3.1 Practice (maxima/minima)

Ex 1 Find all critical pts for $f(x) = x^5 - \frac{25}{3}x^3 + 20x - 1$ on $[-3, 2]$. Identify min + max values.

Critical pts

1. stationary pts
2. singular pts
3. endpts

Ex 2 Find min and max pts, for

$$f(x) = \frac{1}{1+x^2} \quad \text{on} \quad [-3, 1]$$

If $f(c)$ is an extreme value (ie. min/max value), then c is critical pt:

c is either

(i) an endpt of I.

(ii) a stationary pt (where $f'(c) = 0$)

or (iii) a singular pt (where $f'(c)$ DNE)

Ex 3 Under what conditions are we guaranteed min and max pts?

Ex 4 Sketch a graph of a function that meets these conditions.

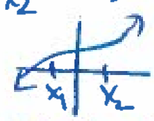
- f continuous
- f is not necessarily differentiable
- domain of $[0, 6]$
- max value of 4 (at $x = 3$)
- min value of 2 (at $x = 1$)
- f has no stationary pts.

3.2–3.3 Practice (Monotonicity / Concavity / Local Extrema)

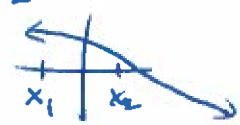
<u>Ex 1</u> For $f(x) = \frac{x^2}{x^2+1}$, find all min/max pts, inflection pts & sketch the graph.

- $f'(x) > 0$ on I
  $\Rightarrow f(x)$ is increasing
- $f'(x) < 0$ on I
  $\Rightarrow f(x)$ is decreasing

<u>increasing</u> means
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



<u>decreasing</u> means
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



<u>concave up</u> means
f' is increasing on I
<u>concave down</u> means
f' is decreasing on I

- $f''(x) > 0$ on I
  $\Rightarrow f(x)$ concave up
- $f''(x) < 0$ on I
  $\Rightarrow f(x)$ concave down

<u>inflection pt</u>: where concavity changes

EX2 Find all max/min pts, inflection pts, where $f(x)$ is increasing/decreasing, where $f(x)$ is concave up/down & sketch graph.

$$y = f(x) = x\sqrt{x-2}$$

ex

First Derivative Sign line:

$$\begin{array}{ccccccccc} + & & - & & - & & + & & - \\ \longleftarrow & a & & b & & c & & d & \longrightarrow f'(x) \end{array}$$

for continuous fn $f(x)$

- $(a, f(a))$ is max pt
- $(c, f(c))$ is min pt
- $(d, f(d))$ is max pt
- $(b, f(b))$ we don't know yet what's happening.

Second Derivative Sign line:

for continuous $f(x)$

$$\begin{array}{ccccccc} + & & - & & + & & + \\ \longleftarrow & A & & B & & C & \longrightarrow f''(x) \end{array}$$

- $(A, f(A))$ and $(B, f(B))$ are inflection pts.
- $(C, f(C))$ is NOT an inflection pt

Ex 3 Sketch graph of a continuous fn that satisfies these conditions:

- $f(0) = f(3) = 3$
- $f(2) = 4$
- $f(4) = 2$
- $f(6) = 0$
- $f'(x) > 0$ on $(0,2)$
- $f'(x) < 0$ on $(2,4) \cup (4,5)$
- $f'(2) = f'(4) = 0$
- $f'(x) = -1$ on $(5,6)$
- $f''(x) < 0$ on $(0,3) \cup (4,5)$
- $f''(x) > 0$ on $(3,4)$

Second Derivative Test:

If $f'(c) = 0$,

then ① $f''(c) < 0$
$\Rightarrow (c, f(c))$ is max

② $f''(c) > 0$
$\Rightarrow (c, f(c))$ is min

Ex 4 Find all min/max pts, inflection pts, increasing/decreasing intervals, concave up/down intervals, for

(a) $f(x) = x^2 - \frac{2}{x}$ and sketch the graph.

Ex 4 (cont)

(b) $f(\theta) = \dfrac{\sin\theta}{1+\cos\theta}$ $\quad \theta \in [0, 2\pi]$

3.4 Practice (Story Problems)

Ex 1 For what number does the principal square root exceed 8 times the number by the largest amt?

Steps

① Draw a picture and/or list info given.

② write down what needs to be optimized.

③ If have more than one input variable, find an eqn to eliminate one of the input vars.

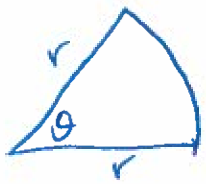
④ Differentiate fn.

