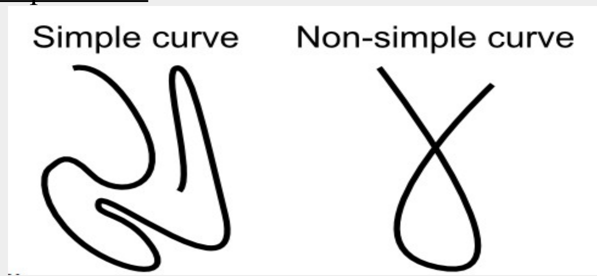
10.4 Parametric Equations

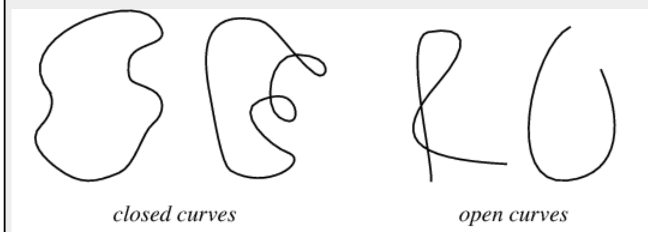
Ex 1: Eliminate the parameter t. Graph the curve and tell if it's simple and closed.

$x = t^2 + 1$, $y = t - 1$, $-2 \leq t \leq 2$

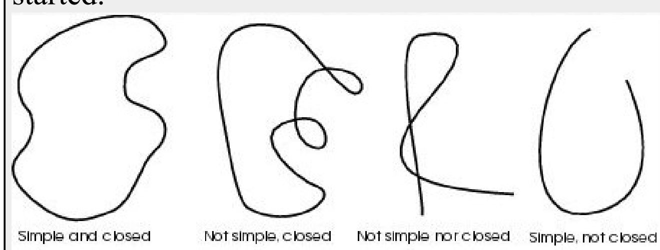
simple curve: A curve that does not cross itself.



closed curve: A curve with no endpoints and which completely encloses an area.



simple, closed curve: A connected curve that does not cross itself and ends at the same point where it started.



If $x = f(t)$, $y = g(t)$ and $f'(t)$, $g'(t)$ exist and are continuous (and $f'(t) \neq 0$ on $\alpha < t < \beta$), then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ and

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} .$$

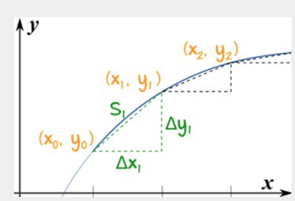
Ex 2: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given $x=\cot(t)-3$, $y=-2\csc(t)$, $t\in(0,\pi)$. (*Do NOT eliminate the parameter first.*)

Ex 3: Find the length of the curve given by $x=t^3$, $y=6t^2$, $t\in[1,4]$.

Length of a Curve:
(From calc 1)

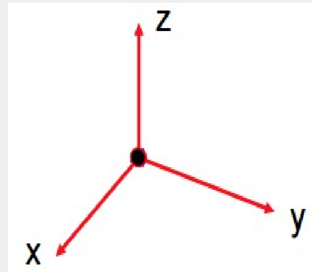
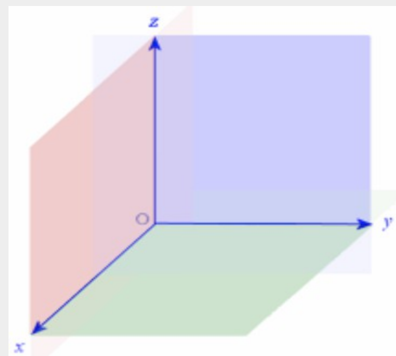
$$L=\int_a^b ds \quad \text{where}$$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$



This is the calculus version of what famous theorem?

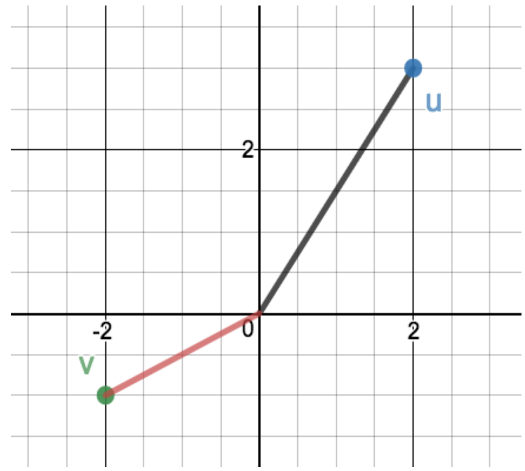
11.1 Cartesian Coordinates in Three-space (3-d)

Ex 1: Plot the following and indicate which type of entity it is. (a) the ___________________ (1, 3, -4)	 (default orientation of 3-d axes in your book)
(b) the ____________________ 3x - 2y + z = 6	
(c) the ______________________ x = 4	(x_1, y_1, z_1) and (x_2, y_2, z_2) are two 3-d points (1) Then distance between the two points is given by $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$.
(d) the _______________________ with center (2, 4, 7) and radius 1.	(2) The midpoint between the two points is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$. (3) A sphere with center (h, k, l) and radius r is given by $(x-h)^2+(y-k)^2+(z-l)^2=r^2$.

