

## Derivatives

$$\begin{aligned}
D_x e^x &= e^x \\
D_x \sin(x) &= \cos(x) \\
D_x \cos(x) &= -\sin(x) \\
D_x \tan(x) &= \sec^2(x) \\
D_x \cot(x) &= -\csc^2(x) \\
D_x \sec(x) &= \sec(x)\tan(x) \\
D_x \csc(x) &= -\csc(x)\cot(x) \\
D_x \sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}}, x \in [-1, 1] \\
D_x \cos^{-1}(x) &= \frac{-1}{\sqrt{1-x^2}}, x \in [-1, 1] \\
D_x \tan^{-1}(x) &= \frac{1}{1+x^2}, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
D_x \sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1 \\
D_x \sinh(x) &= \cosh(x) \\
D_x \cosh(x) &= \sinh(x) \\
D_x \tanh(x) &= \text{sech}^2(x) \\
D_x \coth(x) &= -\text{csch}^2(x) \\
D_x \text{sech}(x) &= -\text{sech}(x)\tanh(x) \\
D_x \text{csch}(x) &= -\text{csch}(x)\coth(x) \\
D_x \sinh^{-1}(x) &= \frac{1}{\sqrt{x^2+1}} \\
D_x \cosh^{-1}(x) &= \frac{1}{\sqrt{x^2-1}}, x > 1 \\
D_x \tanh^{-1}(x) &= \frac{1}{1-x^2}, -1 < x < 1 \\
D_x \text{sech}^{-1}(x) &= \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1 \\
D_x \ln(x) &= \frac{1}{x}
\end{aligned}$$

## Integrals

$$\begin{aligned}
\int \frac{1}{x} dx &= \ln|x| + c \\
\int e^x dx &= e^x + c \\
\int a^x dx &= \frac{1}{\ln a} a^x + c \\
\int e^{ax} dx &= \frac{1}{a} e^{ax} + c \\
\int \frac{1}{\sqrt{1-x^2}} dx &= \sin^{-1}(x) + c \\
\int \frac{1}{1+x^2} dx &= \tan^{-1}(x) + c \\
\int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) + c \\
\int \sinh(x) dx &= \cosh(x) + c \\
\int \cosh(x) dx &= \sinh(x) + c \\
\int \tanh(x) dx &= \ln|\cosh(x)| + c \\
\int \text{sech}(x) dx &= \tan^{-1}(\sinh(x)) + c \\
\int \text{csch}(x) dx &= -\ln|\cosh(x)| + c \\
\int \tan(x) dx &= -\ln|\cos(x)| + c \\
\int \cot(x) dx &= \ln|\sin(x)| + c \\
\int \cos(x) dx &= \sin(x) + c \\
\int \sin(x) dx &= -\cos(x) + c \\
\int \sqrt{a^2-u^2} du &= \frac{1}{2} \left( u\sqrt{a^2-u^2} + a^2 \sin^{-1}\left(\frac{u}{a}\right) \right) + c \\
\int \frac{1}{\sqrt{a^2-u^2}} du &= \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c \\
\int \ln(x) dx &= x \ln(x) - x + c
\end{aligned}$$

## U-Substitution

Let  $u = f(x)$  (can be more than one variable).  
 Determine:  $du = \frac{df(x)}{dx} dx$  and solve for  $x$ .  
 Then, if a definite integral, substitute the bounds for  $u = f(x)$  at each bound. Solve the integral using  $u$ .

## Integration by Parts

$$\int u dv = uv - \int v du$$

## Fns and Identities

$$\begin{aligned}
\cos(\sin^{-1}(x)) &= \sqrt{1-x^2} \\
\sin(\sin^{-1}(x)) &= x
\end{aligned}$$

## Directional Derivatives

Let  $z=f(x,y)$  be a function, (a,b) a point in the domain (a valid input point) and  $\hat{u}$  a unit vector (2D) in the direction of the derivative at the point (a,b) in the direction of  $\hat{u}$ :  
 $D_{\hat{u}}f(a,b) = \hat{u} \cdot \nabla f(a,b)$   
 This will return a scalar. 4-D version:  
 $D_{\hat{u}}f(a,b,c) = \hat{u} \cdot \nabla f(a,b,c)$

## Tangent Planes

let  $F(x,y,z) = k$  be a surface and  $P = (x_0, y_0, z_0)$  be a point on that surface. Equation of a Tangent Plane:  
 $\nabla F(x_0, y_0, z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

## Approximations

let  $z = f(x,y)$  be a differentiable function total differential of  $z = dz$   
 $dz = \nabla f \cdot \langle dx, dy \rangle$   
 This is the approximate change in  $z$ . The actual change in  $z$  is the difference in  $z$  values:  
 $\Delta z = z - z_1$

## Maxima and Minima

**Internal Points**  
 1. Take the Partial Derivatives with respect to X and Y ( $f_x$  and  $f_y$ ) (Can use gradient).  
 2. Set derivatives equal to 0 and use to solve system of equations for x and y.  
 3. Plug back into original equation for z. Use Second Derivative Test for whether points are local max, min, or saddle.

