1.1 Practice (Introduction to limits)

"Calculus is the study of Limits."

Ex 1 Find the limit

(a) $\lim_{t \to -2} (t^2 - 2x^2)$

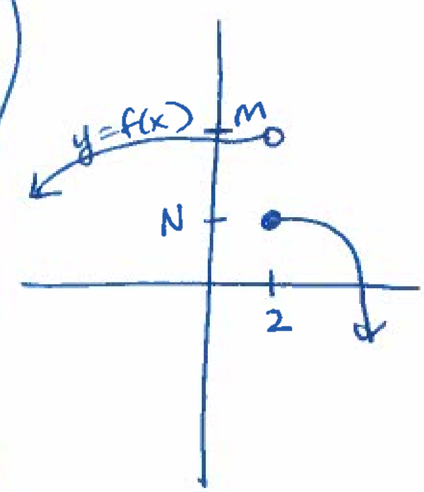
(b) $\lim_{x \to 0} \dfrac{3x^3 + 2x^2 - x^4}{x^2}$

(c) $\lim_{t \to 7^+} \dfrac{\sqrt{(t-7)^5}}{7-t}$

$\lim_{x \to c} f(x) = L$

$\Leftrightarrow$ (iff) $\lim_{x \to c^-} f(x) = L$ $\boxed{\text{and}}$ $\lim_{x \to c^+} f(x) = L$

Note: iff means "if and only if" i.e. implication goes both ways



$\lim_{x \to 2^+} f(x) =$ ______

$\lim_{x \to 2^-} f(x) =$ ______

$\lim_{x \to 2} f(x) =$ ______

(A)

Ex 2 Find the limits.

(a) $\lim_{x \to 3} \frac{[[x]]}{x}$

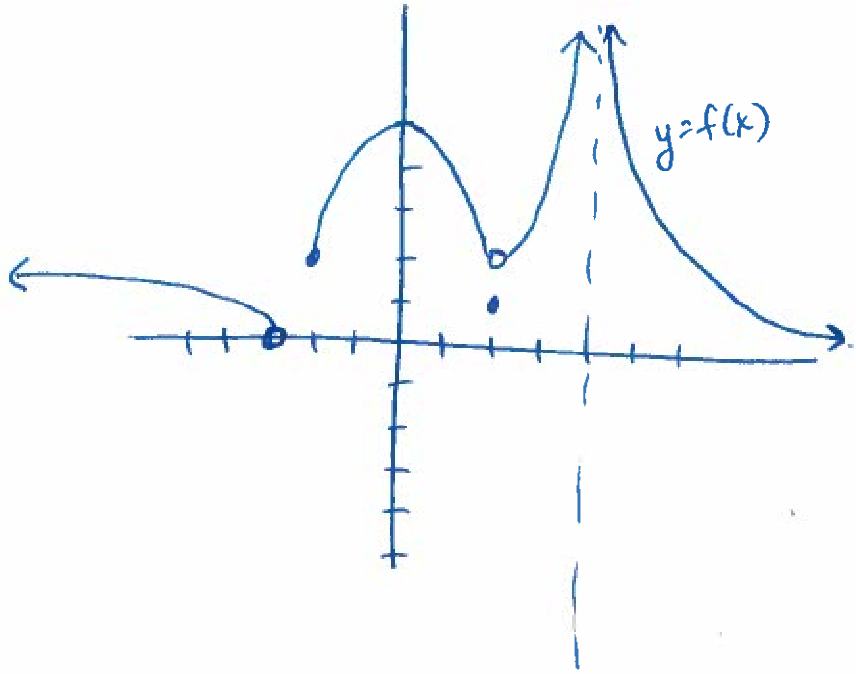
(b) $\lim_{x \to 1.6} \frac{[[x]]}{x}$

(c) $\lim_{x \to 0^+} \frac{[[x]]}{x}$

(d) $\lim_{x \to 0^-} \frac{[[x]]}{x}$

(e) $\lim_{x \to 0} \frac{[[x]]}{x}$

Ex3 For this graph, find the indicated limits.



(a) $\lim_{x \to -3^-} f(x)$

(b) $\lim_{x \to -2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x \to 4^-} f(x)$

(f) $\lim_{x \to 4^+} f(x)$

1.3 Practice (Limit Theorems)

Ex 1 Find these limits.