Ex 2: Describe the graph of $xz = 0$ in 3-space.	Question: In 2-d, I can draw axes and a point and estimate what point it is. Can I do that in 3-d also?
Ex 3: Calculus Hero is at the point (5, 4, -2) and there is a treasure at the point (1, -3, -1). Calculus Hero's lasso is 10 units long. Can they use the lasso to reach the treasure?	Ex 4: Calculus Hero's nemesis lands halfway between Calculus Hero and the treasure. At what point is the nemesis?

Ex 5: Find the center and radius of the following sphere. $x^2+y^2+z^2-2x+18z=-57$

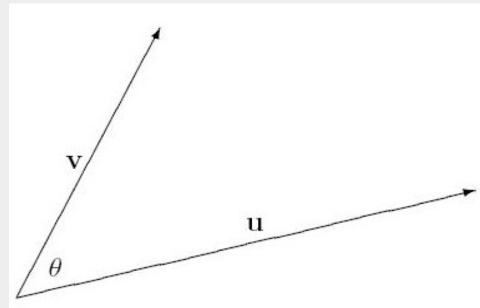
11.2 Vectors

Ex 1: Vectors $\vec{u}$ and $\vec{v}$ are shown. (a) Write $\vec{u}$ and $\vec{v}$ algebraically.	
(b) Find the magnitude of $\vec{u}$ and $\vec{v}$ and the unit vectors, i.e. find $\lVert\vec{u}\rVert$ & $\lVert\vec{v}\rVert$ and $\hat{u}$ & $\hat{v}$.	(c) Calculate (i) $2\vec{u}+\vec{v}$ and (ii) $\vec{u}-\vec{v}$.

Ex 2: The water from a fire hose exerts a force of 150 pounds on the person holding the hose. The nozzle weighs 10 pounds. What is the magnitude and direction of the force exerted by the person holding the hose?

Ex 3: An airplane flies 400 mph in still air. How should the airplane be headed and how fast will it be flying (with respect to the ground) if it flies against a 20 mph wind blowing $N\,40^{\circ}\,W$? Assume the plane wants to travel due north.

11.3 The Dot Product

Ex 1: Let $\vec{u}=<3,-1,1>$ and $\vec{v}<2,1,0>$. Find the following. (a) $\|\vec{u}\|(\vec{u}\cdot\vec{v})$	$\vec{u}=<u_1\ ,\ u_2\ ,\ u_3>$ and $\vec{v}=<v_1\ ,\ v_2\ ,\ v_3>$ angle between the two vectors $\hat{u}$ and $\hat{v}$ = θ 
(b) angle between $\vec{u}$ and $\vec{v}$	(1) $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos(\theta)$ (2) $\vec{u}\cdot\vec{v}=u_1 v_1+u_2 v_2+u_3 v_3=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$
(c) Is the angle between $\hat{u}$ and $\hat{v}$ different or the same as the angle between $\vec{u}$ and $\vec{v}$? Why?	(3) Note: (1) implies that $\hat{u}\cdot\hat{v}=\cos(\theta)$ And notice that (4) $\|\vec{u}\|^2=\vec{u}\cdot\vec{u}$

Question: What is a normal vector?

Ex 2: Let $\vec{u}$ and $\vec{v}$ be vectors and θ be the angle between $\vec{u}$ and $\vec{v}$. Write three more statements that are different but equivalent to the first one.

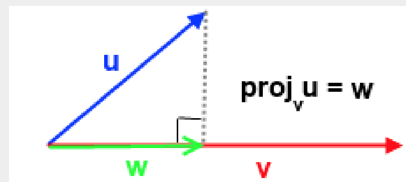
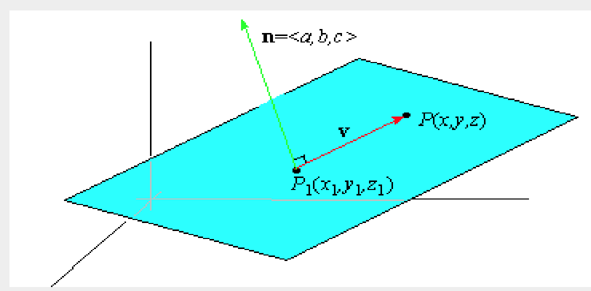
(i) $\vec{u}$ and $\vec{v}$ are orthogonal.

(ii)

(iii)

(iv)

Ex 3: Show that <-1, 2, 0> is parallel to the plane 6x + 3y + z = 2.

