Variations on a Casselman-Osborne theme

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Introduction

This paper is inspired by two classical results in homological algebra of modules over an enveloping algebra – lemmas of Casselman-Osborne and Wigner. They have a common theme: they are statements about derived functors. While the statements for the functors themselves are obvious, the statements for derived functors are not and the published proofs were completely different from each other.

In the first section we give simple, pedestrian arguments for both results based on the same principle. They suggest a common generalization which is the topic of this paper.

In the second section we discuss some straightforward properties of centers of abelian categories and their derived categories. In the third section, we consider a class of functors and prove a simple result about their derived functors which generalizes the first two results.

The original arguments were considerably more complicated and based on different ideas [1], [3] and [5].

1 Classical approach

1.1 Wigner's lemma

Let $\mathfrak{g}$ be a complex Lie algebra, $\mathscr{U}(\mathfrak{g})$ its enveloping algebra and $\mathscr{Z}(\mathfrak{g})$ the center of $\mathscr{U}(\mathfrak{g})$. Denote by $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ the category of $\mathscr{U}(\mathfrak{g})$-modules.

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Let $\chi : \mathscr{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ be an algebra morphism of $\mathscr{Z}(\mathfrak{g})$ into the field of complex numbers. We say that a module V in $\mathscr{M}(\mathfrak{g})$ has an *infinitesimal character* χ if

$$z \cdot v = \chi(z) v \text{ for any } z \in \mathscr{Z}(\mathfrak{g}) \text{ and any } v \in V.$$

Theorem 1.1 *Let U and V be two objects in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ with infinitesimal characters χ_U and χ_V. Then $\chi_U \neq \chi_V$ implies $\operatorname{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V) = 0$ for all $p \in \mathbb{Z}_+$.*

Proof. Clearly, the center $\mathscr{Z}(\mathfrak{g})$ of $\mathscr{U}(\mathfrak{g})$ acts naturally on $\operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(U,V)$ for any two $\mathscr{U}(\mathfrak{g})$-modules U and V, by

$$z(T) = z \cdot T = T \cdot z \text{ for any } z \in \mathscr{Z}(\mathfrak{g}) \text{ and any } T \in \operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(U,V),$$

i.e., we can view it as a bifunctor from the category of $\mathscr{U}(\mathfrak{g})$-modules into the category of $\mathscr{Z}(\mathfrak{g})$-modules. Hence, its derived functors $\operatorname{Ext}^*_{\mathscr{U}(\mathfrak{g})}$ are bifunctors from the category of $\mathscr{U}(\mathfrak{g})$-modules into the category of $\mathscr{Z}(\mathfrak{g})$-modules.

Fix now a $\mathscr{U}(\mathfrak{g})$-module U with infinitesimal character χ_U. Consider the functor $F = \operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(U,-)$ from the category $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ into the category of $\mathscr{Z}(\mathfrak{g})$-modules. Since the infinitesimal character of U is χ_U, any element of $z \in \mathscr{Z}(\mathfrak{g})$ acts on $F(V) = \operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(U,V)$ as multiplication by $\chi_U(z)$ for any object V in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$.

Fix now a $\mathscr{U}(\mathfrak{g})$-module V with infinitesimal character χ_V. Let

$$0 \longrightarrow V \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^n \longrightarrow \dots$$

be an injective resolution of V. Let $z \in \ker \chi_V$. Then we have the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^n & \longrightarrow & \dots \\ & & \downarrow z & & \downarrow z & & & & \downarrow z & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^n & \longrightarrow & \dots \end{array}.$$

We can interpret this as a morphism $\phi^\cdot : I^\cdot \longrightarrow I^\cdot$ of complexes. Clearly, since $H^0(I^\cdot) = V$, we have $H^0(\phi^\cdot) = 0$. Therefore, $\phi^\cdot$ is homotopic to 0. By applying the functor F to this diagram we get

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F(I^0) & \longrightarrow & F(I^1) & \longrightarrow & \dots & \longrightarrow & F(I^n) & \longrightarrow & \dots \\ & & \downarrow F(z) & & \downarrow F(z) & & & & \downarrow F(z) & & \\ 0 & \longrightarrow & F(I^0) & \longrightarrow & F(I^1) & \longrightarrow & \dots & \longrightarrow & F(I^n) & \longrightarrow & \dots \end{array},$$

i.e., a morphism $F(\phi^\cdot) : F(I^\cdot) \longrightarrow F(I^\cdot)$ of complexes. Since $\phi^\cdot$ is homotopic to 0, $F(\phi^\cdot)$ is also homotopic to 0. This implies that all $H^p(\phi^\cdot) : H^p(I^\cdot) \longrightarrow H^p(I^\cdot)$, $p \in \mathbb{Z}$, are equal to 0. Since $H^p(I^\cdot) = R^p F(V) = \operatorname{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V)$, we see that $\operatorname{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V)$ are annihilated by z.

