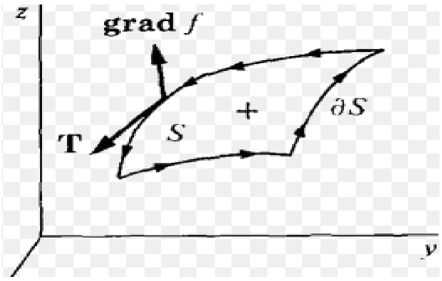
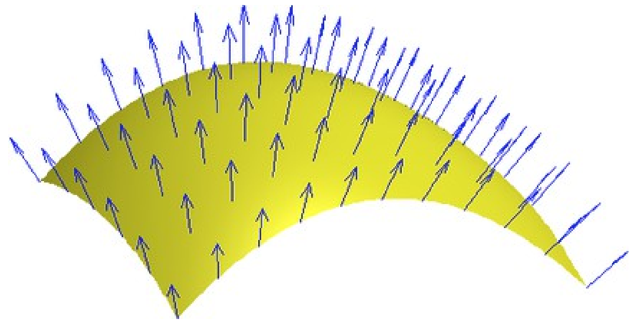
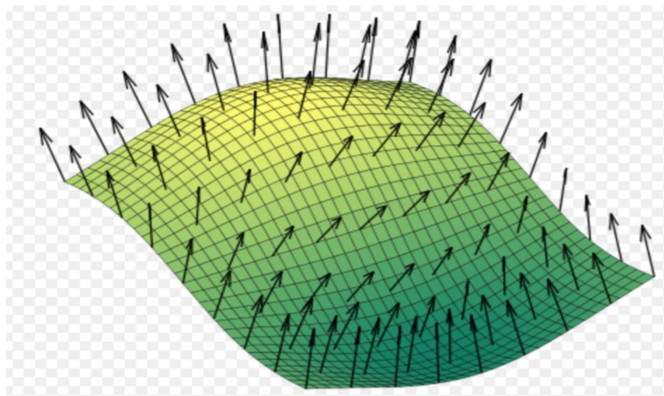
14.1 Vector Fields