Ex 4: Find the equation of the plane through the point (-1, 2, -3) and parallel to the plane $2x+4y-z=6$.	Important vector Facts/Formulas: (1) angle between $\vec{u}$ and $\vec{v}$ $\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)=\cos^{-1}(\hat{u}\cdot\hat{v})$ (2) projection of $\vec{u}$ onto $\vec{v}$ $pr_{\vec{v}}\vec{u}=\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}=(\vec{u}\cdot\hat{v})\hat{v}$ 
	(3) equation of a plane (a) $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ which is equivalent to $<A,B,C>\cdot<x-x_0,y-y_0,z-z_0>=0$ where (x_0,y_0,z_0) is a point on the plane and <A, B, C> is a normal vector to the plane.  (b) The plane equation can also be written as $Ax+By+Cz=D$ where $D=Ax_0+By_0+Cz_0$. (4) The shortest distance from a point (x_0,y_0,z_0) to a plane $Ax+By+Cz=D$ is given by $L=\frac{\|Ax_0+By_0+Cz_0-D\|}{\sqrt{A^2+B^2+C^2}}$.

11.3 (continued)

Ex 5: The planes $2x + 3y - 5z = 2$ and $2x + 3y - 5z = 9$ are parallel. What is the distance between them?

11.4 The Cross Product

<table>
<tr>
<td>

Ex 1: Let $\vec{a}=<3,4,1>$, $\vec{b}=<-2,0,5>$ and $\vec{c}=<2,-1,3>$.

Find the following.

(a) $(\vec{a}+\vec{b})\times\vec{c}$

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<u>Cross Product Facts/Formulas</u>:

Given $\vec{u}=<u_1 , u_2 , u_3>$ and $\vec{v}=<v_1 , v_2 , v_3>$

(1c) $\vec{u}\times\vec{v}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

produces a scalar or vector
(circle one)

(2c) $\vec{u}\times\vec{v}=\vec{0}$ means what is true about the relationship between $\vec{u}$ and $\vec{v}$?

____

(3c) $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\|\sin(\theta)$

==> $\|\hat{u}\times\hat{v}\|=\sin(\theta)$

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<tr>
<td>

(b) $(\vec{a}\times\vec{b})+(\vec{b}\times\vec{a})$

</td>
<td>

Compare the cross product to the dot product:

(1d) $\vec{u}\cdot\vec{v}=u_1 v_1+u_2 v_2+u_3 v_3$

produces a scalar or vector
(circle one)

(2d) $\vec{u}\cdot\vec{v}=0$ means what is true about the relationship between $\vec{u}$ and $\vec{v}$?

____

(3d) $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos(\theta)$

==> $\hat{u}\cdot\hat{v}=\cos(\theta)$

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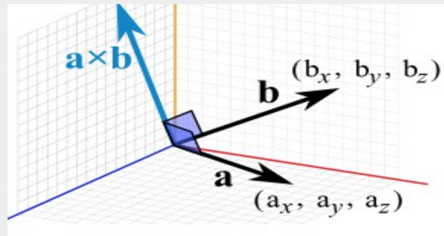
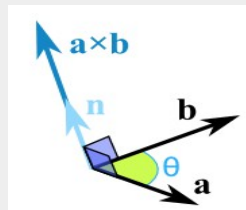
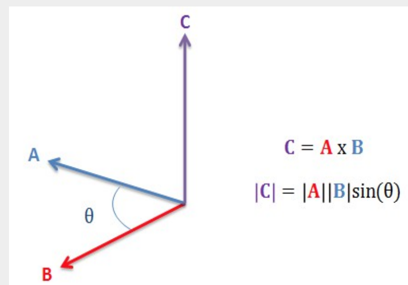
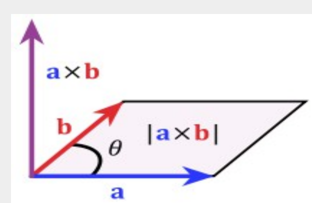
</tr>
</table>

11.4 (continued)

<table>
<tr>
<td>Ex 2: Find all vectors perpendicular/normal to both $\vec{a}=<-2,5,-2>$ and $\vec{b}=3\hat{i}-2\hat{j}+4\hat{k}$.</td>
<td>






</td>
</tr>
<tr>
<td>Ex 3: Find the area of the triangle with vertices (1, 2, 3), (3, 1, 5) and (4, 5, 6).</td>
<td>Given $\vec{w}$, $\vec{u}$, $\vec{v}$ $\in \mathbb{R}^3$ and $c \in \mathbb{R}$
(4c) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$
(5c) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$
(6c) $c(\vec{u}\times\vec{v})=(c\vec{u})\times\vec{v}=\vec{u}\times(c\vec{v})$
(7c) $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$
(8c) $(\vec{u}\times\vec{v})\cdot\vec{w}=\vec{u}\cdot(\vec{v}\times\vec{w})$
(9c) $\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}$
(10c) Area of a parallelogram spanned by $\vec{u}$ and $\vec{v}$:
$Area=A=\|\vec{u}\times\vec{v}\|$


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</table>

Ex 4: Find the equation of the plane through the point (2, -1, 4) that is perpendicular to both the planes $x-3y+2z=7$ and $2x-2y-z=-3$.