**Second Partial Derivative Test**  
 1. Find all (x,y) points such that  $\nabla f(x,y) = \vec{0}$   
 2. Let  $D = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)$  IF (a)  $D > 0$  AND  $f_{xx} < 0$ ,  $f(x,y)$  is local max value.  
 (b)  $D > 0$  AND  $f_{xx}(x,y) > 0$   $f(x,y)$  is local min value.  
 (c)  $D < 0$ ,  $(x,y,f(x,y))$  is a saddle point (d)  $D = 0$ , test is inconclusive  
 3. Determine if any boundary point gives min or max. Typically, we have to parametrize boundary and then reduce to a Calc 1 type of min/max problem to solve.

**The following only apply only if a boundary is given**  
 1. check the corner points  
 2. Check each line ( $0 \leq x \leq 5$  would give  $x=0$  and  $x=5$ )  
 On Bounded Equations, this is the global min and max...second derivative test is not needed.

**Lagrange Multipliers**  
 Given a function  $f(x,y)$  with a constraint  $g(x,y)$ , solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points):  
 $\nabla f = \lambda \nabla g$   
 $g(x,y) = 0$  (or  $k$  if given)

$$\begin{aligned}
\sec(\tan^{-1}(x)) &= \sqrt{1+x^2} \\
\tan(\sec^{-1}(x)) &= \frac{\sqrt{x^2-1}}{x} \\
&= \frac{\sqrt{x^2-1}}{x} \text{ if } x \geq 1 \\
&= \frac{-\sqrt{x^2-1}}{x} \text{ if } x \leq -1 \\
\sinh^{-1}(x) &= \ln|x + \sqrt{x^2+1}| \\
\sinh^{-1}(x) &= \ln|x + \sqrt{x^2-1}|, x \geq -1 \\
\tanh^{-1}(x) &= \frac{1}{2} \ln|x + \frac{1+x}{1-x}|, |x| < 1 \\
\text{sech}^{-1}(x) &= \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|, 0 < x \leq 1 \\
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2}
\end{aligned}$$

## Trig Identities

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
1 + \tan^2(x) &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x) \\
\sin(\pm x) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\
\cos(\pm x) &= \cos(x)\cos(y) \pm \sin(x)\sin(y) \\
\tan(\pm x) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
\cosh(n^2x) - \sinh^2x &= 1 \\
1 + \tan^2(x) &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x) \\
\sin^2(x) &= \frac{1-\cos(2x)}{2} \\
\cos^2(x) &= \frac{1+\cos(2x)}{2} \\
\tan^2(x) &= \frac{1-\cos(2x)}{1+\cos(2x)} \\
\sin(-x) &= -\sin(x) \\
\cos(-x) &= \cos(x) \\
\tan(-x) &= -\tan(x)
\end{aligned}$$

## Calculus 3 Concepts

**Cartesian coords in 3D**  
 given two points:  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .  
 Distance between them:  
 $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$   
 Midpoint:  
 $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$   
 Sphere with center (h,k,l) and radius r:  
 $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

## Vectors

Vector:  $\vec{u}$   
 Unit Vector:  $\hat{u}$   
 Magnitude:  $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$   
 Unit Vector:  $\hat{u} = \frac{\vec{u}}{||\vec{u}||}$   
**Dot Product**  
 $\vec{u} \cdot \vec{v}$   
 Produces a Scalar (Geometrically, the dot product is a vector projection)  
 $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$   
 $\vec{u} \cdot \vec{v} = 0$  means the two vectors are Perpendicular  $\theta$  is the angle between them  
 $\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos(\theta)$   
 NOTE:  
 $\vec{u} \cdot \vec{v} = \cos(\theta)$   
 $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$   
 $\vec{u} \cdot \vec{v} = 0$  when  $\perp$   
 Angle Between  $\vec{u}$  and  $\vec{v}$ :  
 $\cos(\theta) = \frac{|\vec{u} \cdot \vec{v}|}{||\vec{u}|| ||\vec{v}||}$

## Double Integrals

With Respect to the xy-axis, if taking an integral  
 $\int \int f(x,y) dy dx$  is cutting in vertical rectangles,  
 $\int \int f(x,y) dx dy$  is cutting in horizontal rectangles  
**Polar Coordinates**  
 When using polar coordinates,  
 $dA = r dr d\theta$

## Surface Area of a Curve

let  $z = f(x,y)$  be continuous over S (a closed Region in 2D domain)  
 Then the surface area of  $z = f(x,y)$  over S is:  
 $SA = \int \int_S \sqrt{f_x^2 + f_y^2 + 1} dA$

