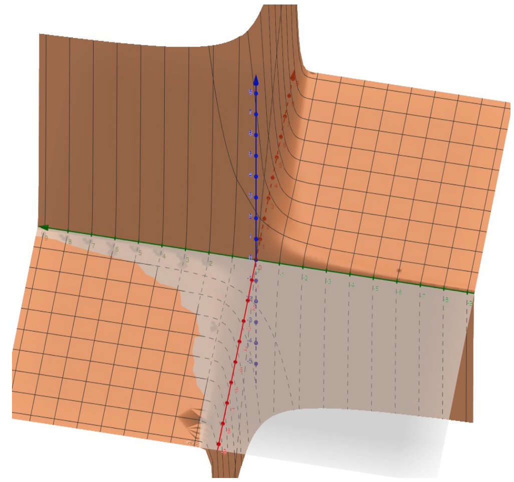
13.1 & 13.2 Double Integrals over Rectangles and Iterated Integration

Ex 1: Given the double integral $\int\int_R (y-x+4)\,dA$ where $R=\{(x,y):0\le x\le 4\ ,\ -4\le y\le 0\}$ (a) Sketch the solid whose volume is given by this integral.	<u>Calculating Signed Volume</u>: Question: What is signed volume?
(b) Calculate the approximate volume.	Let $R=\{(x,y):a\le x\le b\ ,\ c\le y\le d\}$. Draw R in 2d space.
(c) Calculate the exact volume.	For $f(x,y)$, which is continuous over R, the signed volume between the surface $z=f(x,y)$ and the xy-plane (over R) is $\int\int_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx$ OR ___________________________________

Ex 2: Evaluate the integral $\int_0^1 \int_0^2 \frac{y}{1+x^2}\,dy\,dx$	Question: Can we separate this integral into the product of two single integrals? That is, is this a true or false statement? $\int_0^1 \int_0^2 \frac{y}{1+x^2}\,dy\,dx = \left(\int_0^1 \frac{1}{1+x^2}\,dx\right)\cdot\left(\int_0^2 y\,dy\right)$
Ex 3: Evaluate the integral $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx$	Question: Can we separate this integral into the product of two single integrals? If so, how?

In general, what conditions must be met in order for us to split a double integral into a product of two single integrals?

Ex 4: Evaluate the integral. $\int_0^1 \int_0^1 x e^{xy} \, dy \, dx$

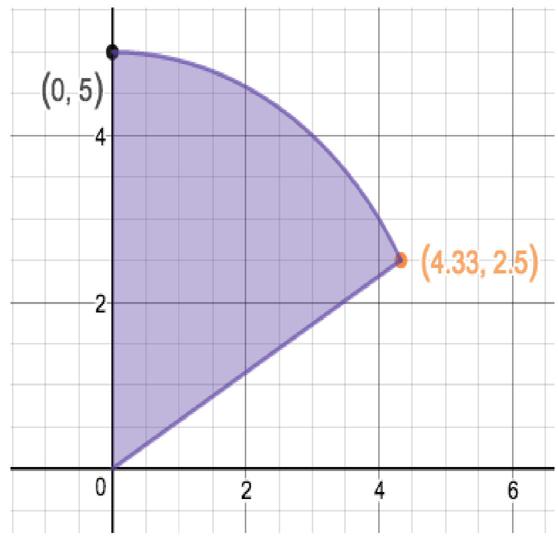


Ex 5: Given the surface $f(x, y) = 1 + x^2 + y^2$ over $R = \{(x, y) : 0 \le x \le 4 \, , \, 0 \le y \le 3\}$, sketch the volume indicated by $\int \int_R f(x, y) \, dA$ and compute that volume.

13.3 Double Integrals Over Non-rectangular Regions

Ex 1: Evaluate the integral. $\int_1^2 \int_0^{x^2} \frac{y^2}{x} \, dy \, dx$	$\int\int_S f(x, y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx$ OR $\int\int_S f(x, y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy$
Ex 2: Evaluate the integral. $\int_1^5 \int_0^{x} \frac{3}{x^2+y^2} \, dy \, dx$	• S is a simple closed curve (not necessarily a rectangle) • dA = dx dy or dy dx • the limits of the innermost integral can be functions of x (or y) (whichever variable hasn't been integrated yet) • Most likely, we can NOT separate the integrals into a product of two single integrals • To switch the order of integration (from dx dy to dy dx, for example) requires some geometric thinking...it's NOT trivial.

Ex 3: Let $z = f(x, y) = x + y$ and S be the region below (a sector of a circle of radius 5).
(Note: $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right) \approx (4.33, 2.5)$)

(a) Determine the limits of integration of the double integral, to calculate the volume of the 3-d region between the surface and the xy-plane, over S. That is, find the limits of integration for $\int\int_S f(x, y)\, dA$. (Note: You must first decide if you want dA = dx dy or dA = dy dx.)	