⑤ Set derivative = 0 or find where derivative is undefined, i.e. look for stationary/singular pts.

⑥ check to ensure it's min or max (whichever you want).

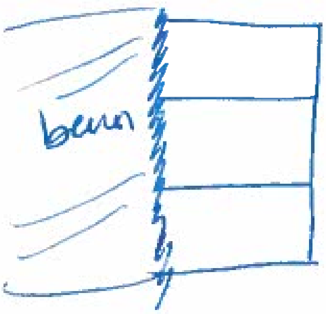
<u>Ex 2</u> Show that the rectangle w/ max perimeter that can be inscribed in a circle is a square.

Ex 3 A flower bed will be in the shape of a sector of a circle of radius r and vertex angle θ. Find r and θ if its area is a constant A and perimeter is a minimum.



Ex 4 Find the volume of the largest open box that can be made from a piece of cardboard that is 24" by 9". Find the dimensions of the box that yields the max volume. (You'll form the box by cutting identical squares from each corner & fold up.)

EX5 A farmer has 80 ft of fencing. He needs to enclose 3 identical pens along one side of the barn. What dimensions for the total enclosure make the area of the pens as large as possible?



3.5 Practice (Graphing Functions)

Ex 1 Analyze & graph.

(a) $f(x) = \tan^2 x$

To analyze a fn's graph:

① Find all VA, HA and SA, domain.

② Find x-intercepts.

③ Find $f'(x)$ + fill in sign-line; find min/max pts.

④ Find $f''(x)$ + fill in sign-line; find inflection pts.

⑤ Graph
- x-intercepts
- min/max pts
- inflection pts
- maybe one or two more pts (as needed)
- all asymptotes

⑥ Fill in rest of graph with knowledge of slope, concavity, pts, and asymptotes.

(b) $f(x) = \dfrac{x^2+x-6}{x-1}$

(c) $f(x) = |x|^3$

(Note:
$D_x(|x|) = \frac{x}{|x|}$)

3.6 Practice (MVT for Derivatives)

Ex 1 Find all possible c given by MVT, if any exist.

(a) $f(x) = x + \frac{1}{x}$ on $[-1, 2]$

(b) $f(x) = x + \frac{1}{x}$ on $[2, 4]$

MVTD

If ① $f(x)$ continuous on $[a,b]$

② $f(x)$ differentiable on (a,b)

Then there is at least one c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

If

$F'(x) = G'(x)$,

then

$F(x) = G(x) + C$

for some arbitrary constant C.

Ex 2 Suppose $F'(x) = 5$ and $F(0) = 4$. Find $F(x)$.

Ex 3 Show that if f is the quadratic fn $f(x) = ax^2 + bx + d$, $a \neq 0$, then the c given by MVT is always the midpt on any given interval $[\alpha, \beta]$.

3.7 Practice (Bisection and Newton's Method)

Ex 1 Use Bisection method to approximate real root of $x-2+2\cos x=0$ on $[1,2]$

<u>EX2</u> Use Newton's method to approximate the smallest root of $2\cos x - \sin x = 0$.

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

use to solve $f(x) = 0$

3.8 Practice (Antiderivatives)

Ex 1 Evaluate.

(a) $\int \left(\frac{\sqrt{2x}}{x} + \frac{3}{x^5} \right) dx$

(b) $\int \frac{x(x+1)^2}{\sqrt{x}}\, dx$

① **Power Rule**

$r \in \mathbb{Q}$, $r \neq -1$,

$\int x^r\, dr = \frac{x^{r+1}}{r+1} + C$

② $\int 1\, dx = x + C$

③ $\int \sin x\, dx = -\cos x + C$

④ $\int \cos x\, dx = \sin x + C$

⑤ Integral / Antiderivative is linear operator:

A) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

$\boxed{\text{and}}$

B) $\int k f(x)\, dx = k \int f(x)\, dx$

for constant k.

Ex 2 Evaluate.

(a) $\int x^2 \sqrt[3]{x^3+5}\, dx$

⑥ Generalized Power Rule (aka u-sub)

$\int (g(x))^r g'(x)\,dx = \frac{(g(x))^{r+1}}{r+1} + C$

(b) $\int \sin x \cos x \sqrt{1+\sin^2 x}\, dx$

3.9 Practice (Intro D.E.s)

Ex 1 Solve

(a) $\frac{dy}{dx} = y^3(x^3 - x)$ $\quad$ $y(x)$ goes thru $(0,4)$

(b) $\frac{dy}{dx} = -y^2 x(x^2+2)^3$ $\quad$ thru $(0,1)$.