## Triple Integrals

$\int \int \int f(x,y,z) dz dy dx = \int_{x_1}^{x_2} \int_{y_1(x,y)}^{y_2(x,y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz dy dx$   
 Note:  $dz$  can be exchanged for  $dx dy dz$  in any order, but you must then choose your limits of integration according to that order

## Jacobian Method

$$\int \int_C f(u,v) |J(u,v)| du dv = \int \int_R f(x,y) dx dy$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Common Jacobians:  
 Rect. to Cylindrical:  $r$   
 Rect. to Spherical:  $\rho^2 \sin(\phi)$

## Vector Fields

let  $f(x,y,z)$  be a scalar field and  $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$  be a vector field.  
 Gradient of  $f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$   
 Divergence of  $\vec{F}$ :  
 $\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$   
 Curl of  $\vec{F}$ :  
 $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

## Line Integrals

C given by  $x = x(t), y = y(t), t \in [a,b]$   
 $\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) ds$   
 where  $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$   
 or  $\sqrt{1 + (\frac{dy}{dx})^2} dx$   
 or  $\sqrt{1 + (\frac{dx}{dy})^2} dy$   
 To evaluate a Line Integral,  
 - get a parameterized version of the line (usually in terms of  $t$ , though in exclusive terms of  $x$  or  $y$  is ok)  
 - evaluate for the derivatives needed (usually  $dy, dx,$  and/or  $dt$ )  
 - plug in to original equation to get in terms of the independent variable  
 - solve integral

Projection of  $\vec{u}$  onto  $\vec{v}$ :  
 $pr_{\vec{v}}\vec{u} = \frac{(\vec{u} \cdot \vec{v})}{||\vec{v}||^2} \vec{v}$   
**Cross Product**  
 $\vec{u} \times \vec{v}$   
 Produces a Vector (Geometrically, the cross product is the area of a parallelogram with sides  $||\vec{u}||$  and  $||\vec{v}||$ )  
 $\vec{u} \times \vec{v} = \langle u_1v_2 - u_2v_1, u_3v_1 - u_1v_3, u_2v_3 - u_3v_2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v} = \vec{0}$  means the vectors are parallel

## Lines and Planes

**Equation of a Plane**  
 $(x_0, y_0, z_0)$  is a point on the plane and  $\langle A, B, C \rangle$  is a normal vector  
 $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$   
 $\langle A, B, C \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$   
 $Ax + By + Cz = D$  where  $D = Ax_0 + By_0 + Cz_0$

**Equation of a line**  
 A line requires a Direction Vector  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and a point  $(x_1, y_1, z_1)$ , then,  
 a parameterization of a line could be:  
 $x = u_1t + x_1$   
 $y = u_2t + y_1$   
 $z = u_3t + z_1$

**Distance from a Point to a Plane**  
 The distance from a point  $(x_0, y_0, z_0)$  to a plane  $Ax+By+Cz=D$  can be expressed by the formula:  
 $d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

## Coord Sys Conv

**Cylindrical to Rectangular**  
 $x = r \cos(\theta)$   
 $y = r \sin(\theta)$   
 $z = z$   
**Rectangular to Cylindrical**  
 $r = \sqrt{x^2 + y^2}$   
 $\tan(\theta) = \frac{y}{x}$   
 $z = z$   
**Spherical to Rectangular**  
 $x = \rho \sin(\phi) \cos(\theta)$   
 $y = \rho \sin(\phi) \sin(\theta)$   
 $z = \rho \cos(\phi)$   
**Rectangular to Spherical**  
 $\rho = \sqrt{x^2 + y^2 + z^2}$   
 $\tan(\theta) = \frac{y}{x}$   
 $\cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$   
**Spherical to Cylindrical**  
 $r = \rho \sin(\phi)$   
 $\theta = \theta$   
 $z = \rho \cos(\phi)$   
**Cylindrical to Spherical**  
 $\rho = \sqrt{r^2 + z^2}$   
 $\theta = \theta$   
 $\cos(\phi) = \frac{z}{\sqrt{r^2 + z^2}}$

## Surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



## Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



## Hyperboloid of Two Sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



## Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



## Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



## Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



## Cylinder

1 of the variables is missing  
 OR  
 $(x-a)^2 + (y-b)^2 = c$   
 (Major Axis is missing variable)

## Partial Derivatives

Partial Derivatives are simply holding all other variables constant (and act like constants for the derivative) and only taking the derivative with respect to a given variable.