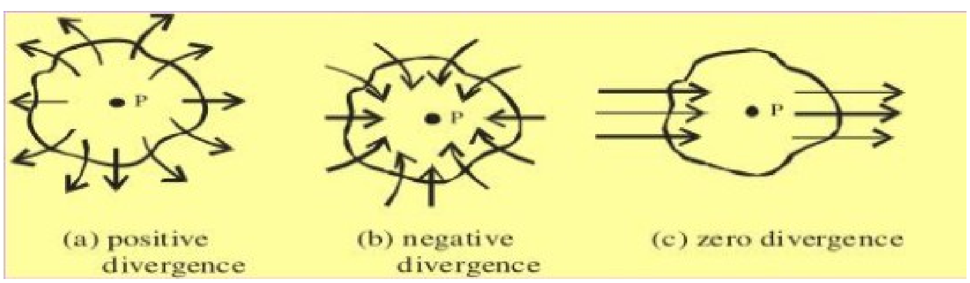
<u>Gradient of 3d surface:</u>

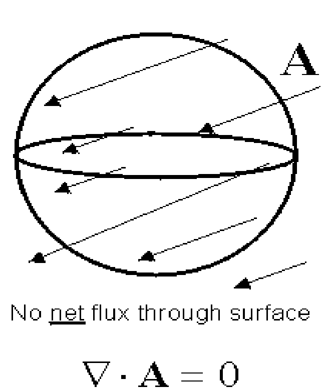


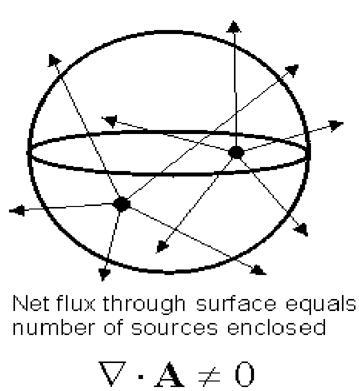




<u>Divergence of a vector field:</u>

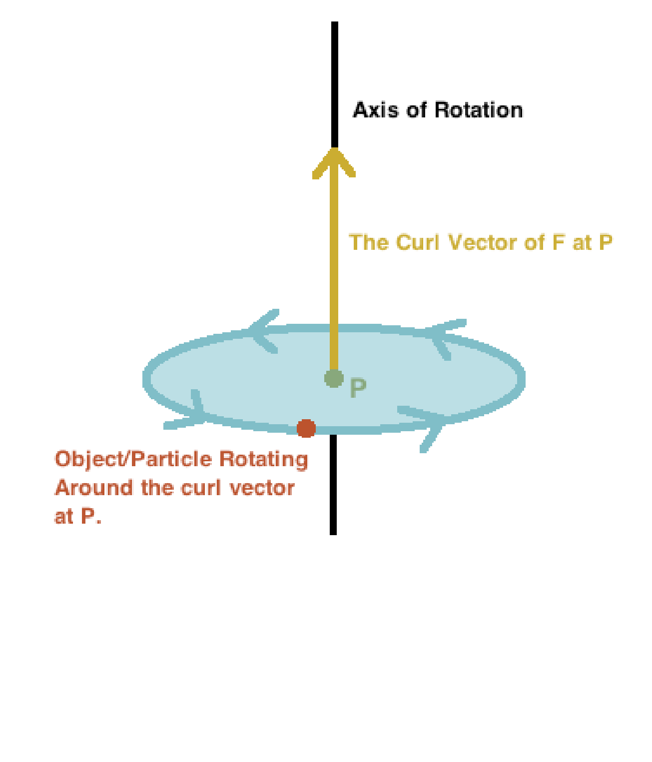


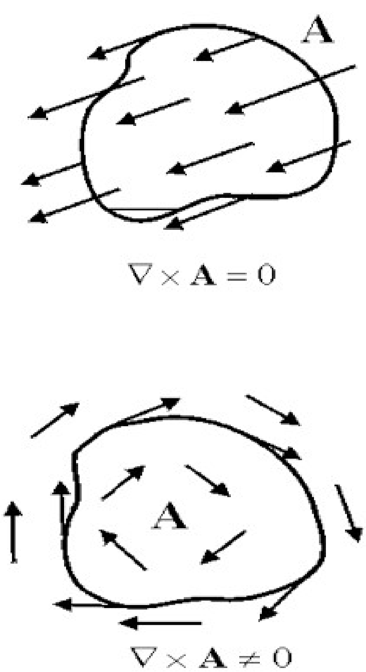




14.1 (continued)

Curl of a vector field:





Ex 1: Fill in the table.

Let $f(x, y, z)$ be a scalar field (i.e. it returns a scalar) and

$\vec{F}(x, y, z) = M(x, y, z)\hat{i} + N(x, y, z)\hat{j} + P(x, y, z)\hat{k}$ be a vector field (i.e. it returns a vector).

	Notation	Formula	Input	Output
Gradient of f				
Divergence of $\vec{F}$				
Curl of $\vec{F}$				

Explanation of divergence and curl:
If $\vec{F}$ is a velocity field for a fluid, then
1. divergence of $\vec{F}$ measures the tendency of the fluid to diverge away from a point ($\operatorname{div} \vec{F} > 0$) or accumulate toward a point ($\operatorname{div} \vec{F} < 0$).

2. Curl of $\vec{F}$ picks out the direction or axis about which the fluid rotates most rapidly with $\|curl\, \vec{F}\|$ = speed of that rotation.

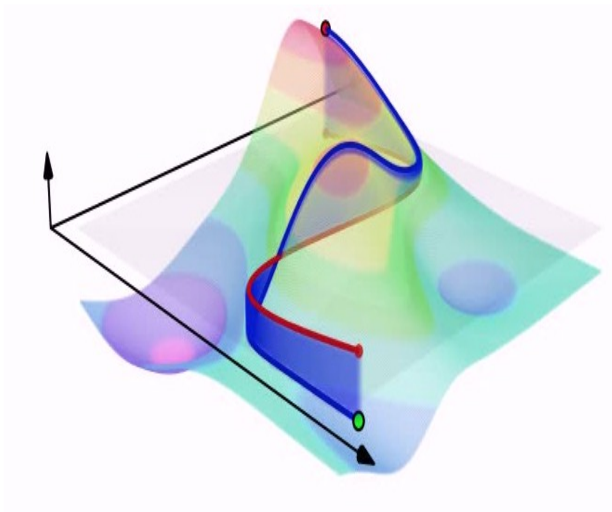
Ex 2: Sketch a sample of vectors for the given vector field $\vec{F}$.