(a) $\lim_{x \to -3} \frac{x^2 - 14x - 51}{x^2 - 4x - 21}$

(b) $\lim_{x \to -1} \frac{x^2 + x}{x^2 + 1}$

(c) $\lim_{x \to -1^+} \frac{\sqrt{1+x}}{5+5x}$

$n \in \mathbb{N}$, k is constant
f & g are fns with limits at $x = c$.

① $\lim_{x \to c} k = k$

② $\lim_{x \to c} x = c$

③ $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$

④ $\lim_{x \to c} [f(x) + g(x)]$
$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

⑤ $\lim_{x \to c} [f(x) - g(x)]$
$= \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

⑥ $\lim_{x \to c} [f(x) \cdot g(x)]$
$= \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$

⑦ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$
as long as $\lim_{x \to c} g(x) \neq 0$

⑧ $\lim_{x \to c} (f(x))^n = \left(\lim_{x \to c} f(x)\right)^n$

⑨ $\lim_{x \to c} \sqrt[n]{f(x)}$
$= \sqrt[n]{\lim_{x \to c} f(x)}$
as long as $\lim_{x \to c} f(x) > 0$
if n is even.

1,3 (cont)

Ex2 Find the limits

(a) $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$

(b) $\lim_{x \to -\pi^+} \frac{\sqrt{\pi^3 + x^3}}{x}$

Squeeze Thm

f, g, h fns s.t.

$f(x) \leq g(x) \leq h(x)$

$\forall x$ near c,

except possibly at c.

If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$,

then $\lim_{x \to c} g(x) = L$.

Ⓔ

1.5 Practice (on Squeeze Theorem/Limits)

Ex 1 Find the limits.

(a) ① $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

② $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{x}\right)$

(b) ① $\lim_{x \to -\infty} \frac{3\sqrt[3]{x^5} + x - 7}{\sqrt[3]{5x} + 3x + 1}$

② $\lim_{x \to \infty} \frac{3\sqrt[3]{x^5} + x - 7}{\sqrt[3]{5x + 3x + 1}}$

(c) ① $\lim_{x \to \infty} \sqrt[5]{\frac{2x^3 - 5x + 1}{\pi x^3 + 3}}$

② $\lim_{x \to -\infty} \sqrt[5]{\frac{2x^3 - 5x + 1}{\pi x^3 + 3}}$

(d) $\lim_{x\to\infty} \sin\left(x+\frac{1}{x}\right)$

(e) $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

1.6 Continuity Practice

Ex 2 Redefine this fn so it's continuous.

(a) $f(x)=\frac{\sqrt{x}-1}{x-1}$

(b) $f(x)=\begin{cases} x & \text{if } x\le 0 \\ x^2 & \text{if } 0<x\le 1 \\ 3-x & \text{if } x>1 \end{cases}$

Is this continuous? If not, where and why does it fail continuity?

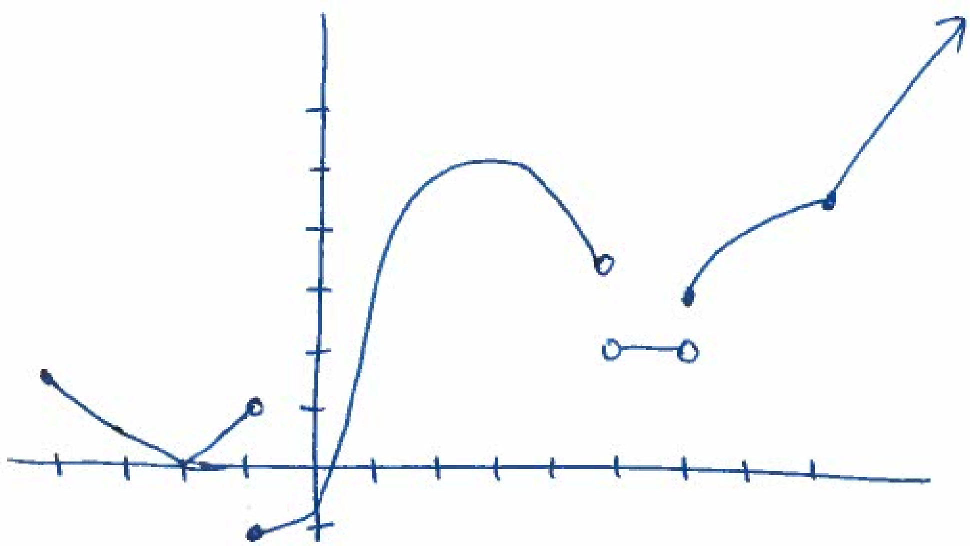
Continuity at $x=c$

① $\lim_{x\to c} f(x)$ exists

② $f(c)$ exists

③ $\lim_{x\to c} f(x) = f(c)$

Ex 3 Is this fn continuous? If not, tell where it's discontinuous & why. On what intervals is it continuous?