## Surface Integrals

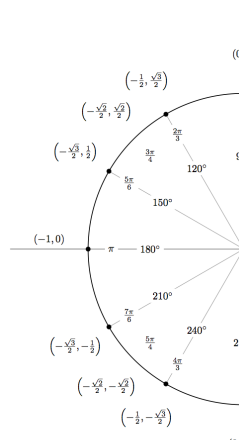
Let  $R$  be closed, bounded region in  $xy$ -plane  
 $f$  be a fn with first order partial derivatives on  $R$   
 $G$  is surface  $f(x,y,z) = z$   
 $\vec{n}$  is upward unit normal on  $G$ .  
 $f(x,y,z)$  has continuous  $1^{st}$  order partial derivatives

## Flux of $\vec{F}$ across $G$

$\int \int_G \vec{F} \cdot \vec{n} dS = \int \int_R (-Mf_x - Nf_y + P) dx dy$   
 where:  $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$   
 $G$  is surface  $f(x,y,z) = z$   
 $\vec{n}$  is upward unit normal on  $G$ .  
 $f(x,y,z)$  has continuous  $1^{st}$  order partial derivatives

## Unit Circle

(cos, sin)



Given  $z=f(x,y)$ , the partial derivative of  $z$  with respect to  $x$  is:  
 $f_x(x,y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$   
 likewise for partial with respect to  $y$ :  
 $f_y(x,y) = z_y = \frac{\partial f(x,y)}{\partial y}$   
**Notation**  
 For  $f_{xy}$ , work "inside to outside"  $f_x$  then  $f_{xy}$ , then  $f_{xyy}$   
 $f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x}$   
 For  $\frac{\partial^3 f}{\partial y^2 \partial x}$ , work right to left in the denominator

## Gradients

The Gradient of a function in 2 variables is  $\nabla f = \langle f_x, f_y \rangle$   
 The Gradient of a function in 3 variables is  $\nabla f = \langle f_x, f_y, f_z \rangle$

## Chain Rule(s)

Take the Partial derivative with respect to the first-order variables of the function times the partial (or normal) derivative of the first-order variable to the ultimate variable you are looking for summed with the same process for other first-order variables this makes sense for Example:  
 let  $x = x(s,t), y = y(t)$  and  $z = z(x,y)$ .  
 $z$  then has first partial derivative:  
 $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$   
 (Major Axis:  $z$  because it is the variable NOT squared)  
 $x$  has the partial derivatives:  
 $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$   
 and  $y$  has the derivative:  
 $\frac{dy}{dt}$

In this case (with  $z$  containing  $x$  and  $y$  as well as  $x$  and  $y$  both containing  $s$  and  $t$ ), the chain rule for  $\frac{\partial z}{\partial s}$  is  $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$   
 The chain rule for  $\frac{\partial z}{\partial t}$  is  $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$   
 Note: the use of "d" instead of "∂" with the function of only one independent variable

## Limits and Continuity

Limits in 2 or more variables  
 Limits taken over a vectorized limit just evaluate separately for each component of the limit.  
**Strategies to show limit exists**  
 1. Plug in Numbers, Everything is Fine  
 2. Algebraic Manipulation -factoring/dividing out -use trig identities  
 3. Change to polar coords  
 $\vec{r}(x,y) = (0,0) \Leftrightarrow r \rightarrow 0$   
**Strategies to show limit DNE**  
 1. Show limit is different if approached from different paths  
 $(x=y, x=y^2, \text{etc.})$   
 2. Switch to Polar coords and show the limit DNE.  
**Continuity**  
 A fn,  $z = f(x,y)$ , is continuous at (a,b) if  
 $f(a,b) = \lim_{(x,y) \rightarrow (a,b)} f(x,y)$   
 Which means:  
 1. The limit exists  
 2. The fn value is defined  
 3. They are the same value

## Other Information

$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$   
 Where a Cone is defined as  $z = \sqrt{a^2x^2 + b^2y^2}$ .  
 In Spherical Coordinates,  
 $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$   
 Right Circular Cylinder:  
 $V = \pi r^2 h, SA = \pi r^2 + 2\pi r h$   
 $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e^1$   
 Law of Cosines:  
 $a^2 = b^2 + c^2 - 2bc \cos(\theta)$

## Stokes Theorem

Let:  
 $S$  be a 3D surface  
 $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$   
 $\vec{n}$  is surface normal on  $S$   
 $f(x,y,z)$  has continuous  $1^{st}$  order partial derivatives

$$\oint_C \vec{F} \cdot d\vec{s} = \int \int_S (\nabla \times \vec{F}) \cdot \vec{n} dS = \int \int_S (\nabla \times \vec{F}) \cdot \vec{n} dx dy$$

$$\text{Remember: } \oint_C \vec{F} \cdot d\vec{s} = \int (M dx + N dy + P dz)$$

Originally Written By Daniel Kenner for MATH 2210 at the University of Utah. Source code available at: <https://github.com/kenner/m2210-CheatSheet> Thanks to Kelly Macarthur for Teaching and Providing Notes.