12.1 Functions of Two or More Variables

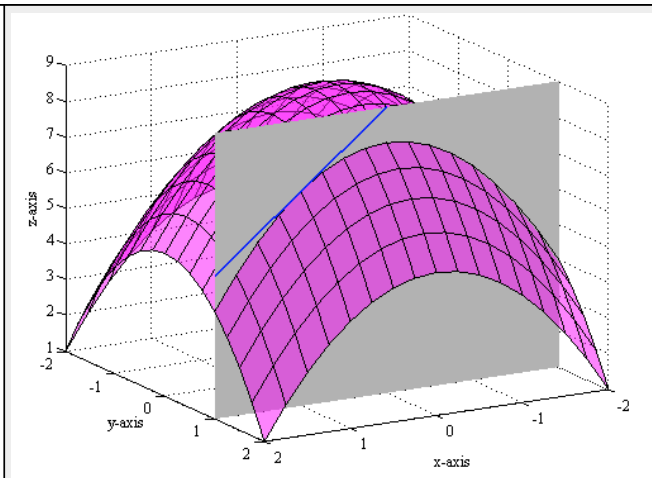
Ex 1: Let $f(x, y, z) = \sqrt{x \cos y} + z^2$ (a) What is the domain of this function? (b) What dimension space does the graph of this function live in? (c) Find $f\left(2, \frac{\pi}{3}, -1\right)$.	Note: $y = f(x)$ is a function of one input variable and its graph lives in 2-d (one input + one output = 2-d) space $z = f(x, y)$ is a function of two input variables and its graph lives in 3-d (two inputs + one output = 3-d) space $w = f(x, y, z)$ is a function of three input variables and its graph lives in 4-d (three inputs + one output = 4-d) space
Ex 2: Find the domain of the function $f(x, y, z) = z \ln y$	Domain still asks for the set of allowed variable values for the input variables.
Ex 3: Sketch the level curves for $z = f(x, y) = 2 - \frac{x^2}{4} - y^2$ for at least three z-values.	Ex 4: Sketch the graph of $z = f(x, y) = x^2 + y^2 - 4$

12.2 Partial Derivatives

Ex 1: Find f_x and f_y given $f(x,y)=\ln(x^2-y^2)$.	(Note: wrt means "with respect to".) Given $z=f(x,y)$ the partial derivative of z wrt x is $f_x(x,y)=z_x=\frac{\partial z}{\partial x}=\frac{\partial f(x,y)}{\partial x}$. Likewise for the partial derivative of z wrt y. $f_y(x,y)=z_y=\frac{\partial z}{\partial y}=\frac{\partial f(x,y)}{\partial y}$
Ex 2: Find the four second order partial derivatives for $f(x,y)=2x^3\cos(4y)$.	For f_{xyy} , work "inside to outside" (or left to right) for this notation, i.e. find f_x then differentiate wrt y $(f_x)_y=f_{xy}$ and then differentiate again wrt y $(f_{xy})_y=f_{xyy}$. Notation: $f_{xyy}=\frac{\partial^3 f}{\partial^2 y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$ In Leibniz notation, we read the partial derivative from right to left to enact it.

Ex 3: Imagine you are on the surface $z = \dfrac{5\sqrt{16 - x^2}}{4}$ at point $\left(2, 3, \dfrac{5\sqrt{3}}{2}\right)$.

(a) Name and sketch the graph of this surface.



This shows a tangent line to a 3-d surface that's in the x-direction, i.e. a way to visualize the partial derivative wrt x.

(b) Find the slope of the tangent line to this point that lies in the x = 2 plane (i.e. tangent to the intersection curve of the surface with the x = 2 plane).

(c) Find the slope of the tangent line to this point that lies in the y = 3 plane (i.e. tangent to the intersection curve of the surface with the y = 3 plane).