2.1 Practice (The Derivative)

Derivative defn

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(instantaneous slope at any point of a curve $y = f(x)$)

Ex 1 Find derivative of $f(x) = 4x^2 - 3$.

Ex 2 Find the slope formula of $y = \sqrt{x-1}$ at any point x. Then use that formula to find the slope of the curve at $x = 10$.

<u>Ex 3</u> Find the equation of the tangent line to $y = \frac{3}{x^2}$ at $x = -1$.

Ex 4 What is a secant line?
What is a tangent line?
What is formula for secant and tangent lines for some function $f(x)$?

Ex 5 If a particle moves along a coordinate line so that its directed distance from the origin after t seconds is $(-t^2+4t)$ feet, when did the particle come to a momentary stop?

2.2 Practice (More Derivatives)

Use the defn of derivative to find these derivatives.

Ex 1 derivative for $f(x) = \frac{x-1}{x+1}$

Defns of derivative

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

or

$f'(x) = \lim_{w \to x} \frac{f(x)-f(w)}{x-w}$

or

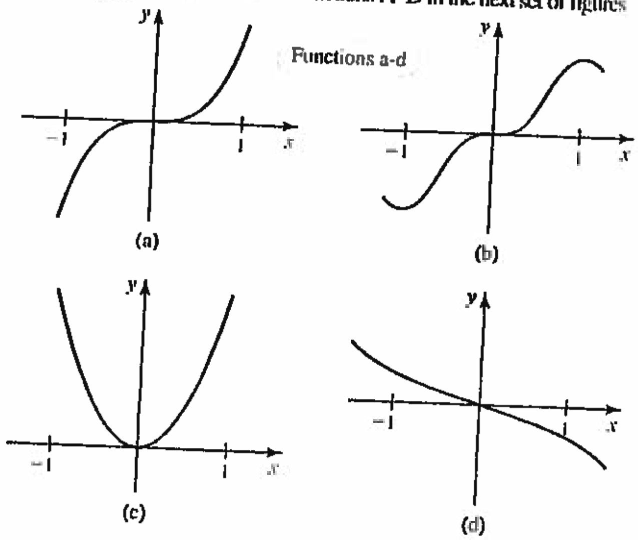
$f'(x) = \lim_{w \to x} \frac{f(w)-f(x)}{w-x}$

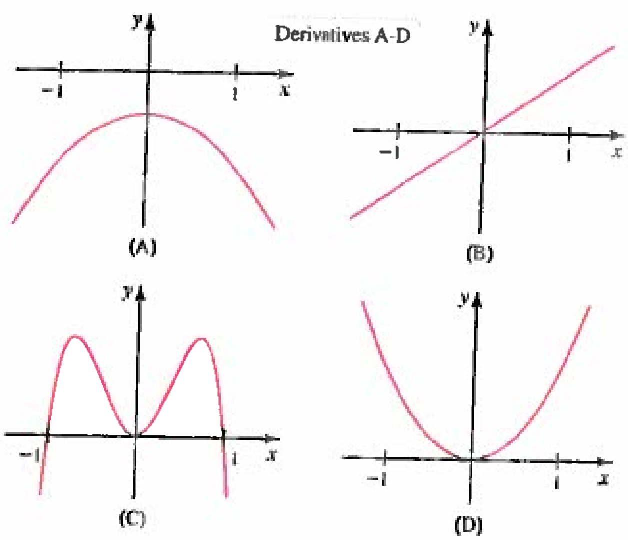
True or False?

1. Differentiability $\Rightarrow$ Continuity
2. Continuity $\Rightarrow$ Differentiability

Can you defend your answer with a graphical representation?

Matching functions with derivatives Match the functions a–d in the first set of figures with the derivative functions A–D in the next set of figures.





8A

Daily Quiz 07

⚠ This is a preview of the published version of the quiz

Started: Sep 7 at 2:48pm

Quiz Instructions

This quiz covers section 2.2. You have 30 minutes to complete the quiz.