(a) $\vec{F}(x, y) = x\hat{i} - y\hat{j}$	(b) $\vec{F}(x, y) = -2\hat{j}$
(c) $\vec{F}(x, y, z) = 2\hat{j} + z\hat{k}$ (try to draw vectors with starting points in xy, yz and xz-planes.	

Ex 3: Let $\vec{F}(x,y,z)=xyz\,\hat{i}+2y^2\,\hat{j}-3x^2z\,\hat{k}$. (a) Find $\operatorname{div}\vec{F}$.	(c) Find $\operatorname{grad}(\operatorname{div}\vec{F})$.
(b) Find $\operatorname{curl}\vec{F}$.	(d) Find $\operatorname{div}(\operatorname{curl}\vec{F})$.

14.2 Line Integrals

Line integral of a scalar field.



If C is given by $x=x(t),\, y=y(t),\, t\in[a,b]$,

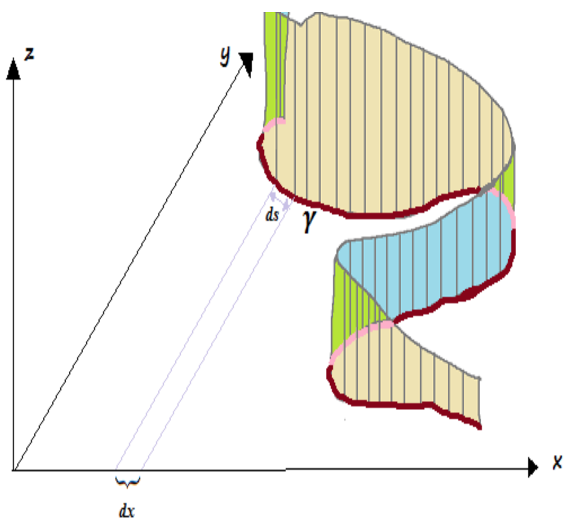
then $\int_C f(x,y)\,ds=\int_a^b f(x(t),y(t))\,ds$

where

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \quad .$$

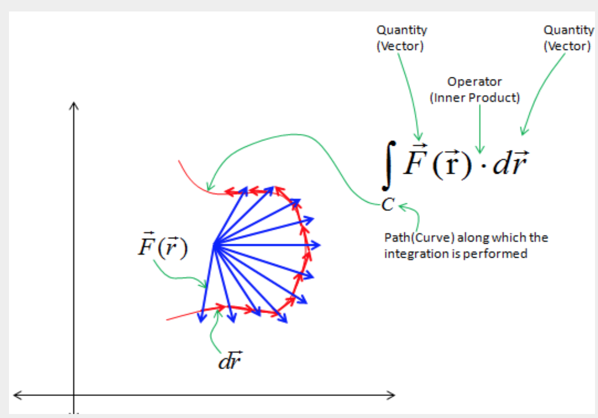
What does this line integral measure?

We can think of it as the area of the curved vertical curtain along that curve under the surface $z=f(x,y)$



