos u + C$

$= -\frac{1}{m}\cos(mx+b) + C$

$\displaystyle\int \cos(mx+b)\,dx$

$= \frac{1}{m}\sin(mx+b) + C$

$\displaystyle\int \sin(mx+b)\,d(mx+b)$

$= -\cos(mx+b) + C$

$\displaystyle\int \sin(\heartsuit)\,d\heartsuit$

$= -\cos\heartsuit + C$

17h) $\int x^3 e^x\,dx$

u	dv	
x^3	e^x	
$3x^2$	e^x	$+$
$6x$	e^x	$-$
6	e^x	$+$
0	e^x	$-$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

11) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int_2^3 \frac{1}{u^2}\,du = 2\left(\frac{-1}{u}\right)\Big|_2^3 = 2\left(\frac{-1}{3} - \frac{-1}{2}\right) = \frac{1}{3}$

$u = 1+\sqrt{x}$ $\qquad x=1,\ u=1+\sqrt{1}=2$

$du = \frac{1}{2\sqrt{x}}\,dx$ $\qquad x=4,\ u=1+\sqrt{4}=3$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

correct

$\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int \frac{1}{u^2}\,du = 2\left(\frac{-1}{u}\right) + C = \frac{-2}{1+\sqrt{x}} + C$

$\displaystyle\Rightarrow \int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = \frac{-2}{1+\sqrt{x}}\Big|_1^4 = \frac{-2}{3} - \frac{-2}{2} = \frac{-2}{3} + 1 = \frac{1}{3}$

6.2

(a) lower bound

68
53
61

(b) upper bound

100
86
88
78

$\Rightarrow$ area is in $[68, 78]$

(c) she says area $\simeq 168$ units^2

- more than double the actual area
- $168-78=90$ error ≥ 90 un^2
- $\frac{90}{78} \cong 115\%$

(d) $73=$ avg un^2

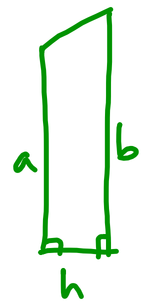
area $= 73 \pm 5$ un^2

$\frac{5}{73} \cong 0.068$

$\simeq 6.8\%$

1. Trapezoid Method

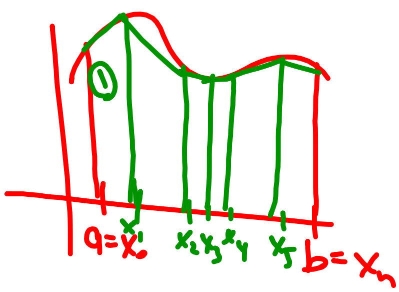
$\Delta x_1 = x_1 - x_0$



$$A_1 = \frac{1}{2}(\Delta x_1)(f(x_0)+f(x_1))$$

$$A_2 = \frac{1}{2}(\Delta x_2)(f(x_1)+f(x_2))$$

$$\vdots$$



$$A_n = \frac{1}{2}(\Delta x_n)(f(x_{n-1})+f(x_n))$$

$$\Rightarrow A = \frac{1}{2}\sum_{j=1}^{n}(\Delta x_j)(f(x_{j-1})+f(x_j))$$

If x_j are uniformly distributed, $\Delta x_j = \Delta x = \frac{b-a}{n}$

$$A = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{b-a}{n}\right)(f(x_j)+f(x_{j-1})) = \frac{b-a}{2n}\sum_{j=1}^{n}(f(x_j)+f(x_{j-1}))$$

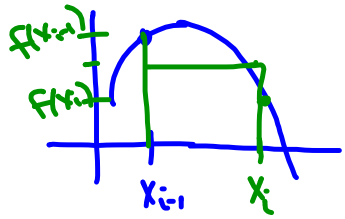
$$= \frac{b-a}{2n}\Big[(f(x_0)+f(x_1))+(f(x_1)+f(x_2))+(f(x_2)+f(x_3)) + \dots + (f(x_{n-2})+f(x_{n-1}))+(f(x_n)+f(x_{n-1}))\Big]$$

$x_0 = a$
$x_n = b$

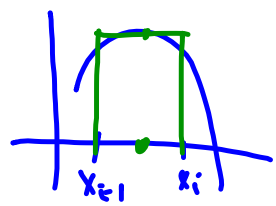
$$\text{Area} = \frac{b-a}{2n}\left[f(a)+f(b)+2\sum_{j=1}^{n-1}f(x_j)\right] \qquad x_j = a+\Delta x\, j$$

2. Rectangular method

(c) $A \simeq \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}$ (trapezoid method)

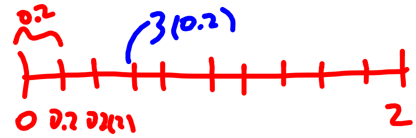


? or $\boxed{\frac{b-a}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)}$



if $y=f(x)$ is below x-axis, we get "signed" area.