Question 1 — 6 pts

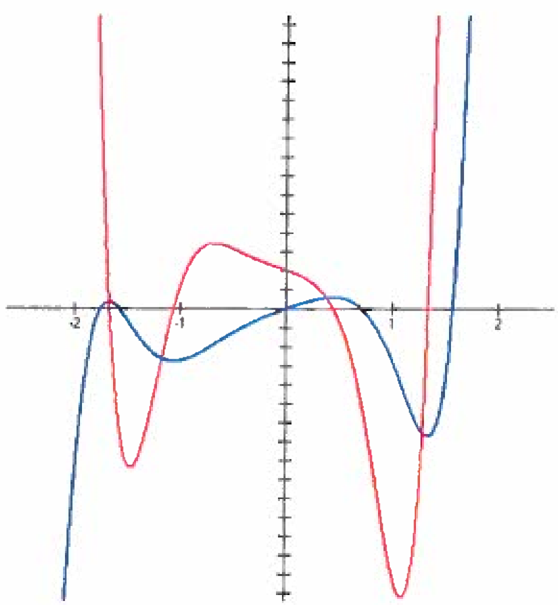
Find the equation of the tangent line to the graph of $f(x) = \frac{2}{x} - 1$ when x = -1.

Note: To answer this question, put your answer in slope-intercept form, y=mx+b, and just enter the m and b values in the boxes below, as shown. Use NO spaces. Give exact, not approximated, values.

Tangent Line: y = ______ x + ______

Question 2 — 4 pts

In this picture, both the graph of the function and the graph of its derivative are on the same coordinate axes. Which one is which?



(88)

Ex 2 derivative for $f(x) = \dfrac{x+2}{x}$

Defns of derivative

① $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

② $f'(x) = \lim\limits_{t\to x} \dfrac{f(t) - f(x)}{t-x}$

Ex3 given this derivative, what's the fn?

$$\lim_{x\to y} \frac{\sin x - \sin y}{x-y} = \underline{f'(?)} \quad \text{for } f(x) = ?$$

2.3 Practice (Derivative Rules)

Use "shortcuts" to find these derivatives, for the given fns.

Ex 1 (a) $y = x^{12} + 5x^{-2} - \pi x^{-10}$

(b) $y = (3x^{-2} + 2x)(x^4 - 3x + 7)$

(c) $y = \frac{3}{x^5} - x^{-1} + \frac{\pi}{x^6}$

(d) $y = \frac{5x^2 + 6x - 3}{x^{-2} + 1}$

Derivative Rules

① $D_x(k) = 0$

② $D_x(x^n) = nx^{n-1}$

③ Derivative is a linear operator:

A) $D_x(f(x) + g(x)) = D_x(f(x)) + D_x(g(x))$

$\boxed{\text{and}}$

B) $D_x(kf(x)) = k\,D_x(f(x))$

Product Rule

$(f \cdot g)' = f'g + g'f$

Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$\left(\frac{\text{low} \cdot \text{d-hi} - \text{hi} \cdot \text{d-low}}{\text{low}^2}\right)$

Ex 2 Find the derivative of these fns.

(a) $y = \dfrac{5}{36 - x^2}$

(b) $y = 3x^2(x^9 + x^8 - 100)$

EX 3 Find eqn of tangent line to $y = \dfrac{1}{x^2+4}$ at $x = 1$.

Ex 4 Find all points of $y = \frac{1}{3}x^3 + x^2 - x$ graph where tangent line has slope of 1.

1.4 Practice (Trigonometric Limits)

Ex 1 $\lim\limits_{\theta \to 0} \dfrac{\sin(5\theta)}{\theta}$

Ex 2 $\lim\limits_{x \to 0} \dfrac{3x \tan x}{\sin x}$

Special Trig Limits

① $\lim\limits_{\heartsuit \to 0} \dfrac{\sin \heartsuit}{\heartsuit}$

$= \lim\limits_{\heartsuit \to 0} \dfrac{\heartsuit}{\sin \heartsuit} = 1$

② $\lim\limits_{\star \to 0} \dfrac{1-\cos(\star)}{\star} = 0$

<u>Ex 3</u> $\lim_{x \to 0} \frac{\sin(4x) - 2x}{x \cos x}$

<u>Ex 4</u> $\lim_{x \to 0} \frac{\cos x}{3x}$

2.4 Practice (Trigonometric Derivatives)

Ex 1 (a) Find $D_x(\sin x)$ using definition of derivative.

(b) Use quotient rule to find derivative of $f(x) = \csc x$.