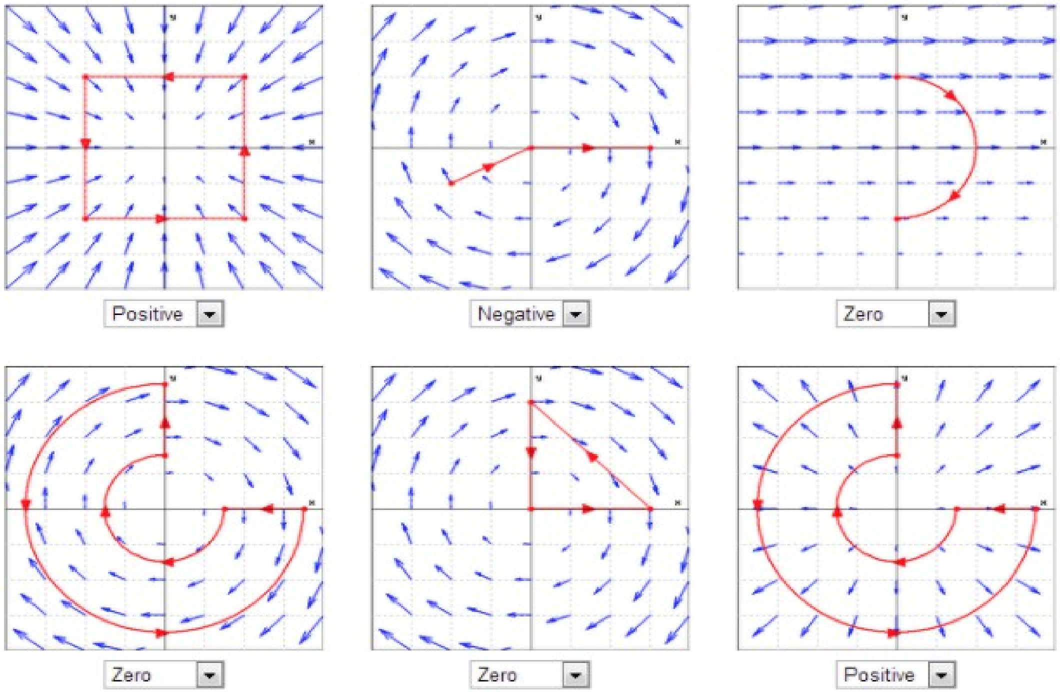
Let $\vec{F}(x,y,z)=M\,\hat{i}+N\,\hat{j}+P\,\hat{k}$ where $M=M(x,y,z), N=N(x,y,z),$ and $P=P(x,y,z)$.

Then $d\vec{r}=dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$ and $W=\int_C \vec{F}\cdot d\vec{r}$ calculates the work done in moving a particle over curve C with force $\vec{F}$.



<u>14.2 (continued)</u>

More cool pictures: In each example, the answer below the picture shows whether the line integral of each vector field (in blue) along the oriented path (in red) is positive, negative or zero.



Ex 1: Evaluate $\int_C x e^y \, ds$ where C is the line segment from (-1, 2) to (1, 1).

Ex 2: Evaluate $\int_C (xz\,dx+(y+z)\,dy+x\,dz)$ where C is the curve $x=e^t$, $y=e^{-t}$, $z=e^{2t}$ for $t\in[0,1]$

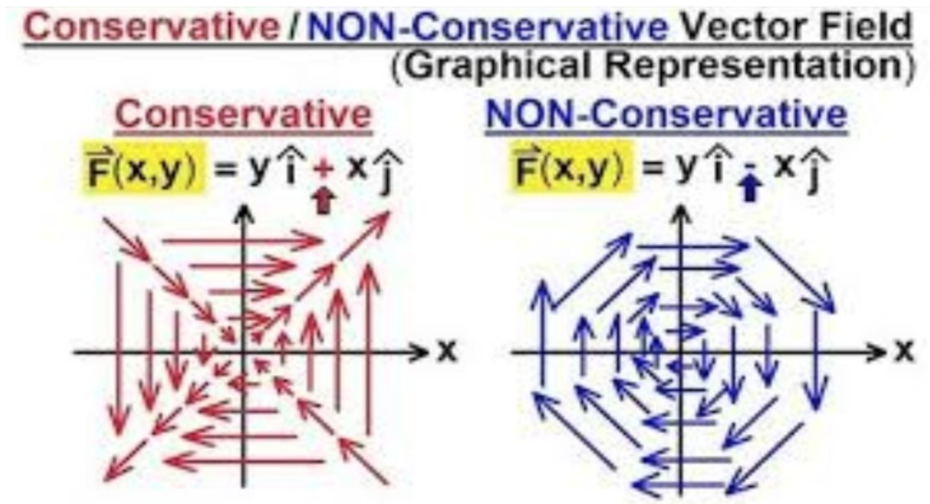
Ex 3: Find the work done by the force field $\vec{F}(x,y,z)=(2x-y)\hat{\imath}+(2z)\hat{\jmath}+(y-z)\hat{k}$ when moving a particle along the line segment from (0, 0, 0) to (1, 4, 5).