On the other hand, by the first remark in the proof, z must act on $\mathrm{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V)$ as multiplication by $\chi_U(z)$.

Since $\chi_U \neq \chi_V$, there exists $z \in \ker \chi_V$ such that $\chi_U(z) \neq 0$. This implies that $\mathrm{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V)$ must be zero for all $p \in \mathbb{Z}_+$. □

1.2 Casselman-Osborne lemma

Now we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, R the root system of $(\mathfrak{g},\mathfrak{h})$ and R^+ a set of positive roots. Let $\mathfrak{n}$ be the nilpotent Lie algebra spanned by root subspaces of positive roots. Let $\gamma : \mathscr{Z}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{h})$ be the Harish-Chandra homomorphism, i.e., the algebra morphisms such that $z - \gamma(z) \in \mathfrak{n}\mathscr{U}(\mathfrak{g})$ [2, Ch. VIII, §6, no. 4].

Let V be a $\mathscr{U}(\mathfrak{g})$-module. Since $\mathfrak{h}$ normalizes $\mathfrak{n}$, the quotient $V/\mathfrak{n}V = \mathbb{C} \otimes_{\mathscr{U}(\mathfrak{n})} V$ has a natural structure of $\mathscr{U}(\mathfrak{h})$-module. Also, $\mathscr{Z}(\mathfrak{g})$ acts naturally on $V/\mathfrak{n}V$, and this action is given by the composition of γ and the $\mathscr{U}(\mathfrak{h})$-action.

We can consider $F(V) = V/\mathfrak{n}V$ as a right exact functor F from the category of $\mathscr{U}(\mathfrak{g})$-modules into the category of $\mathscr{U}(\mathfrak{h})$-modules. Let $\mathrm{For}_{\mathfrak{g}}$ denote the forgetful functor from the category of $\mathscr{U}(\mathfrak{g})$-modules into the category of $\mathscr{U}(\mathfrak{n})$-modules. Let $\mathrm{For}_{\mathfrak{h}}$ denote the forgetful functor from the category of $\mathscr{U}(\mathfrak{h})$-modules into the category of linear spaces. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{M}(\mathscr{U}(\mathfrak{g})) & \xrightarrow{F} & \mathscr{M}(\mathscr{U}(\mathfrak{h})) \\ {\scriptstyle \mathrm{For}_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle \mathrm{For}_{\mathfrak{h}}} \\ \mathscr{M}(\mathscr{U}(\mathfrak{n})) & \xrightarrow[H_0(\mathfrak{n},-)]{} & \mathscr{M}(\mathbb{C}) \end{array} .$$

By the Poincaré-Birkhoff-Witt theorem, a free $\mathscr{U}(\mathfrak{g})$-module is a free $\mathscr{U}(\mathfrak{n})$-module, hence we can use free left resolutions in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ to calculate Lie algebra homology $H_p(\mathfrak{n},-)$ of $\mathscr{U}(\mathfrak{g})$-modules, i.e., we get the commutative diagram

$$\begin{array}{ccc} \mathscr{M}(\mathscr{U}(\mathfrak{g})) & \xrightarrow{L_pF} & \mathscr{M}(\mathscr{U}(\mathfrak{h})) \\ {\scriptstyle \mathrm{For}_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle \mathrm{For}_{\mathfrak{h}}} \\ \mathscr{M}(\mathscr{U}(\mathfrak{n})) & \xrightarrow[H_p(\mathfrak{n},-)]{} & \mathscr{M}(\mathbb{C}) \end{array} ,$$

for any $p \in \mathbb{Z}_+$. Therefore, Lie algebra homology groups $H_p(\mathfrak{n},-)$ of $\mathscr{U}(\mathfrak{g})$-modules have the structure of $\mathscr{U}(\mathfrak{h})$-modules.