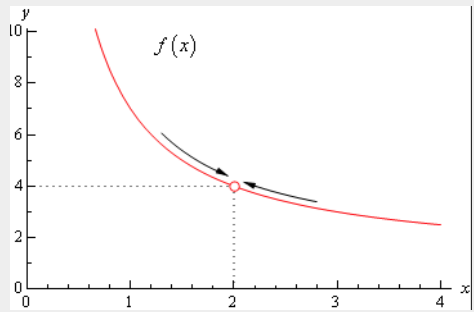
12.3 Limits and Continuity

Ex 1: Find the limit or justify that it does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

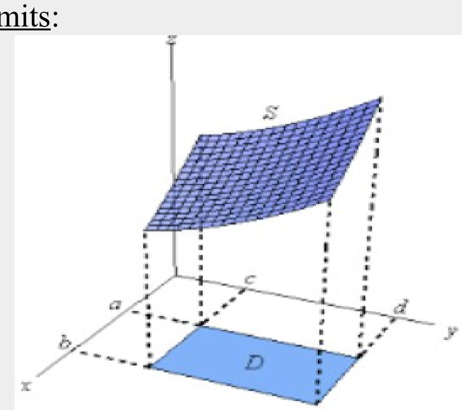
(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\,y^2}{x^2+y^4}$

<u>2-d Limits</u>:



To decide if $\lim\limits_{x\to c} f(x)$ exists, we have to only check both the right and left-hand limits, because we can only approach c from two sides (since x is 1-d).

<u>3-d Limits</u>:



To decide if $\lim\limits_{(x,y)\to(c,d)} f(x,y)$ exists, it's infinitely harder, because we can approach the point (x, y) from infinitely many directions, not just two. So we cannot exhaustively check all those infinitely many directions, like we can for the 2-d case.

A function $z=f(x,y)$ is continuous at (a, b) if $f(a,b)=\lim\limits_{(x,y)\to(a,b)} f(x,y)$, that is if

(1) the limit exists, and
(2) the function value is defined and
(3) they are the same.

12.3 (continued)

<table>
<tr>
<td rowspan="2">Ex 2: Find the limit or justify that it does not exist.
$$\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right)$$</td>
<td>Strategies to show the limit exists (in 3-d, i.e. for $z=f(x,y)$):

1. Case 1: Plug in numbers, and everything is fine, then you're done.

2. Case 2: You get some indeterminate form.
(a) Do algebraic manipulation, like factoring and dividing out or using trigonometric identities, etc.

OR

(b) Change to polar coordinates, if $(x,y)\to(0,0)$ then we can replace that with $r\to 0$.</td>
</tr>
<tr>
<td>Strategies to show the limit DNE (in 3-d, i.e. for $z=f(x,y)$):

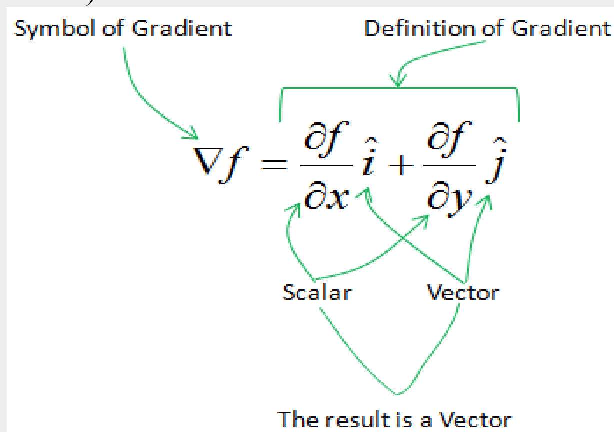
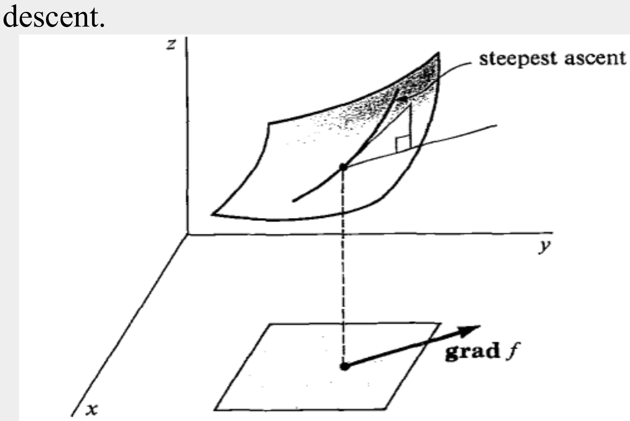
1. Show the limit yields different finite answers depending on how you approach the point. (That is, try two different approaches to the point and show you get different answers.)