(b) Calculate the volume.

Ex 4: Sketch the solid in the first octant bounded by the coordinate planes, $2x+y-4=0$ and $8x+y-4z=0$. Then calculate its volume using iterated integration.

Ex 5: For the triangular region in the xy-plane bounded by the vertices (1, 7), (4, 1) and (-2, 1), set up $\int\int_S f(x,y)\,dx\,dy$ and $\int\int_S f(x,y)\,dy\,dx$. (S is the inside of the triangle.)

13.4 Double Integrals in Polar Coordinates

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Ex 1: Evaluate these integrals, if it's possible. (Hint: One of these integrals is "illegal"...which one and why?)

(a) $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\,dr\,d\theta$
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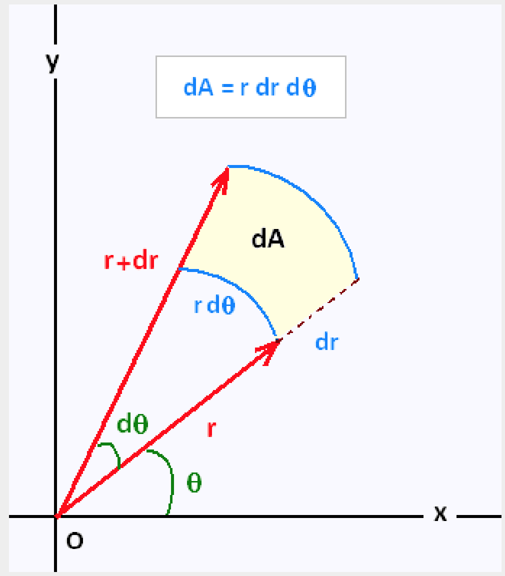
Volume of a solid between surface $z=f(x,y)$ and the xy-plane, over a simple closed region S is given by
$\int\int_S f(r,\theta) r\,dr\,d\theta$

(Note: $dA=r\,dr\,d\theta$ or $dA=r\,d\theta\,dr$)



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<td>
(b) $\int_0^{\sin\theta} \int_0^{\frac{\pi}{2}} r\,d\theta\,dr$
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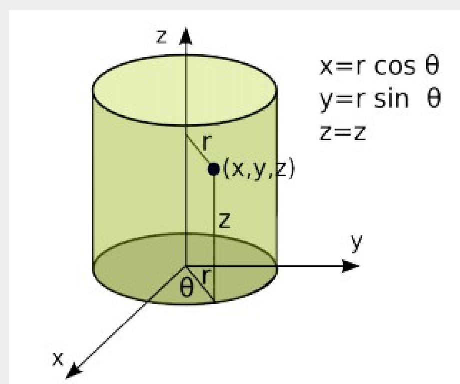

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Ex 2: Sketch the region, convert to polar coordinates and evaluate the volume represented by this integral.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left(x^2 y^2 + y^4\right) dy\, dx$$

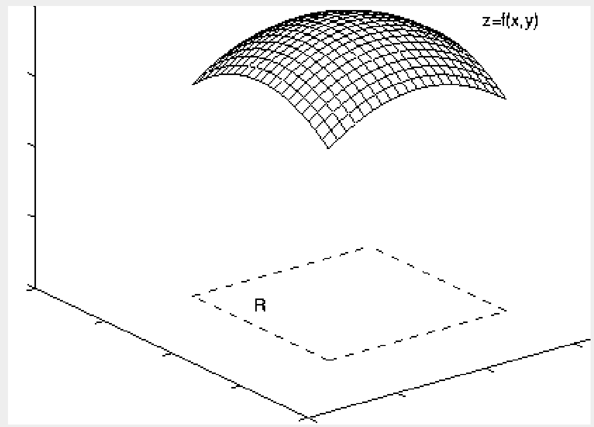
Ex 3: Consider the solid inside the paraboloid $z = 4 - x^2 - y^2$, outside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. Sketch the solid and calculate the volume.

13.6 Surface Area

Ex 1: Make a sketch and find the area of the part of the surface $z=\sqrt{4-y^2}$ in Octant 1 that is directly above the circle $x^2+y^2=4$ in the xy-plane.