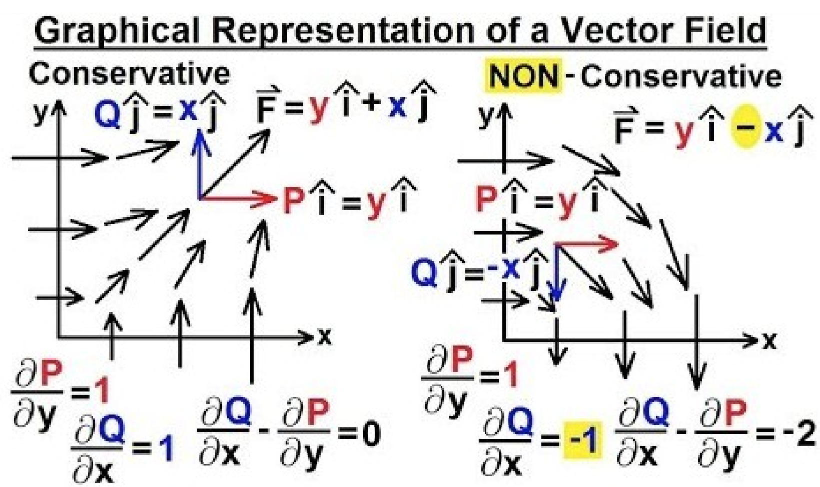
14.3 Independence of Path

<u>Fundamental Theorem of Line Integrals</u> C is a curve given by $\vec{r}(t), t \in [a,b]$. $\vec{r}\,'(t)$ exists. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then $\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$.	Ex 1: Answer these questions. (a) What does it mean to be independent of path?
<u>Equivalent Conditions:</u> If $\vec{F}(\vec{r})$ is continuous on an open connected set D, then the following conditions are equivalent. • $\vec{F} = \nabla f$ for some function f (i.e. $\vec{F}$ is conservative). • $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of path in D. • $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0 \ \forall$ closed paths in D.	(b) If $\vec{F}$ is conservative, what is it conserving? (c) Why does D need to be open and simply connected?
<u>Theorem:</u> If $\vec{F}(x,y,z) = M\hat{i} + N\hat{j} + P\hat{k}$ is continuously differentiable on an open, simply connected set D, then $\vec{F}$ conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$ (3d) (Note: In 2d, this becomes $M_y = N_x \Leftrightarrow \nabla \times \vec{F} = \vec{0}$.)	(d) Why didn't the Theorem get grouped with the equivalent conditions?

14.3 (continued)

Conservative Vector Fields:





Ex 2: Determine if the given field is conservative. If so, find f such that $\vec{F} = \nabla f$.

(a) $\vec{F}(x, y) = \left(x + \frac{1}{(x+y)^2}\right)\hat{i} + \left(3 + \frac{1}{(x+y)^2}\right)\hat{j}$	(b) $\vec{F}(x, y) = (4y^2\cos(xy^2))\hat{i} + (8x\cos(xy^2))\hat{j}$

Ex 3: Use $\vec{F}(x,y)=\left(x+\frac{1}{(x+y)^2}\right)\hat{i}+\left(3+\frac{1}{(x+y)^2}\right)\hat{j}$ to answer the following questions.

(a) What is the largest open, connected set on which $\vec{F}(x,y)$ is continous?

(b) Evaluate $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ using the Fundamental Theorem of Line Integrals., if C is the curve given by $\vec{r}=t^2\hat{i}+2t^3\hat{j}$, $t\in[1,2]$. (Why are we sure we can use FTLI?)

(c) How would you calculate $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ without FTLI?

Ex 4: Show that the line integral is independent of path.

$$\int_{(0,0,0)}^{(\pi,\pi,0)} \left((\cos x+2yz)\,dx+(\sin y+2xz)\,dy+(z+2xy)\,dz\right)$$. Then evaluate it.

14.4 Green's Theorem

<u>Theorem:</u>
Let R be a region in the xy-plane, C is a simple, closed curve enclosing R (with parameterization $\vec{r}(t)$), $\vec{F}(x, y)=M(x, y)\hat{i}+N(x, y)\hat{j}$ be continuously differentiable over $R \cup C$.