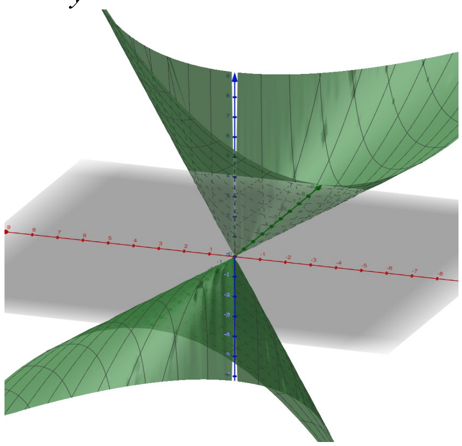
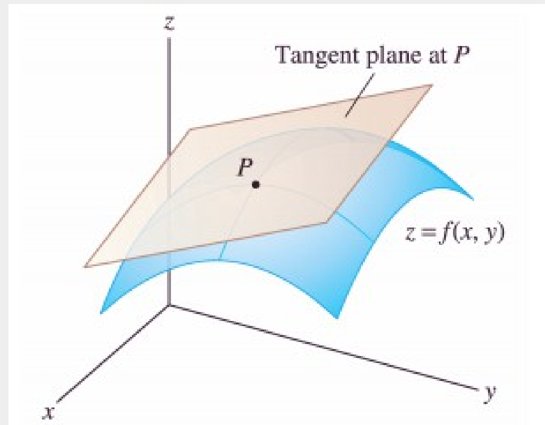
OR

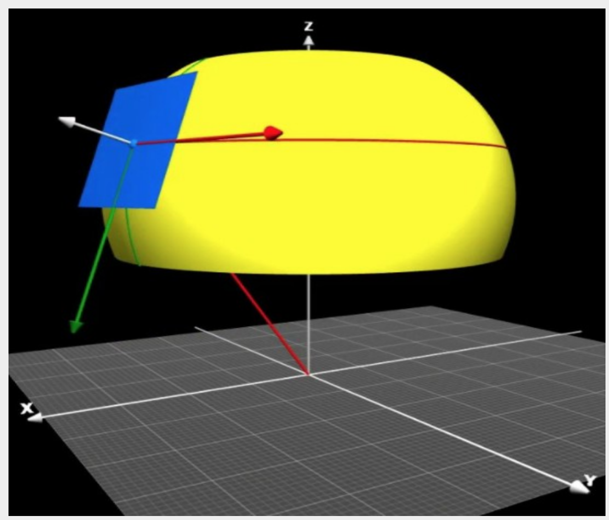
2. Switch to polar coordinates and show the limit DNE.</td>
</tr>
<tr>
<td colspan="2">Ex 3: Describe the largest set S on which f is continuous.
(a) $f(x,y)=\frac{1}{\sqrt{1+x+y}}$
(b) $f(x,y,z)=\ln(4-x^2-y^2-z^2)$</td>
</tr>
</table>

12.4 Differentiability

<table>
<tr>
<td>Ex 1: Find the gradient, ∇f, of $f(x,y)=4x e^{9xy}$.</td>
<td>Let $z=f(x,y)$ be a function and (a,b) be a point in the domain of f.

<u>Gradient of f at point (a,b) :</u>
$\nabla f=<f_x(a,b), f_y(a,b)>$
Important Note: The gradient is a vector (not a number).

</td>
</tr>
<tr>
<td>Ex 2: Find the gradient of $f(x,y)=\frac{x^2}{y}$ at input point (2, -1).</td>
<td>The gradient points in the direction of steepest ascent on the surface at that point. The opposite of the gradient points in the direction of the steepest descent.