Ex 2 Find derivatives

(a) $y = 1 - \cos^2 x$

(b) $y = \dfrac{\sin x + \cos x}{\tan x}$

(c) $y = 4x^5 \csc x$

$D_x(\sin x) = \cos x$

$D_x(\cos x) = -\sin x$

$D_x(\tan x) = \sec^2 x$

$D_x(\cot x) = -\csc^2 x$

$D_x(\sec x) = \sec x \tan x$

$D_x(\csc x) = -\csc x \cot x$

Limit & Derivative Practice

EX1 Find the limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^3 - 2x^2 + \sqrt[5]{4x^{19}} - 1}{2x^2 - 4x^4}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{3x^3 - 4x^2 + 5x}{1 - 7x^3}$

(c) $\displaystyle\lim_{x\to\infty} \frac{-5x^9 - 8x^7 + 3}{x^2 + x + 1}$

Ex 2 Find the limits

(a) $$\lim_{x\to 0}\left(\frac{\sin(3x)\tan(5x) + 2x}{x\cos x}\right)$$

(b) $$\lim_{x\to 4}\frac{3x^2-5x-28}{x^2-16}$$

Ex 3 Find the limits.

(a) $$\lim_{x \to 1} \frac{(x+2)(x-5)}{(x-1)\,x^2(x+3)}$$

(b) $$\lim_{x \to 3^+} \frac{x+2}{\sqrt{x-3}}$$

Ex 4 Describe discontinuities for

$$f(x) = \frac{x^3(4x-1)(x+2)}{(x+2)(x-5)(x^2)}$$

Ex 5 Use defn of derivative to find $f'(x)$.

(a) $f(x) = \dfrac{1}{\sqrt{2x-1}}$

(b) $f(x) = x^3 + 5$

Ex6 Find the derivatives. (Don't bother to simplify.)

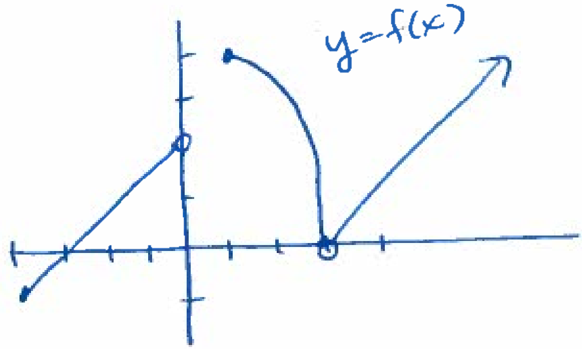
(a) $y = \sin x \sec x + \cos x$

(b) $y = (3x^{-5} + \pi x^{2} - 7)(x^{-6} + 9)$

(c) $y = \dfrac{4x^2+3x-8}{(x+1)(x^2+1)}$

Ex 7 Find the eqn of the tangent line to $y = \dfrac{1+x}{x^2-3}$ at $x = 1$.

Ex 8



Discuss continuity + limits.

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

$\lim_{x \to 3} f(x) =$

$f(3) =$

$f(0) =$

1. Find each limit, if it exists.

(a) $\lim_{x \to -1} (-4x^5+3x^3+5x^2-1)$

(b) $\lim_{x \to 4} \frac{2x^2-10x+8}{3x-12}$

(c) $\lim_{x \to \infty} \frac{-5x^3+x^2-5x+4}{8+3x+9x^3}$

(d) $\lim_{x \to -\infty} \frac{-4x^{\frac{6}{5}}+x-100}{10x+8}$

(e) $\lim_{x \to 5} \frac{(x-1)(x+3)}{x^2-25}$

(f) $\lim_{x \to 0} \frac{\sin(2x)\tan(5x)-2x\cos x}{4x}$

2. Find the equation of the tangent line to the curve $f(x)=3x^3-4x^2+5$ at $x=1$.

3. For $f(x)=\frac{(x+2)(3x-5)(-x^2)}{4(3x^2+x-10)(x-1)}$,

(a) Find the x-values where $f(x)$ is discontinuous and categorize the type of discontinuity.

(b) For what x-values is this function "patchable?"

4. Use the definition of the derivative, namely $f'(x)=\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, to find the derivative for $f(x)=\frac{4}{\sqrt{x-2}}$.

5. For this piecewise function.

$$g(x)=\begin{cases} x+5 & -5 \le x < 0 \\ x^2-1 & 0 < x \le 3 \\ -x+5 & x > 3 \end{cases}$$

(a) Graph this function.

Now, evaluate the following, or state that the answer does not exist.