Ex 5: Given the points A(-1, 0, 5), B(2, 1, 3) and C(4, 2, 9), find the equation of the plane through them.

Strategy Used:

11.5 Vector-Valued Functions and Curvilinear Motion

<table>
<tr>
<td>Ex 1: Find the limit, if it exists.
$\lim_{t\to -2}\left(\frac{2t^2-10t-28}{t+2}\hat{i}-\frac{7t^3}{t-3}\hat{j}\right)$</td>
<td>f , g , h are $\mathbb{R}-$valued functions (i.e. input = a real number and output = a real number)

vector-valued function F is given by
$\vec{F}(t)=f(t)\hat{x}+g(t)\hat{y}+h(t)\hat{z}$
$= f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$
$= \langle f(t)\ ,\ g(t)\ ,\ h(t)\rangle$
From the above notation, we can see that

input for F = ____________________________

output for F = ____________________________</td>
</tr>
<tr>
<td>Ex 2: What is the domain of the function
$\vec{r}(t)=\ln(t-1)\hat{i}+\sqrt{20-t}\,\hat{j}$?</td>
<td>Assuming t represents time, for physical applications

$\vec{r}(t)$ is ____________________________

$\vec{v}(t)$ is ____________________________

$\vec{a}(t)$ is ____________________________

Question: What is the difference between speed and velocity?</td>
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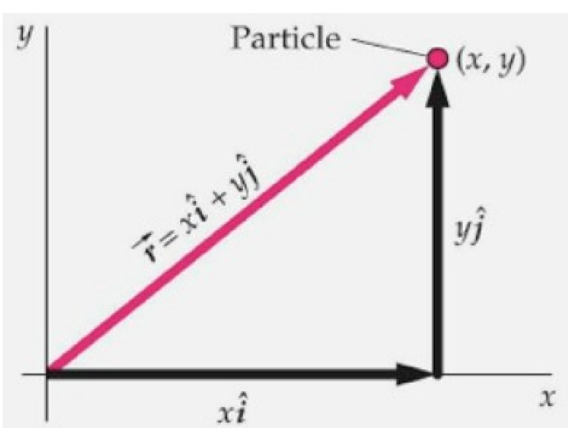
<td>
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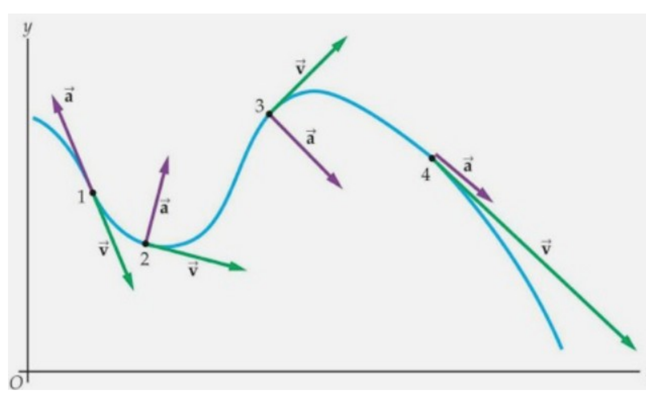
<td>
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</table>

11.5 (continued)

Ex 3: Given the position vector

$\vec{r}(t)=t^6\hat{i}+(6t^2-5)^6\hat{j}+t\hat{k}$

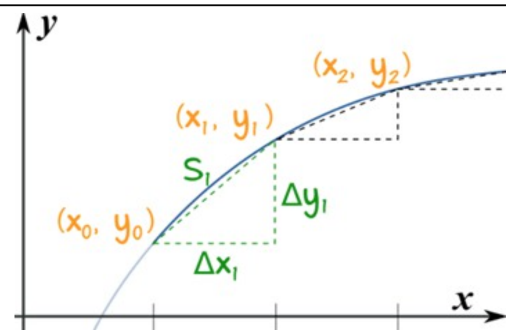
(a) Find $\vec{v}(t)$.