</td>
</tr>
</table>

<table>
<tr>
<td>
Ex 3: For the gradient and function in Ex 2, find the equation of the tangent plane at the given point.
$f(x,y)=\frac{x^2}{y}$ at (2, -1)

</td>
<td>
Tangent plane to $z=f(x,y)$ at point (a,b) :
$z=f(a,b)+\nabla f(a,b)\cdot\langle x-a, y-b\rangle$




</td>
</tr>
</table>

Questions:

1. Does the gradient always exist for any surface?

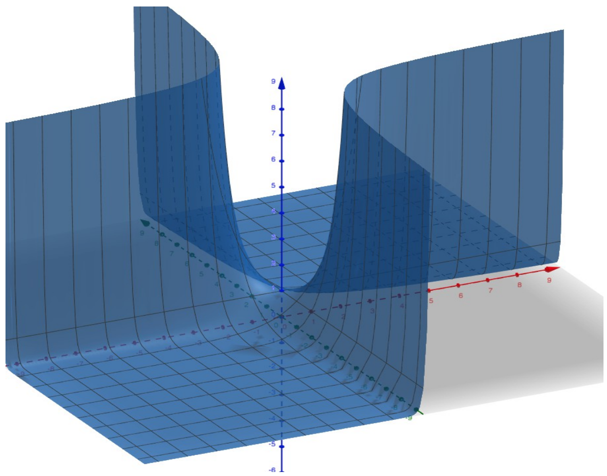
2. Does the tangent plane at a point on the surface always exist?

3. How is the gradient related to differentiability?

Ex 4: Find the equation of the tangent "hyperplane" to
$w = f(x, y, z) = 2y\cos(2\pi x) + 4x\cos(\pi y) + xz$ at input point (1, 1/2, 3).

12.5 Directional Derivatives

Ex 1: Find the directional derivative of $f(x, y) = e^{-xy}$ at the input point (1, -1) in the direction of $\vec{u} = -\hat{i} + \sqrt{3}\,\hat{j}$.
(Hint: Is $\vec{u}$ a unit vector?)



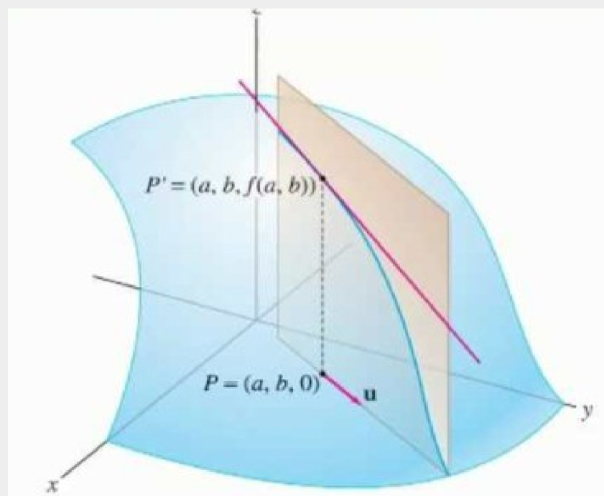
Let $z = f(x, y)$ be a function, (a, b) an input point in the domain of f, and $\hat{u}$ a unit vector (2-d).

Theorem: If f is differentiable, then f has a directional derivative at (a, b) in the direction of $\hat{u} = u_1\hat{i} + u_2\hat{j}$.

The directional derivative is given by

$$D_{\hat{u}} f(a, b) = \hat{u} \cdot \nabla f(a, b)$$
$$= \langle u_1, u_2 \rangle \cdot \langle f_x(a, b), f_y(a, b) \rangle$$
$$= u_1 f_x(a, b) + u_2 f_y(a, b)$$

Note: This returns a number/scalar. It represents the slope of the surface in the direction of $\hat{u}$.