<u>Form (1):</u> (Flux across a boundary)

$$\oint_C \vec{F}\cdot\vec{n}\,ds=\iint_R \nabla\cdot\vec{F}\,dA$$

$\vec{n}$ = unit normal vector to C

$$\Leftrightarrow \oint_C (M\,dy-N\,dx)=\iint_R (M_x+N_y)\,dx\,dy$$

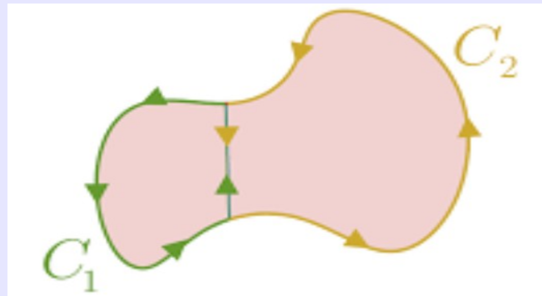
(Think of flux as flow.)

Idea of proof of Green's Theorem:

(1) Observation:

$$\int_{(a,b)}^{(c,d)} f\,ds=-\int_{(c,d)}^{(a,b)} f\,ds$$

(2) Subdivide regions such that $C=C_1 \cup C_2$



$\Rightarrow \oint_C f\,ds=\oint_{C_1} f\,ds+\oint_{C_2} f\,ds$ (because the integrals over the overlapping up and down curve piece above cancel each other out)

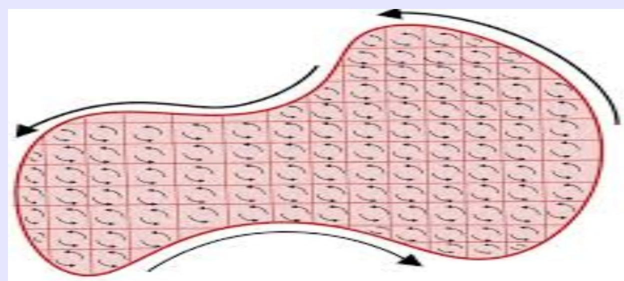
<u>Form (2):</u> (Circulation along a boundary)

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_R \nabla\times\vec{F}\cdot\hat{k}\,dA$$

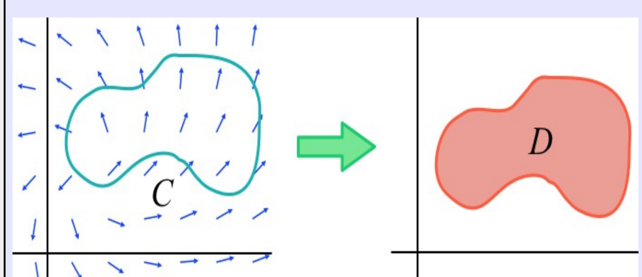
$$\Leftrightarrow \oint_C (M\,dx+N\,dy)=\iint_R (N_x-M_y)\,dx\,dy$$

Big Idea: We can exchange the line integral for a double integral.

(3) Subdivide into infinitely many subregions.



This means that the line integral becomes a double integral over a closed 2d region.



Ex 1: Given $\oint_C (\sqrt{y}\, dx + \sqrt{x}\, dy)$ where C is the closed curve formed by $y=0$, $x=2$, $y=\frac{1}{2}x^2$,

(a) Draw C.

(b) Calculate the integral using Green's Theorem.

Ex 2: Given the vector field $\vec{F}(x, y) = x\hat{i} + 2y\hat{j}$ and curve C given by $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi)$.

(a) Draw the vector field, curve C and make predictions about the flux and circulation.

(b) Calculate $\oint_C \vec{F} \cdot \vec{n}\, ds$ (flux across the boundary).	(c) Calculate $\oint_C \vec{F} \cdot d\vec{r}$ (circulation along the boundary).

Ex 3: Find the area (using Green's Theorem), between $y=\sqrt{x}$ and $y=\frac{x}{4}$. Check your answer with a different method.