Ex 1 $\int_0^2 x^3\,dx$ $n=10$



$\Delta x = \frac{2-0}{10} = 0.2$

$x_i = 0 + \Delta x\, i$
$= 0.2i$

Rt endpts: $A = \sum_{i=1}^{10} f(x_i)\,\Delta x = 0.2 \sum_{i=1}^{10} x_i^3 = 0.2 \sum_{i=1}^{10} (0.2i)^3$

$= 0.2^4 \sum_{i=1}^{10} i^3 = 0.2^4\left(\frac{10^2(11^2)}{4}\right) = \frac{1}{5^{\cancel{4}\,2}}\,\frac{\cancel{5^2}(\cancel{2^2})11^2}{\cancel{4}} = \frac{121}{25}$

left endpts: $A = 0.2 \sum_{i=0}^{9} (0.2i)^3 = 0.2^4 \sum_{i=0}^{9} i^3 = 0.2^4 \sum_{i=1}^{9} i^3$

$= \frac{1}{5^4}\left(\frac{9^2(10^2)}{4}\right) = \frac{\cancel{5}^2\cdot\cancel{2}^2\cdot 9^2}{5^{\cancel{4}\,2}\cdot\cancel{4}} = \frac{81}{25}$

trapezoid: gives answer $\frac{101}{25}$

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx \qquad x_0 = a,\ x_n = b$$

Ex 1 $\quad f(x) = x^2+2$

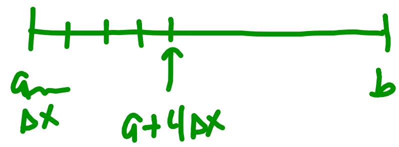
$$\int_{-1}^{3}(x^2+2)\,dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad (\text{right endpts})$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + (\Delta x)i$$

$$\Delta x = \frac{4}{n}$$

$$x_i = -1 + \frac{4}{n}i$$



$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(-1+\frac{4}{n}i\right)^2+2\right]\frac{4}{n} = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[1-\frac{8}{n}i+\frac{16}{n^2}i^2+2\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[3-\frac{8}{n}i+\frac{16}{n^2}i^2\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[\sum_{i=1}^{n}3-\frac{8}{n}\sum_{i=1}^{n}i+\frac{16}{n^2}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[3n-\frac{8}{n}\frac{(n)(n+1)}{2}+\frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty}\left[12-\frac{16(n+1)}{n}+\frac{32(n+1)(2n+1)}{3n^2}\right]$$

$$= \lim_{n\to\infty}\left[12-\frac{16\cancel{n}}{\cancel{n}}+\frac{32(2\cancel{n^2})}{3\cancel{n^2}}\right] = 12-16+\frac{64}{3} = -4+\frac{64}{3}$$

$$= \frac{52}{3}$$

$D_x\left(\frac{x^3}{3}\right)=x^2$

note: $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c \quad \forall n\neq -1$