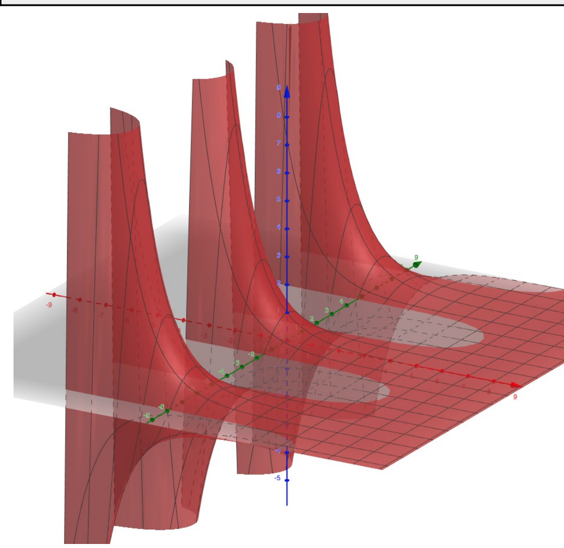
4-d version (and from there we can continue to extrapolate):

$$D_{\hat{u}} f(a, b, c) = \hat{u} \cdot \nabla f(a, b, c)$$
$$= \langle u_1, u_2, u_3 \rangle \cdot \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle$$

Ex 2: Find a unit vector in the direction in which $f(x, y, z) = 4xyz^2$ decreases most rapidly at the point (2, -1, 1). What is the rate of change in that direction?

Theorem:
At the input point (a, b), the function $z = f(x, y)$ increases most rapidly in the direction of the gradient, at a rate of $\|\nabla f(a, b)\|$,
and decreases most rapidly in the direction of the opposite of the gradient, at a rate of $-\|\nabla f(a, b)\|$.

Ex 3: Find the directional derivative of $f(x, y) = e^{-x}\cos y$ at the input point $\left(0, \frac{\pi}{3}\right)$ in the direction toward the origin.



12.6 Chain Rule(s)

Ex 1: Find $\frac{dw}{dt}$ using a chain rule.

$w = xy + yz + xz$ and

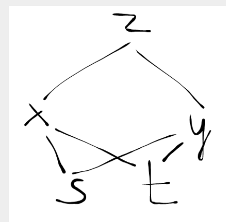
$x = t^2$, $y = 1 - t^2$, $z = 1 - t$

Let $z = f(x, y)$ and $x = x(s, t)$, $y = y(s, t)$ with first partial derivatives at $(x(s,t), y(s,t))$. Then z has first partial derivatives given by

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\text{—}\right) + \frac{\partial z}{\partial y}\left(\text{—}\right) \quad \text{and}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\text{—}\right) + \frac{\partial z}{\partial y}\left(\text{—}\right)$$

Flow chart for this chain rule:

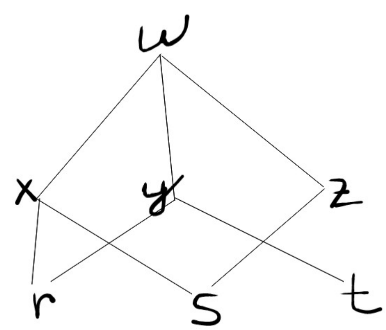


Ex 2: Find $\frac{\partial w}{\partial t}$ using a chain rule.

$w = x^2 - y \ln x$

$x = \frac{s}{t}$, $y^2 = s^2 t$

Question: If the flow chart is given as follows



then find the following:

$\frac{\partial w}{\partial r} =$ ______________________________

$\frac{\partial w}{\partial s} =$ ______________________________

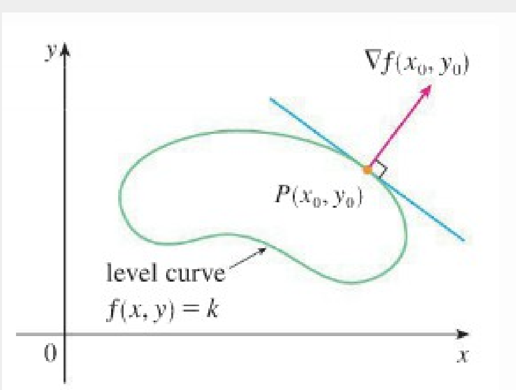
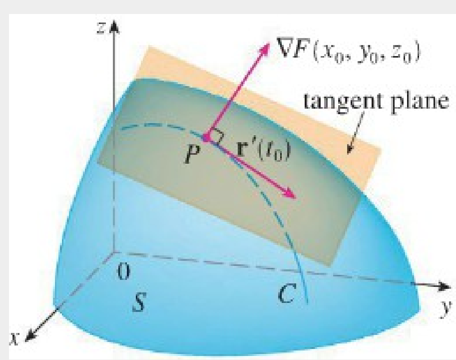
$\frac{\partial w}{\partial t} =$ ______________________________