Area of R: (Just another cool way to calculate the area of a closed region in xy-plane.)

$$A=\frac{1}{2}\oint_C (-y\,dx+x\,dy)$$

14.5 Surface Integrals

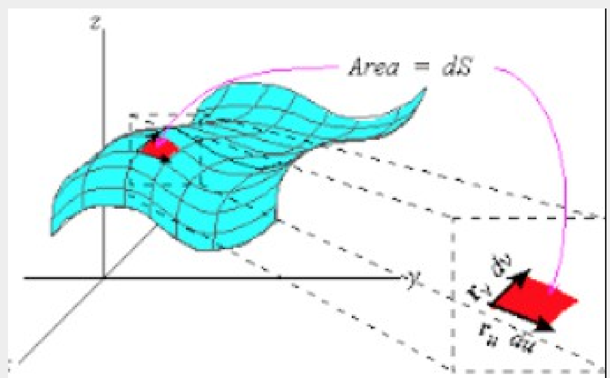
Ex 1: Evaluate the surface integral $\iint_G g(x,y,z)\,dS$ given $g(x,y,z)=y$ where G is the surface $z=4-y^2$ over the region $\{(x,y): 0\le x\le 3\ ,\ 0\le y\le 2\}$

Theorem:
Let
- R be a closed, bounded region in the xy-plane
- $z=f(x,y)$ be a function with continuous first-order partial derivatives on R
- G be the surface over R given by $z=f(x,y)$
- $g(x,y,z)=g(x,y,f(x,y))$ be a continuous function on R.

Then the surface integral is given by

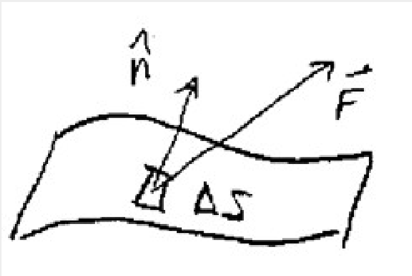
$$\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dy\,dx.$$

Note: $dS=\sqrt{f_x^2+f_y^2+1}\ dA$.



Ex 2: Evaluate $\iint_G 3z\,dS$ where G is the top of the tetrahedron bounded by all three coordinate planes and the plane $2x+6y+3z=6$.

14.5 (continued)



Flux of $\vec{F}$ across G:

$$\iint_G \vec{F}\cdot\hat{n}\,dS=\iint_R(-M f_x-N f_y+P)\,dx\,dy$$ where

- $\vec{F}(x,y,z)=M(x,y,z)\hat{i}+N(x,y,z)\hat{j}+P(x,y,z)\hat{k}$
- G is the surface $z=f(x,y)$
- $\hat{n}$ is the upward unit normal on G.
- $z=f(x,y)$ has continuous first order partial derivatives.

Ex 3: Evaluate the flux across G where $\vec{F}(x,y,z)=2\hat{i}+5\hat{j}+3\hat{k}$, G is the part of the cone $z=\sqrt{x^2+y^2}$ outside the cylinder $x^2+y^2=1$ and inside the cylinder $x^2+y^2=4$.

14.6 Gauss' Divergence Theorem

Ex 1: Let $\vec{F}(x, y, z) = 4z\hat{k}$ and S be the upper hemisphere with radius 3 and center (0, 0, 0). (a) Calculate $\iint_{\partial S} \vec{F} \cdot \hat{n}\, dS$ as a surface integral.	Theorem: Let • $\vec{F}(x, y, z)$ be a vector field continuously differential in solid S. • S is a 3-d solid. • ∂S be the boundary of the solid S (i.e. ∂S is a surface). • $\hat{n}$ be the unit outer normal vector to ∂S . Then $\iint_{\partial S} \vec{F}(x, y, z) \cdot \hat{n}\, dS = \iiint_{S} \text{div}\, \vec{F}\, dV$ (Note: Remember that $dV = dx\, dy\, dz$ in some order.) This is the 3d analog of Green's Theorem!! We can think of this integral as measuring the flux across the boundary of the surface (as opposed to one form of Green's theorem that measured the flux across the boundary of a curve). One big idea is that we can replace a surface integral with a regular triple integral.
	(b) Calculate $\iint_{\partial S} \vec{F} \cdot \hat{n}\, dS$ using Gauss' Theorem.

Ex 2: Calculate $\iint_{\partial S} \vec{F}\cdot\hat{n}\,dS$ where $\vec{F}(x,y,z)=xy\,\hat{i}+e^x\,\hat{j}+z^3\,\hat{k}$ over the box $\{(x,y,z): x\in[0,3],\, y\in[1,2],\, z\in[0,1]\}$.