$D_x\left(\frac{x^2}{2}\right)=x$

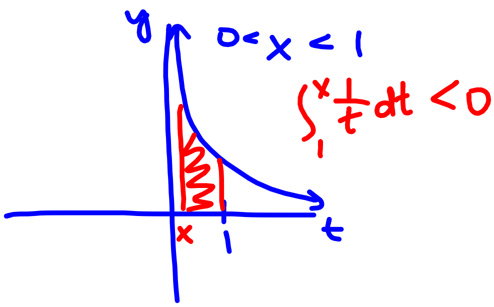
$D_x\left(\frac{x}{1}\right)=1$

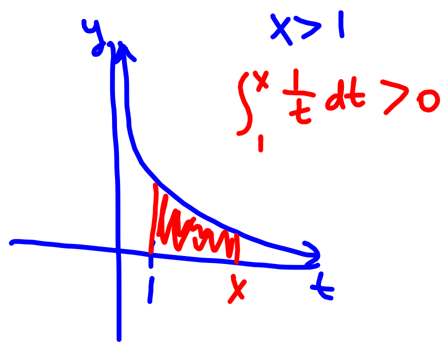
let $F(x)=\int_1^x \frac{1}{t}\,dt$

$D_x\left(??\right)=x^{-1}$

$\Rightarrow F'(x)=\frac{1}{x}$

$D_x\left(\frac{-1}{x}\right)=x^{-2}$





$\boxed{x} = \int_1^x \frac{1}{t}\,dt \qquad \Rightarrow D_x\left(\boxed{x}\right)=\frac{1}{x} \quad \forall\ x>0$

Claim: $D_x\left(\boxed{|x|}\right)=\frac{1}{x}, \ x\neq 0$

Pf: ① if $x>0$, $|x|=x$, $D_x\left(\boxed{|x|}\right)=D_x\left(\boxed{x}\right)=\frac{1}{x}$

② if $x<0$, $D_x\left(\boxed{|x|}\right)=D_x\left(\boxed{-x}\right)=\frac{1}{-x}(-1)=\frac{1}{x}$ //

$\Rightarrow \int \frac{1}{x}\,dx = \boxed{|x|}+c \qquad x\neq 0$

This fills in gap, $D_x(??)=\frac{1}{x} \Rightarrow ?? = \boxed{|x|} \quad \forall x\neq 0$

Properties $\quad a,b \in \mathbb{R}^+,\ r \in \mathbb{Q}$

(i) $\ln 1 = 0$ $\qquad$ (ii) $\ln ab = \ln a + \ln b$

(iii) $\ln \frac{a}{b} = \ln a - \ln b$ $\qquad$ (iv) $\ln a^r = r \ln a$

Pf (i) $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

(ii) If $x>0$, then $D_x\left(\ln ax\right) = \frac{1}{ax}(a) = \frac{1}{x}$.

And we know $D_x\left(\ln x\right) = \frac{1}{x}$

$\Rightarrow \ln ax = \ln x + c$ $\quad$ (c = some constant)

plug in $x=1$, to solve for c.

$\ln a = \ln 1 + c \Rightarrow c = \ln a$

$\Rightarrow \ln ax = \ln x + \ln a$

(iii) $\ln \frac{1}{b}(b) = \ln \frac{1}{b} + \ln b$ $\quad$ but $\frac{1}{b}(b) = 1$

$\Rightarrow \ln \frac{1}{b} + \ln b = \ln 1 = 0$

$-\ln b = \ln \frac{1}{b}$

$\Rightarrow \ln \frac{a}{b} = \ln a\left(\frac{1}{b}\right) = \ln a + \ln \frac{1}{b} = \ln a - \ln b$

(iv) For $x>0$, then $D_x\left(\ln x^r\right) = \frac{1}{x^r}\left(rx^{r-1}\right) = \frac{r}{x}$

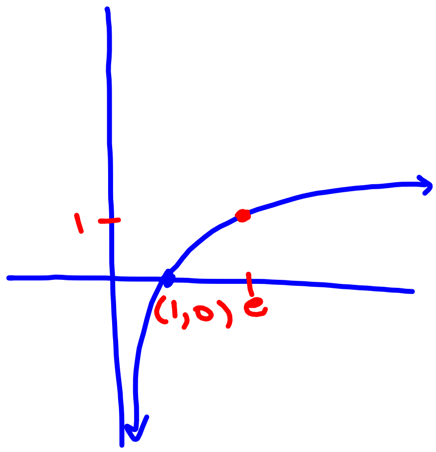
$= r\left(\frac{1}{x}\right) = r\,D_x\left(\ln x\right) = D_x\left(r \ln x\right)$

Since $D_x\left(\ln x^r\right)$ and $D_x\left(\ln x\right)$ differ only by a constant

$\ln x^r = r \ln x + C \quad \forall x>0$

Let $x=1$. $\ln 1^r = r \ln 1 + C \Rightarrow C = 0$

$\ln x^r = r \ln x$



$y = \sqrt{x}$ $(1,0)$

$\sqrt{x} > 0$ if $x > 1$

$\sqrt{x} < 0$ if $0 < x < 1$

$x \neq 0$ VA: $x = 0$

$\Rightarrow$ inverse exists: $\boxed{x} = f^{-1}(x)$ if $f(x) = \sqrt{x}$.