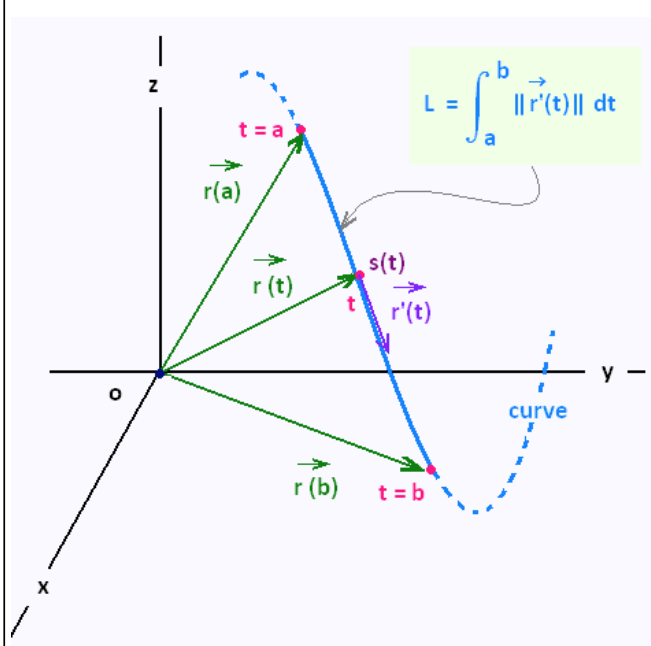
(b) Find $\vec{a}(t)$.

(c) What is the speed when t = 1?

Ex 4: Find the length of the curve given by the vector equation

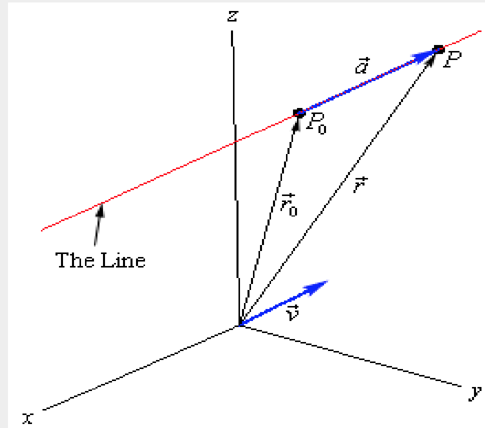
$\vec{r}(t)=t\cos t\,\hat{i}+t\sin t\,\hat{j}+\sqrt{2}\,t\hat{k}$, $t\in[0,2]$





11.6 Lines and Tangent Lines

Ex 1: Find the equations of the line through the point (-1, 3, 6) and parallel to the vector <2, 0, 5>.



<u>In 2-d</u>:
If we have a point (x_0, y_0) and a slope (direction information), then the equation of the line is given as

$$y - y_0 = m(x - x_0) \Leftrightarrow y - y_0 = \frac{\Delta y}{\Delta x}(x - x_0)$$

$$\Leftrightarrow \frac{y - y_0}{\Delta y} = \frac{x - x_0}{\Delta x}$$

<u>In 3-d</u>:
If we have a point $(x_0, y_{0,} z_0)$ and a direction vector $\vec{v} = <a, b, c>$, then the equation of the line is given by
$\vec{r} = \vec{r}_0 + \vec{v}\,t$ where $\vec{r}_0 = <x_{0,} y_{0,} z_0>$ (i.e. $\vec{r}_0$ is the vector from the origin to the point on the line) and t is a parameter.
This can also be written as
$<x, y, z> = <x_0, y_{0,} z_0> + <a, b, c>t$
or equivalently, the
<u>Parametric Equations of the 3-d Line</u>:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

We can write these parametric equations instead as one compound set of
<u>Symmetric Equations of the Line</u>:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