Theorem 1.2 *Let V be an object in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$. Let $z \in \mathscr{Z}(\mathfrak{g})$ be an element which annihilates V. Then $\gamma(z)$ annihilates $H_p(\mathfrak{n},V)$, $p \in \mathbb{Z}_+$.*

Proof. Let $z \in \mathscr{Z}(\mathfrak{g})$. Let

$$\dots \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

be a projective resolution of V in $\mathcal{M}(\mathcal{U}(\mathfrak{g}))$. Multiplication by z gives the following commutative diagram:

$$\begin{array}{ccccccccc} \dots \longrightarrow & P_n & \longrightarrow \dots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ & {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z} & & \\ \dots \longrightarrow & P_n & \longrightarrow \dots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \end{array}$$

We can interpret this diagram as a morphism $\psi_\cdot : P_\cdot \longrightarrow P_\cdot$ of complexes of $\mathcal{U}(\mathfrak{h})$-modules. Applying the functor F we get the diagram

$$\begin{array}{ccccccccc} \dots \longrightarrow & F(P_n) & \longrightarrow \dots \longrightarrow & F(P_1) & \longrightarrow & F(P_0) & \longrightarrow & 0 \\ & {\scriptstyle F(z)}\downarrow & & \downarrow{\scriptstyle F(z)} & & \downarrow{\scriptstyle F(z)} & & \\ \dots \longrightarrow & F(P_n) & \longrightarrow \dots \longrightarrow & F(P_1) & \longrightarrow & F(P_0) & \longrightarrow & 0 \end{array}$$

representing $F(\psi_\cdot)$, where $F(z)$ is the multiplication by $\gamma(z)$.

Now, assume that $z \in \mathcal{Z}(\mathfrak{g})$ annihilates V. Then we have $H^0(\psi_\cdot) = 0$. It follows that $\psi_\cdot$ is homotopic to 0. This in turn implies that $F(\psi_\cdot)$ is homotopic to 0. Hence, the multiplication by $\gamma(z)$ on $F(P_\cdot)$ is homotopic to zero. Therefore, the multiplication by $\gamma(z)$ annihilates the cohomology groups of the complex $F(P_\cdot)$, i.e., $\gamma(z) \cdot H_p(\mathfrak{n}, V) = 0$ for $p \in \mathbb{Z}_+$. □

2 Centers of derived categories

2.1 Center of an additive category

Let $\mathcal{A}$ be an additive category. This implies that for any object V in $\mathcal{A}$, all its endomorphisms form a ring $\operatorname{End}(V)$ with identity id_V.

An endomorphism z of the identity functor on $\mathcal{A}$ is an assignment to each object U in $\mathcal{A}$ of an endomorphism z_U of U such that for any two objects U and V in $\mathcal{A}$ and any morphism $\varphi : U \longrightarrow V$ we have $z_V \circ \varphi = \varphi \circ z_U$.

Lemma 2.1 *Let z be an endomorphism of the identity functor on $\mathcal{A}$ and V an object in $\mathcal{A}$. Then z_V is in the center of the ring* $\operatorname{End}(V)$.

Proof. Let $e : V \longrightarrow V$ be an endomorphism of V. Then, $z_V \circ e = e \circ z_V$, i.e., z_V commutes with e. This implies that z_V is in the center of $\operatorname{End}(V)$. □

All endomorphisms of the identity functor on $\mathcal{A}$ form a commutative ring with identity which is called the *center* $Z(\mathcal{A})$ of $\mathcal{A}$.

Let $\mathcal{B}$ be the full additive subcategory of $\mathcal{A}$. Then, by restriction, any element of the center of $\mathcal{A}$ determines an element of the center of $\mathcal{B}$. Clearly, the induced map

$r : Z(\mathscr{A}) \longrightarrow Z(\mathscr{B})$ is a ring homomorphism. If the inclusion functor $\mathscr{B} \longrightarrow \mathscr{A}$ is an equivalence of categories, the morphism of centers is an isomorphism.