Ex 3: If $w = x^2 y + z^2$, $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, $z = \rho \cos\phi$

find $\left. \frac{\partial w}{\partial \theta} \right|_{\rho=2\ ,\ \theta=\pi\ ,\ \phi=\frac{\pi}{2}}$.

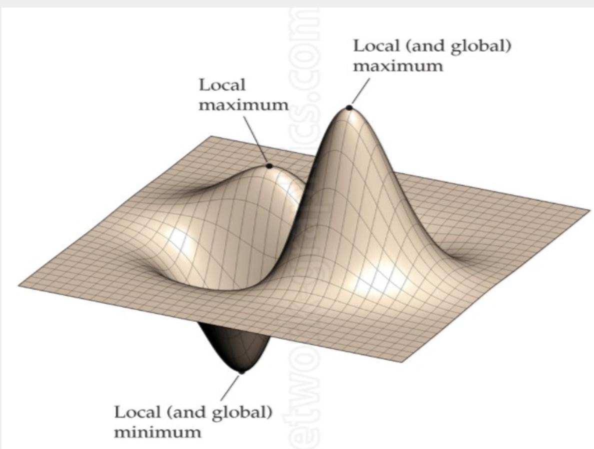
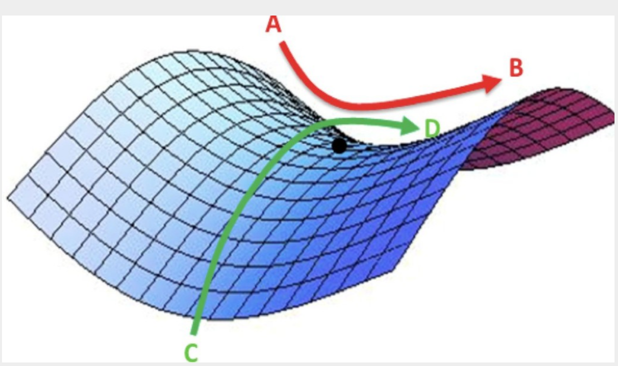
Ex 4: Airplanes A and B depart from point P at the same time. Plane A flies due east and plane B flies $N\,50^{o}\,E$. At a certain time, plane A is 200 miles from P flying 450 mph and plane B is 150 miles from P flying 400 mph. How fast are they separating at that instant?

12.7 Tangent Planes

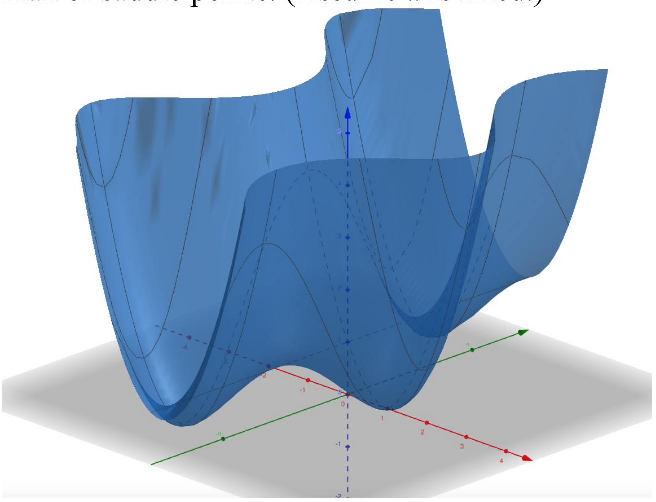
Ex 1: Find the equation of the tangent plane to $x^2+y^2-z^2=4$ at the point (2, 1, 1).	<u>Definition</u>: Let $F(x,y,z)=k$ be a surface in 3-d (k is a constant) and $P_0=(x_0, y_0, z_0)$ be a point on that surface. If F is differentiable AND $\nabla F(x_0, y_0, z_0)\neq\vec{0}$, then the tangent plane to F at P_0 exists. It is the plane orthogonal to $\nabla F(x_0, y_0, z_0)$ that passes through the point P_0 . <u>Equation of the Tangent Plane to F at</u> P_0 : $\nabla F(x_0, y_0, z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle=0$ (Think about what the equation of a tangent plane says: the vector normal to the surface at P_0 is orthogonal to any vector in the plane.)
2-d gradient of an explicit function $z=f(x,y)$ is orthogonal to the level curves of the surface  	3-d gradient of an implicit (or explicit) function $F(x,y,z)=k$ is orthogonal to the surface. 