Ex 3: Calculate $\iint_{\partial S} \vec{F}\cdot\hat{n}\,dS$ where $\vec{F}(x,y,z)=3x\,\hat{i}+2\,\hat{j}+2z^2\,\hat{k}$ and S is the solid between the paraboloid $z=4-x^2-y^2$, cylinder $x^2+y^2=1$ and the xy-plane.

14.7 Stokes Theorem

Ex 1: Use Stokes' Theorem to calculate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$ where $\vec{F}(x,y,z)=xy\,\hat{i}+yz\,\hat{j}+xz\,\hat{k}$, S is the triangular surface (part of a plane) with vertices (0,0,0), (1,0,0) and (0,2,1), and $\hat{n}$ is an upper normal.

Stokes' Theorem:

Let

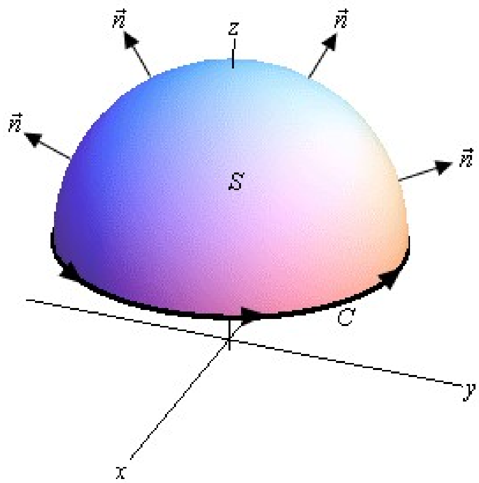
- S be a 3d surface.
- $\vec{F}(x,y,z)=M\,\hat{i}+N\,\hat{j}+P\,\hat{k}$ where $M=M(x,y,z)$, $N=N(x,y,z)$ and $P=P(x,y,z)$.
- M, N, and P have continuous first order partial derivatives.
- C is a piece-wise smooth, simple, closed curve, positively oriented.
- $\hat{T}$ is a unit tangent vector to C.

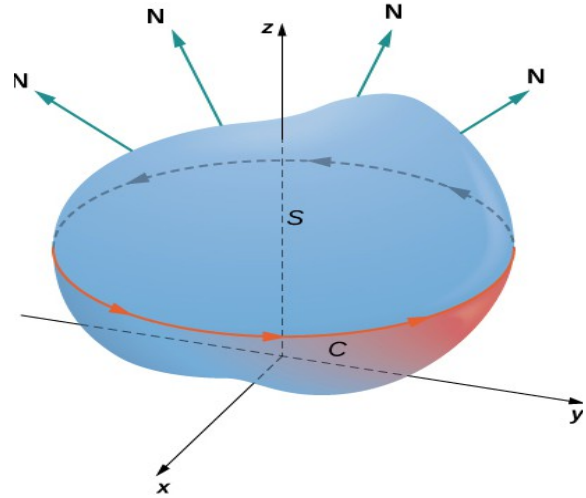
Then,

$$\oint_C \vec{F}\cdot\hat{T}\,ds=\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$
$$=\iint_R (\nabla\times\vec{F})\cdot\vec{n}\,dx\,dy$$

(This essentially says that the circulation along the boundary surface is the same as the circulation on the boundary curve.)

Remember $\oint_C \vec{F}\cdot\hat{T}\,ds=\int_C (M\,dx+N\,dy+P\,dz)$.





Ex 2: Use Stokes' Theorem to calculate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ where $\vec{F} = \langle z-y, z+x, -x-y \rangle$, S is the part of the paraboloid $z = 2 - x^2 - y^2$ above the z = 1 plane, and $\hat{n}$ is the upward normal.

14.7 (continued)

Ex 3: Use Stokes' Theorem to calculate $\oint_C \vec{F}\cdot\hat{T}\,ds$ where $\vec{F}=\left(x^2+y^2\right)\hat{i}-x\left(x^2+y^2\right)\hat{j}+0\,\hat{k}$ and C is the rectangular path from (0,0,0) to (1,0,0) to (1,1,1) to (0,1,1) to (0,0,0).