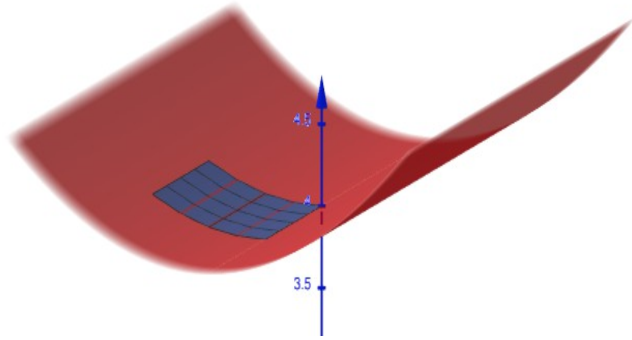
Given $z=f(x,y)$ is a continuous surface/function over S (a closed region in the xy-plane), then the surface area of $z=f(x,y)$ over S is

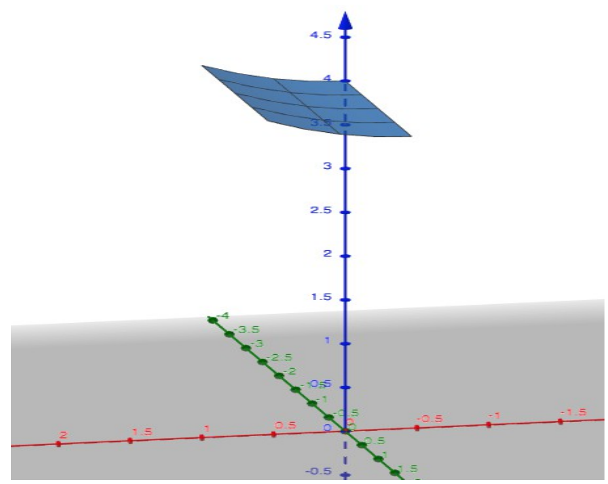
$$SA=\iint_S \sqrt{f_x^2+f_y^2+1}\ dA$$



13.6 (continued)

Ex 2: Find the surface area of the surface $z = \frac{x^2}{4} + 4$ that is cut off by the planes x = 0, x = 1, y = 0 and y = 2.





Ex 3: Find the surface area for the surface that is part of the sphere $x^2+y^2+z^2=a^2$ inside (or synonymously, cut off by) the cylinder $x^2+y^2=ay$ where $a>0$ is fixed.

(Sketch the region first.)

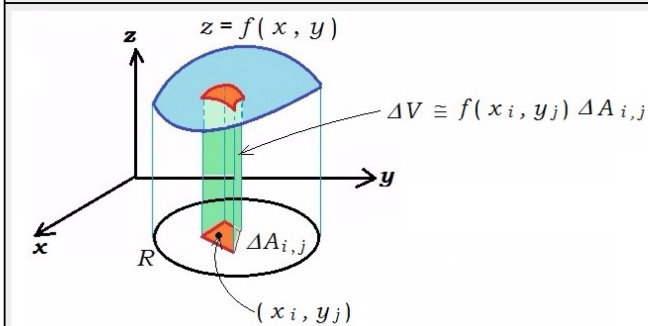
13.7 Triple Integrals in Cartesian (Rectangular) Coordinates

Ex 1: Write the iterated integral (i.e. triple integral) $\iiint_S xyz\,dV$ where $S=\{(x,y,z) : 0 \le x \le 5 ,\ z^2 \le y \le 9 ,\ 0 \le z \le 3\}$. (Hint: You'll need to sketch the S solid first.)

$$\iiint_S f(x,y,z)\,dV = \int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Note: dV can be exchanged for dx dy dz in any order, but you must then choose your limits of integration according to that order!!!

How many orders of the differentials is possible?



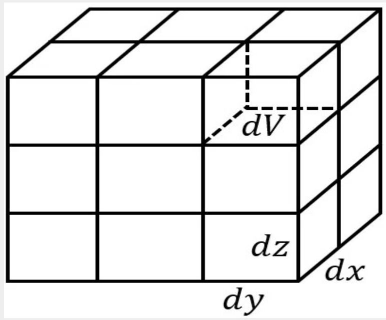
Ex 2: Find the volume of the solid bounded by the cylinder $y = x^2 + 2$ and the planes $y = 4$, $z = 0$ and $3y - 4z = 0$.

13.7 (continued)

Ex 3: Rewrite the integral $\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x, y, z)\, dx\, dz\, dy$ with order dy dx dz.

13.8 Triple Integrals in Cylindrical and Spherical Coordinates

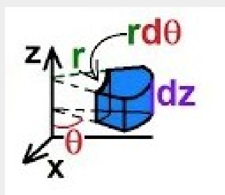
Cartesian/Rectangular:

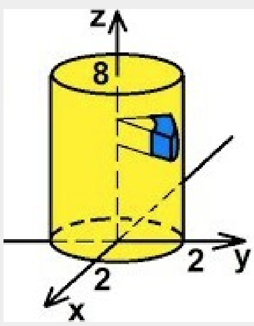


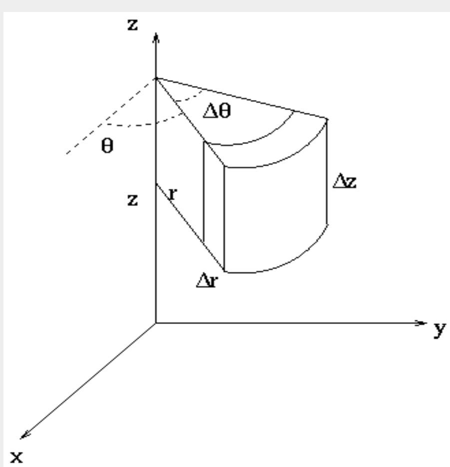
Note:

Rectangular Coords	$dV = dx\,dy\,dz$
Cylindrical Coords	$dV = r\,dr\,d\theta\,dz$
Spherical Coords	$dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

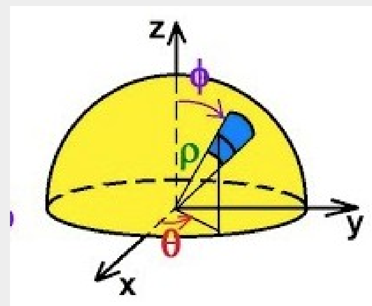
Cylindrical:

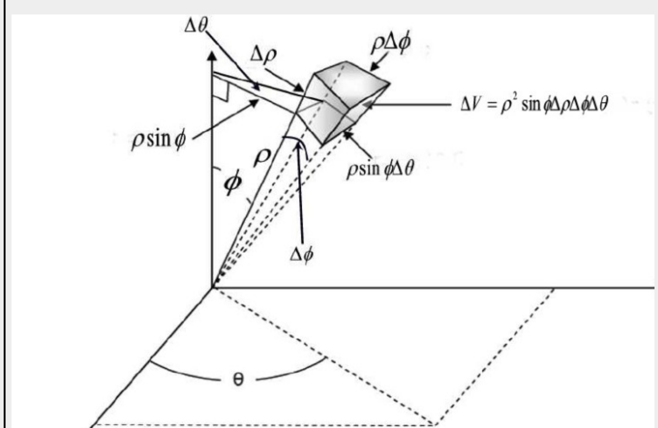






Spherical:





Ex 1: Sketch the region of integration and evaluate the integral.

$$\int_0^{\pi} \int_0^{\sin\theta} \int_0^{2} r\, dz\, dr\, d\theta$$

Ex 2: Sketch the region bounded above by the plane z = y + 4, below by the xy-plane, and laterally by the right circular cylinder having radius 4 and whose axis is the z-axis. Then, find its volume.

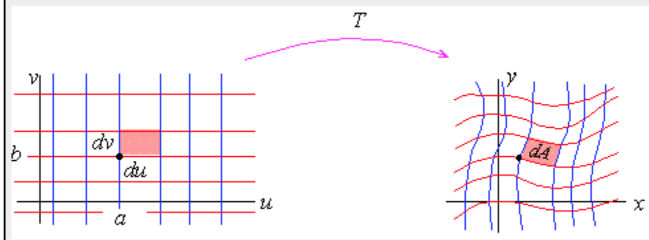
13.8 (continued)

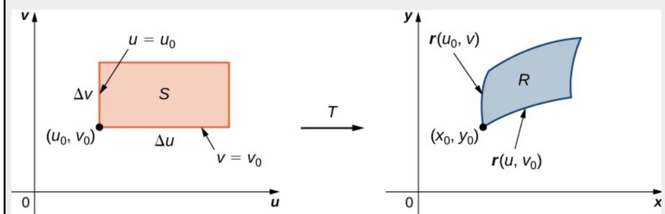
Ex 3: Change this integral to spherical coordinates and evaluate that integral (Hint: You'll need to sketch the integration region first.)

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dy\,dx$$

Ex 4: The plane z = 1 divides the sphere $x^2+y^2+z^2=2$ into two parts. Use spherical coordinates to find the volume of the smaller part.

13.9 Change of Variables (Jacobi Method)

<table>
<tr>
<td rowspan="2">Ex 1: Find the image of the rectangle with (u,v) corners of (0, 0), (3, 0), (3, 1) and (0, 1) under the transformation of x = 2u + 3v and y = u - v .
Then find the Jacobian, J(u,v).</td>
<td>Idea:
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<td>For $x=g(u,v)$ and $y=h(u,v)$
$$\int_G\int f(g(u,v),h(u,v))\,|J(u,v)|\;du\,dv = \int_R\int f(x,y)\,dx\,dy$$
where
$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}$$
Note: In 3-d, we get
$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$</td>
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</table>

13.9 (continued)

Ex 2: Use a transformation to evaluate $\iint_R (2x-y)\cos(y-2x)\,dA$ over R where R is the triangle with vertices (1, 0), (4, 0) and (4, 3).

13.9 (continued)

Ex 3: Find the area of the ellipse $x^2+\frac{y^2}{36}=1$ using a double integral (with a transformation).