$\boxed{y} = x \iff y = \sqrt{x}$ $\left(\sqrt{x} = \int_1^x \frac{1}{t}\,dt \right)$

$\boxed{\sqrt{x}} = x = \sqrt{\boxed{x}}$

Defn $e \in \mathbb{R}^+$ s.t. $\sqrt{x} = 1$ i.e. $\sqrt{e} = \int_1^e \frac{1}{t}\,dt = 1$

$\boxed{1} = e$

$\Rightarrow e^r = \boxed{\sqrt{e^r}} = \boxed{r\sqrt{e}} = \boxed{r}$

$\Rightarrow$ "hunger of r" $\boxed{r}$ is none other than our well-known exponential fn!!!

For $a>0$, $a\neq 1$, $a^r = \sqrt{a^r} = e^{\sqrt{a}} = e^{r\sqrt{a}}$

Let $f(x)=a^x$. Then we have exp. fn we learned about in algebra.

$$x=a^y \iff \log_a x = y$$

but we know (from here), $e^y = x \iff y = \sqrt{x}$

$$\Rightarrow \sqrt{x} = \ln x$$

$$D_x(??) = \frac{1}{x} \iff D_x(\sqrt{x}) = \frac{1}{x} \iff D_x(\ln x) = \frac{1}{x},\ x>0$$

<u>Claim</u>: $D_x(e^x) = e^x$

<u>Pf</u> know $D_x(\ln x) = \frac{1}{x}$ and $x = \ln y \iff e^x = y$

$$x = \ln y$$

$$D_x(x) = D_x(\ln y)$$

$$1 = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = y = e^x \iff D_x(e^x) = e^x \ \#$$

(1) $e = \lim_{h\to 0}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Pf

$\lim_{h\to 0}(1+h)^{1/h} = \lim_{h\to 0} e^{\ln(1+h)^{1/h}}$

(1^∞ case)

indeterminate case

$= \lim_{h\to 0} e^{\frac{\ln(1+h)}{h}}$

$= e^{\lim_{h\to 0}\frac{\ln(1+h)}{h}}$

$\overset{L}{=} e^{\lim_{h\to 0}\frac{\frac{1}{1+h}}{1}} = e^{\lim_{h\to 0}\frac{1}{1+h}} = e^1 = e$ #

<u>aside</u>

$\lim_{h\to 0^-}(0.99)^{1/h} = \infty$

$\lim_{h\to 0}(1^{1/h}) = 1$

$\lim_{h\to 0^+}(1.01)^{1/h} = \infty$

2) claim $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Pf Remember Taylor Series. We know that T.S. is unique for every fn that has a Taylor Series. And a fn has a T.S. if

$$R_n(x) = \frac{f^{(n+1)}(c)\,x^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

(and assuming fn has all order of derivatives.)

T.S.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

if $a=0$, it's called Maclaurin series

$$\Leftrightarrow f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

ex $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$

(geometric series)

(a) we know $D_x(\ln x) = \frac{1}{x}$ and $x = \ln y \Leftrightarrow e^x = y$

and $D_x(e^x) = e^x$

$\Rightarrow f(x) = e^x$ has derivatives of <u>all</u> orders

(b) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ (I chose $a=0$.)

$f^{(n)}(0) = e^0 = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

(c) We still need to show that the remainder term goes to zero to be convinced that the T.S. converges. $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ $c \in \mathbb{R}$

$$R_n(x) = \frac{e^c x^{n+1}}{(n+1)!}$$

use Abs. Ratio Test: $\lim_{n\to\infty} \frac{\cancel{e^c} |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{\cancel{e^c} |x|^n}$

$$= |x| \left(\lim_{n\to\infty} \frac{1}{n+1} \right) = 0$$

$\Rightarrow e^x$ T.S. converges $\forall x \in \mathbb{R}$.

(d) Let $x=1$. We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (assume $x^0 = 1$)

Then $\boxed{e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}}$