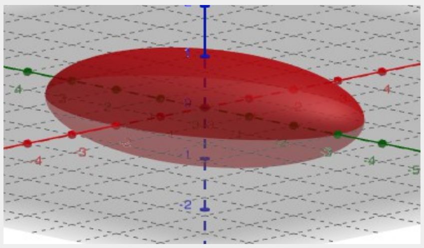
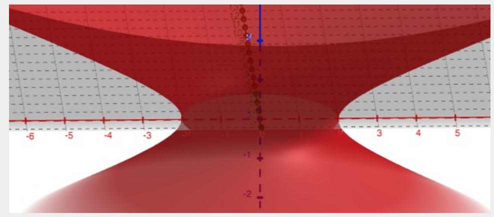
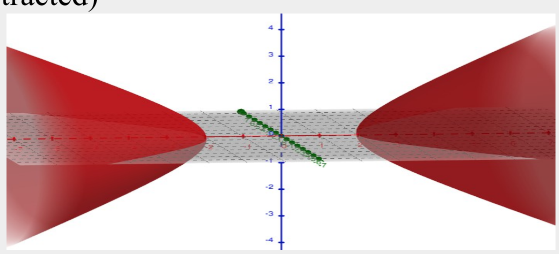
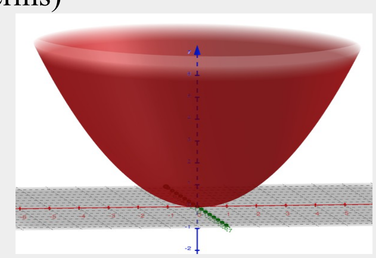
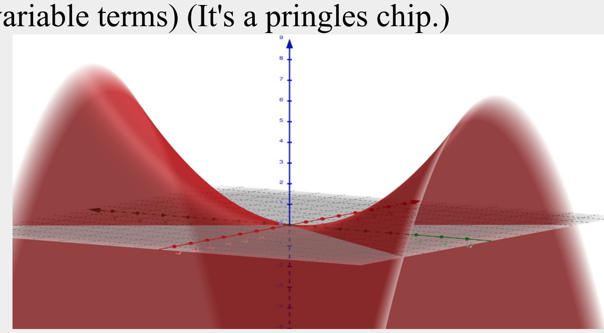
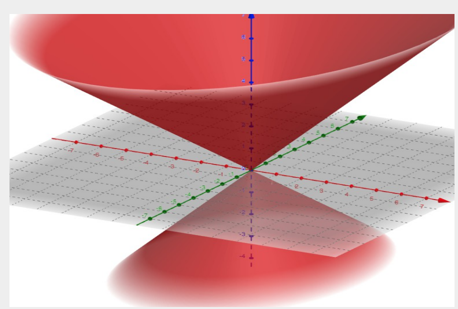
Ex 2: Find the line equations for the line through the point (2, -4, 5) that is parallel to the plane 3x + y - 2z = 5 and perpendicular to the line $\frac{x+8}{2}=\frac{y-5}{3}=\frac{z-1}{-1}$.	To find the equations of the Line: 1. Given a point $(x_0, y_{0,} z_0)$ and a direction vector $\vec{v}$, the line is computed as $\vec{r}=\vec{r_0}+\vec{v}\,t$ where $\vec{r_0}=<x_{0,} y_{0,} z_0>$. OR 2. Given two points, (a) compute $\vec{v}$ = vector from one of the points to the other point. Then (b) compute $\vec{r}=\vec{r_0}+\vec{v}\,t$ where $\vec{r_0}=<x_{0,} y_{0,} z_0>$ (and $(x_0, y_{0,} z_0)$ is one of the two given points).
Ex 3: Find the equation of the plane containing the line $\begin{cases} x=3 \\ y=1+t \\ z=2t \end{cases}$ and perpendicular to the intersection of the planes 2x - y + z = 0 and y + z + 1 = 0.	

Ex 4: Find the line of intersection between the two planes (1) $x - 3y + z = 5$ and (2) $6x - 5y + 4z = 3$.

Ex 5: Find the line that is tangent to the curve given parametrically by

$$\begin{cases} x = 2t^2 \\ y = 4t \\ z = t^3 \end{cases}, \text{ at } t = 2 .$$

11.8 Surfaces in 3-space

<table>
<tr>
<td>(1) Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
(all variables are squared and everything is added together, giving 1 on the RHS)
</td>
<td>(2) Hyperboloid of One Sheet
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
(major axis: z- axis, because it follows the subtraction sign)
</td>
</tr>
<tr>
<td>(3) Hyperboloid of Two Sheets
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
(major axis: x-axis, because it's the only one not subtracted)
</td>
<td>(4) Elliptic Paraboloid
$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
(major axis: z-axis, because it's the variable NOT squared and there's addition between the other variable terms)
</td>
</tr>
<tr>
<td>(5) Hyperbolic Paraboloid
$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$
(major axis: z-axis, because it's the variable NOT squared and there's subtraction between the other variable terms) (It's a pringles chip.)
</td>
<td>(6) Elliptic Cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
(major axis: z-axis, because it's the only variable term being subtracted)
</td>
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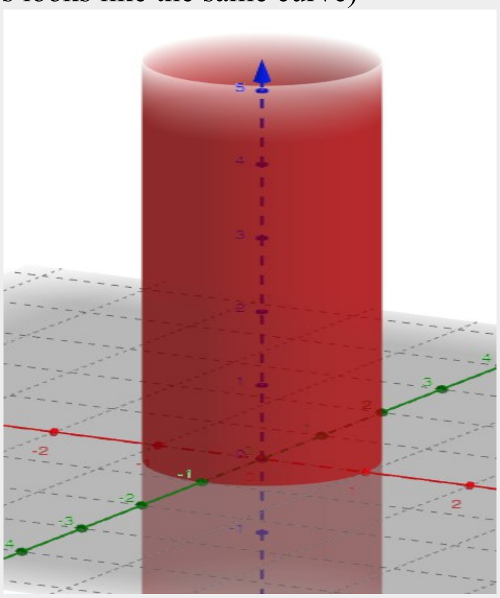
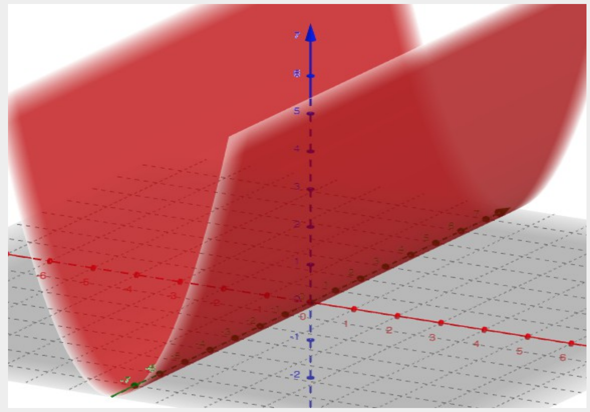
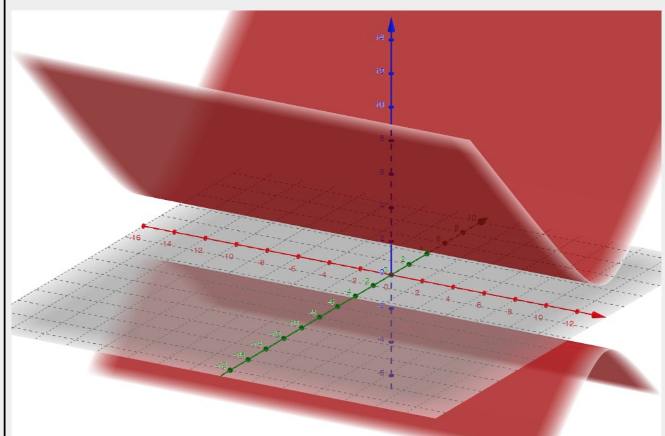
</table>