Ex 2: Find a point on the surface $z=2x^2+3y^2$ where the tangent plane is parallel to the plane $8x-3y-z=0$.	Differentials: Let $z=f(x,y)$ be a differentiable function. The total differential of f is *dz.* $dz=\nabla f\cdot\langle dx,dy\rangle$ (This *dz* is the approximate change in *z*.) The actual change in z is given by $\Delta z=z_2-z_1$ (i.e. the difference in the z-values).
Ex 3: Use differentials to approximate the change in $z=\tan^{-1}(xy)$ from P(2, -0.5) to Q(-2.03, -0.51). Then find Δz (the actual change).	Ex 4: An object's weight in air is 36 lbs = A and its weight in water is W = 20 lbs, with a possible error in each measurement of 0.02 lbs. Find (by approximating) the maximum possible error in calculating its specific gravity from $S(A,W)=\dfrac{A}{A-W}$.

12.8 Maxima and Minima

Ex 1: Draw the domain given by $\{(x, y): 1 < x \leq 4\}$.	<u>Types of critical points</u>: 1. Stationary points (surface has a tangent plane parallel to xy-plane) 2. Singular points (surface is continuous but not differentiable) 3. Boundary points (from given boundary) (Notice that these are basically the same as we saw for 2-d curves.)
Min/Max:  Saddle:  	Vocabulary: For $z = f(x, y)$, 1. Global max 2. Global min 3. Local max 4. Local min 5. Saddle point

Ex 2: For
$f(x, y)=x^2+a^2-2ax\cos y$, $y\in[-\pi, \pi]$
find all critical points. Determine if they are min, max or saddle points. (Assume *a* is fixed.)



Strategy to find max/min & saddle points for $z=f(x, y)$:

1. Find all (x, y) points such that $\nabla f(x, y)=\vec{0}$.

2. Let $D=f_{xx}(x, y)f_{yy}(x, y)-f_{xy}^2(x, y)$.

(a) If $D>0$ AND $f_{xx}(x, y)<0$, then $f(x, y)$ is a local maximum value.

(b) If $D>0$ AND $f_{xx}(x, y)>0$, then $f(x, y)$ is a local minimum value.