Let U and V be two objects in $\mathscr{A}$. Then the center $Z(\mathscr{A})$ acts naturally on $\mathrm{Hom}(U,V)$ by

$$z(\varphi) = z_V \circ \varphi = \varphi \circ z_U$$

for $z \in Z(\mathscr{A})$. Therefore, $\mathrm{Hom}(U,V)$ has a natural structure of a $Z(\mathscr{A})$-module. Clearly, in this way $\mathrm{Hom}(-,-)$ becomes a bifunctor from $\mathscr{A}^\circ \times \mathscr{A}$ into the category of $Z(\mathscr{A})$-modules.[1]

Assume that $\mathscr{C}$ is a triangulated category and T its translation functor. Let z be an element of the center of $\mathscr{C}$. Let U and V be two objects in $\mathscr{C}$ and $\varphi : U \longrightarrow V$ a morphism. Then $T^{-1}(\varphi) : T^{-1}(U) \longrightarrow T^{-1}(V)$ is a morphism and we have

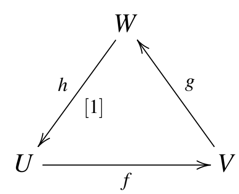
$$z_{T^{-1}(V)} \circ T^{-1}(\varphi) = T^{-1}(\varphi) \circ z_{T^{-1}(U)}.$$

By applying T to this equality we get

$$T(z_{T^{-1}(V)}) \circ \varphi = \varphi \circ T(z_{T^{-1}(U)}).$$

Since $\varphi : U \longrightarrow V$ is arbitrary, we conclude that the assignment $U \longmapsto T(z_{T^{-1}(U)})$ is an element of the center of $\mathscr{A}$, which we denote by $T(z)$. It follows that T induces an automorphism of the center $Z(\mathscr{C})$ of $\mathscr{C}$. The elements of the center $Z(\mathscr{C})$ fixed by this automorphisms form a subring with identity which we call the *t-center* of $\mathscr{C}$ and denote by $Z_0(\mathscr{C})$.

Let



be a distinguished triangle in $\mathscr{C}$ and z an element of the t-center $Z_0(\mathscr{C})$ of $\mathscr{C}$. Clearly, since z is in the center, we have the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{f} & V & \xrightarrow{g} & W \\ {\scriptstyle z_U}\downarrow & & \downarrow{\scriptstyle z_V} & & \downarrow{\scriptstyle z_W} \\ U & \xrightarrow[f]{} & V & \xrightarrow[g]{} & W \end{array}.$$

Moreover, since z is in the t-center of $\mathscr{C}$, we have $T(z_U) = z_{T(U)}$ and the diagram

[1] $\mathscr{A}^\circ$ is the category opposite to $\mathscr{A}$.

$$\begin{array}{ccc} W & \xrightarrow{h} & T(U) \\ {\scriptstyle z_W}\downarrow & & \downarrow{\scriptstyle T(z_U)} \\ W & \xrightarrow[h]{} & T(U) \end{array}$$

commutes. Therefore,

$$\begin{array}{ccccccc} U & \xrightarrow{f} & V & \xrightarrow{g} & W & \xrightarrow{h} & T(U) \\ {\scriptstyle z_U}\downarrow & & \downarrow{\scriptstyle z_V} & & \downarrow{\scriptstyle z_W} & & \downarrow{\scriptstyle T(z_U)} \\ U & \xrightarrow[f]{} & V & \xrightarrow[g]{} & W & \xrightarrow[h]{} & T(U) \end{array}$$

is an endomorphism of the above distinguished triangle. It follows that the elements of the t-center induce endomorphisms of distinguished triangles in $\mathscr{C}$.

Let X be another object of $\mathscr{C}$. The above remark implies that the distinguished triangle determines long exact sequences

$$\cdots \to \mathrm{Hom}(X,U) \to \mathrm{Hom}(X,V) \to \mathrm{Hom}(X,W) \to \mathrm{Hom}(X,T(U)) \to \ldots$$

and

$$\cdots \to \mathrm{Hom}(T(U),X) \to \mathrm{Hom}(W,X) \to \mathrm{Hom}(V,X) \to \mathrm{Hom}(U,X) \to \ldots$$

of $Z_0(\mathscr{C})$-modules.

2.2 Center of a derived category

Let $C^*(\mathscr{A})$ (where $*$ is b, $+$, $-$ or nothing, respectively) be the category of (bounded, bounded from below, bounded from above or unbounded) complexes of objects of $\mathscr{A}$. Then $C^*(\mathscr{A})$ is also an additive category.