11.8 (continued)

(7) Cylinder
examples: $z=y^2$, $x^2+y^2=1$, $z^2-y^2=4$
(We can tell it's a cylinder because one of the variables is missing in the surface equation, which means the variable that's missing behaves like an axis upon which the perpendicular cross-section always looks like the same curve)







Ex 1: Name the shape and sketch the graph of the following surfaces.

(a) $z^2+x^2=9$

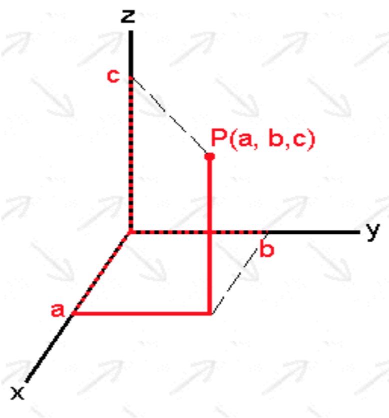
(b) $y^2+z^2-4x^2+4=0$

Ex 2: Name the shape and sketch the graph of the following surfaces. (a) $9x^2+25y^2+9z^2=225$
(b) $y^2-x^2+z=0$
(c) $2x^2-6z^2=0$

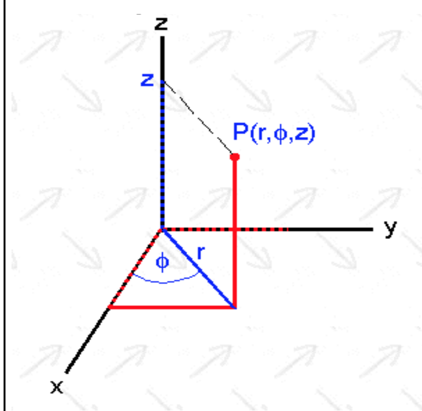
Ex 3: What surface results when the curve z = 2y in the yz-plane is revolved around the z-axis? Sketch it and find the equation for it.

11.9 Cylindrical and Spherical Coordinates

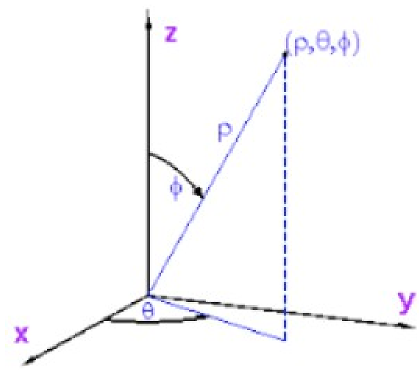
Cartesian Coordinates



Cylindrical Coordinates



Spherical Coordinates



Ex 1: Change the point $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$ from spherical to rectangular coordinates.

C = Cylindrical
S = Spherical
R = Rectangular

<u>Cylindrical Coordinates</u>:

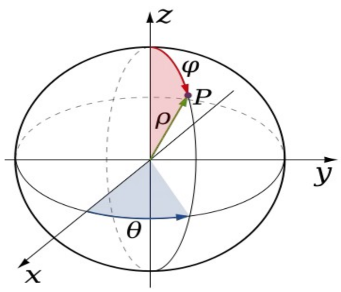
From C to R:	From R to C:
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\tan\theta = \frac{y}{x}$ $z = z$ Note: $r \geq 0$, $\theta \in [0, 2\pi)$

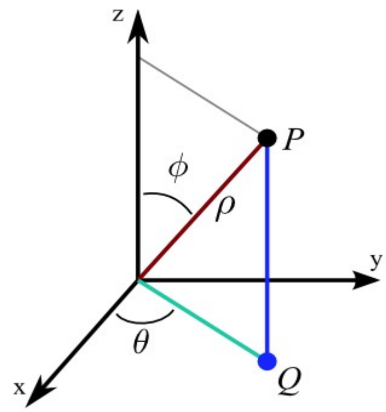
Ex 2: Convert the point (1, -3, 2) from rectangular to spherical coordinates.