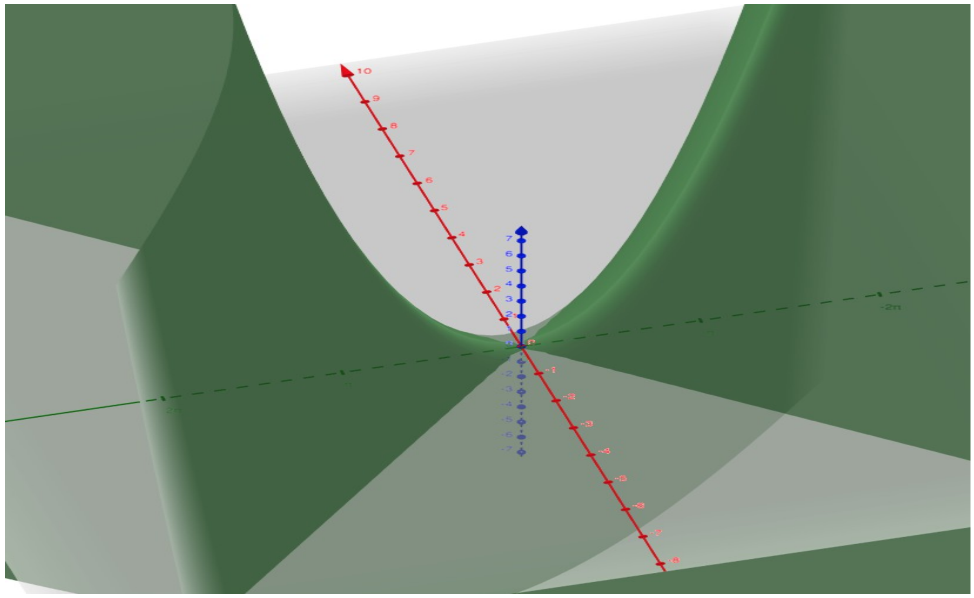
(c) If $D<0$, then the point $(x, y, f(x, y))$ is a saddle point.

(d) If $D=0$, then the test is inconclusive.

3. Determine if any boundary point gives a min or max z-value. (To do this, typically we have to parameterize the boundary and then reduce it to a Calc 1 type of min/max problem to solve.)

12.8 (continued)

Ex 3: Find the min and max values of $z = y^2 - x^2$ (hyperbolic paraboloid...a Pringles chip) on the closed triangle with vertices (0, 0), (1, 2) and (2, -2).



(z = blue, x = red, y = green axes)

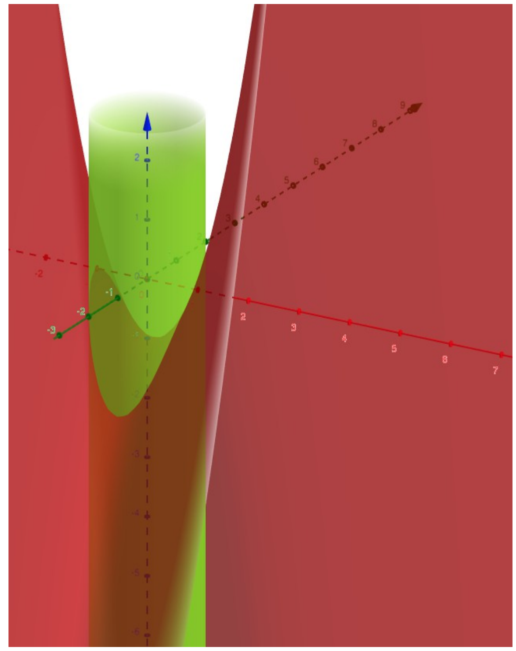
12.8 (continued)

Ex 4: Find the point on the plane $x + 2y + 3z = 12$ that is closest to the origin. What is that minimum distance? (You might want to set up this problem and then take it home to solve.)

12.9 Lagrange Multipliers

Ex 1: Find the minimum of $f(x, y)=x^2+4xy+y^2$ subject to the constraint $x-y-6=0$.	Using Lagrange Multipliers Setup (i.e. when to use Lagrange Multipliers method): Given $z=f(x, y)$, we want points that produce a global min/max AND satisfy an additional constraint/boundary given by $g(x, y)=0$. To find these max/min points: Solve (a) $\nabla f=\lambda \nabla g$ AND (b) $g(x, y)=0$ simultaneously. Note: We can easily expand this to functions of more variables than two.

Ex 2: Use Lagrange Multipliers method to find the min and max values for $f(x,y)=x^2-y^2-1$ on $S=\{(x,y)\mid x^2+y^2\leq 1\}$. (Question: can we use only Lagrange method here? Why or why not?)



12.9 (continued)

Ex 3: Find the 3-d vector of length 9 with the largest possible sum of its components.

Ex 4: Find the point on the plane $x + 2y + 3z = 12$ that is closest to the origin. (This is the same problem as 12.8 Ex 4, but this time, see how much faster it is to solve with Lagrange multipliers.)