(b) $\lim_{x \to 3} g(x)$

(c) $\lim_{x \to 5} g(x)$

(d) $\lim_{x \to 0^+} g(x)$

(e) $\lim_{x \to 0^-} g(x)$

(f) $g(3)$

(g) $g(0)$

6. Find the derivatives of the following functions.

Don't simplify!

(a) $f(x) = \dfrac{2x^3 + 6x + 1}{x^5 + x^3}$

(b) $f(x) = (2x^8 + 4x^{-2} - 6)(x^2 + 3x^5)$

(c) $f(x) = \pi^2 + \pi x^2$

2.5 Practice (Chain Rule)

Ex 1 $D_x\left(\cos^2(\cos(\cos(\sin(2x))))\right)$

Chain Rule

$(f(g(x)))'$

$= f'(g(x))g'(x)$

(work from outside in)

Ex 2 Find y'.

(a) $y = (2x^{7/2} - 4x^2)^3 + \tan^2(3x-1)$

(b) $y = \left(\dfrac{\sin(5x)}{\sqrt{x} - 1/x^2}\right)^4$

Ex 3 Find $D_x\left(F\left(x^2-\frac{1}{x^2}\right)\right)$ if $F(x)$ is some differentiable fn.

2.6 Practice (Higher Order Derivatives)

Ex 1 Find $\frac{d^3(x^{-3})}{dx^3}$

Ex 2 Find $D_x^{14}(96x^{14}-81x^9)$

Notation for $y=f(x)$

$D_x(f(x)) = f'(x) = \frac{dy}{dx}$

$= \frac{df(x)}{dx} = y' = D_x(y)$

$f''(x) = y'' = D_x^2(f)$

$= \frac{d^2f}{dx^2}$

$f'''(x) = y''' = D_x^3(f)$

$= \frac{d^3f}{dx^3}$

$f^{(n)}(x) = y^{(n)} = D_x^n(f)$

$= \frac{d^nf}{dx^n}$

Ex 3 If $s = \frac{1}{10}(t^4 - 14t^3 + 60t^2)$, find the velocity of the moving object when its acceleration is zero.

Ex 4 Find $f''(2)$. $f(x) = \frac{(x+1)^2}{x-1}$

2.7 Practice (Implicit Differentiation)

Ex 1 Find $\frac{dy}{dx}$.

(a) $x^2 + 2x^2y + 3xy = 0$

(b) $\cos(xy^3) = y^3 + x$

Ex 3 Find y'' at $(3,4)$ if $x^2+y^2=25$.

Ex2 Find $\frac{dy}{dx}$.

(a) $y = \sqrt[3]{x^2 \cos x}$

(b) $\sqrt{xy} + \cos x = 3y^2$

2.8 Practice (Related Rates)

Ex 1 The vertex angle θ opposite the base of an isosceles triangle w/ equal sides of length 100 cm is increasing at $\frac{1}{10}$ radian per minute. How fast is the area of the triangle increasing when the vertex angle measures $\pi/6$ radians?

(#13)

Ex 2 A metal disk expands during heating. If its radius increases at a rate of 0.02 in/sec, how fast is the area of one of its faces increasing when its radius is 8.1 inches?

Ex 3 A woman on a dock is pulling in a rope fastened to the bow of a small boat. If the woman's hands are 10 ft higher than the point where the rope is attached to the boat and if she is retrieving the rope at a rate of 2 ft/sec, how fast is the boat approaching the dock when 25 ft of rope is still out?

2.9 Practice (Differentials)

$\Delta y \approx dy = f(x+\Delta x) - f(x)$

$dy = f'(x)\, dx$

$f(x+\Delta x) \approx f(x) + dy$

$= f(x) + f'(x)\, dx$

this is because

$f'(x) \approx \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Ex 1 Find dy.

(a) $y = (x^{10} + \sin^2 2x)^{1/2}$

(b) Let $y = f(x) = x^3$. $x = 0.5$, $dx = 1$.

Ex 2 Let $y = \frac{1}{x^3}$. Find dy, if

(a) $x = 1$, $dx = 0.5$

(b) $x = -2$, $dx = 3/4$

Ex 3 For $y = x^2 - 3$, find Δy and dy for $x = 3$ and $dx = \Delta x = -0.12$

Ex 4 All six sides of a cubical metal box are $1/4$ inch thick, and the interior volume is 40 in^3. Use differentials to find the approximate volume of metal used to make the box.