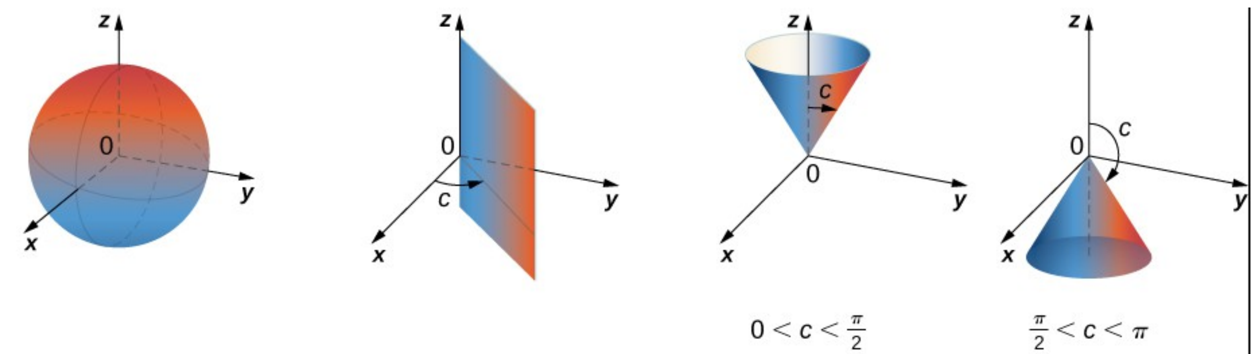
<u>Spherical Coordinates</u>:

From S to R:	From R to S:
$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\tan\theta = \frac{y}{x}$ $\cos\phi = \frac{z}{\rho}$ Note: $\theta \in [0, 2\pi)$, $\rho \geq 0$, $\phi \in [0, \pi]$

Spherical Coordinates:

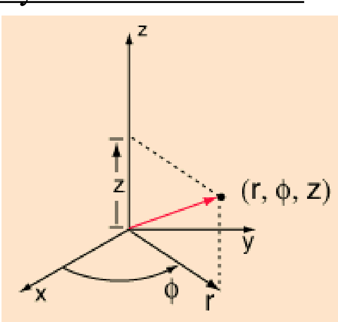


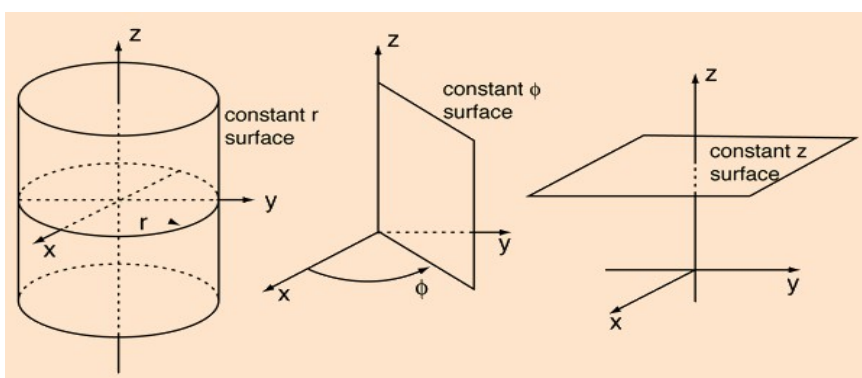




Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

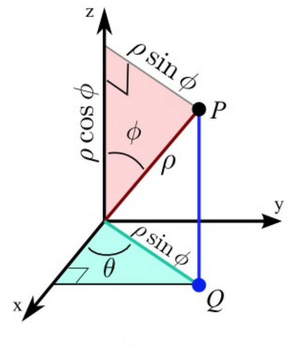
Cylindrical Coordinates:

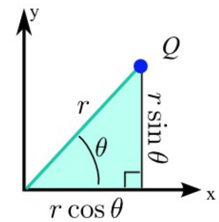




11.9 (continued)

Ex 3: Compute formulas to go from C to S and from S to C coordinates.





Ex 4: Sketch the graph of the given cylindrical or spherical equation. (a) $r=2\sin(2\theta)$	Ex 5: Change the equation to the indicated coordinate system. (a) $\rho \sin \phi = 2$ to Cylindrical
(b) $\rho = \sec \phi$	(b) $r^2+6z^2=7$ to Spherical
(c) $\theta = \frac{\pi}{3}$	(c) $x^2+y^2+z^2=16$ to Spherical