Let z be an element of the center of $\mathscr{A}$. If

$$\ldots \longrightarrow V^0 \longrightarrow V^1 \longrightarrow \ldots \longrightarrow V^n \longrightarrow \ldots$$

is an object in $C^*(\mathscr{A})$, we get the commutative diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & V^0 & \longrightarrow & V^1 & \longrightarrow & \ldots & \longrightarrow & V^n & \longrightarrow & \ldots \\ & & {\scriptstyle z_{V^0}}\downarrow & & {\scriptstyle z_{V^1}}\downarrow & & & & \downarrow{\scriptstyle z_{V^n}} & & \\ \ldots & \longrightarrow & V^0 & \longrightarrow & V^1 & \longrightarrow & \ldots & \longrightarrow & V^n & \longrightarrow & \ldots \end{array}$$

which we can interpret as an endomorphism $z_{V^\cdot}$ of $V^\cdot$.

Let $\varphi^\cdot : U^\cdot \longrightarrow V^\cdot$ be a morphism in $C^*(\mathscr{A})$. Then $z_{V^p} \circ \varphi^p = \varphi^p \circ z_{U^p}$ for any $p \in \mathbb{Z}$, i.e., $z_{V^\cdot} \circ \varphi^\cdot = \varphi^\cdot \circ z_{U^\cdot}$. Therefore, the assignment $V^\cdot \longmapsto z_{V^\cdot}$ defines an element $C^*(z)$ of the center of $C^*(\mathscr{A})$. Moreover, we have the following trivial observation.

Lemma 2.2 *The map $z \longmapsto C^*(z)$ defines a homomorphism of the center $Z(\mathscr{A})$ of $\mathscr{A}$ into the center $Z(C^*(\mathscr{A}))$ of $C^*(\mathscr{A})$.*

Let $K^*(\mathscr{A})$ be the corresponding homotopic category of complexes. Let $[z_{V^\cdot}]$ be the homotopy class of endomorphism $z_{V^\cdot}$ of $V^\cdot$ in $C^*(\mathscr{A})$. Then it defines an endomorphism of $V^\cdot$ in $K^*(\mathscr{A})$. Clearly, the assignment $V^\cdot \longmapsto [z_{V^\cdot}]$ is an endomorphism $K^*(z)$ of the identity functor in $K^*(\mathscr{A})$. Moreover, the category $K^*(\mathscr{A})$ is triangulated and the translation functor is given by $T(U^\cdot)^p = U^{p+1}$ for any $p \in \mathbb{Z}$ for any object $U^\cdot$ in $K^*(\mathscr{A})$. If a morphism $\varphi : U^\cdot \longrightarrow V^\cdot$ is the homotopy class of a morphism of complexes $f^\cdot : U^\cdot \longrightarrow V^\cdot$, the morphism $T(\varphi) : T(U^\cdot) \longrightarrow T(V^\cdot)$ is the homotopy class of the morphism of complexes given by $f^{p+1} : T(U^\cdot)^p \longrightarrow T(V^\cdot)^p$ for $p \in \mathbb{Z}$. This immediately implies that $T([z_{U^\cdot}]) = [z_{T(U^\cdot)}]$ for any element z of the center of $\mathscr{A}$. It follows that $K^*(z)$ is in the t-center $Z_0(K^*(\mathscr{A}))$ of $K^*(\mathscr{A})$.

Therefore, we have the following observation.

Lemma 2.3 *The map $z \longmapsto K^*(z)$ defines a homomorphism of the center $Z(\mathscr{A})$ of $\mathscr{A}$ into the t-center $Z_0(K^*(\mathscr{A}))$ of $K^*(\mathscr{A})$.*

Finally, assume that $\mathscr{A}$ is an abelian category and let $D^*(\mathscr{A})$ be the corresponding derived category of $\mathscr{A}$, i.e., the localization of $K^*(\mathscr{A})$ with respect to all quasiisomorphisms. Clearly, for any $z \in Z(\mathscr{A})$, $[z_{V^\cdot}]$ determines an endomorphism $[[z_{V^\cdot}]]$ of $V^\cdot$ in $D^*(\mathscr{A})$.

Let $U^\cdot$ and $V^\cdot$ be two complexes in $D^*(\mathscr{A})$ and $\varphi : U^\cdot \longrightarrow V^\cdot$ a morphism of $U^\cdot$ into $V^\cdot$ in $D^*(\mathscr{A})$. We can represent φ by a roof (see, for example [4]):

$$\begin{array}{ccccc} & & W^\cdot & & \\ & \overset{s}{\underset{\sim}{\swarrow}} & & \overset{f}{\searrow} & \\ U^\cdot & & & & V^\cdot \end{array}$$

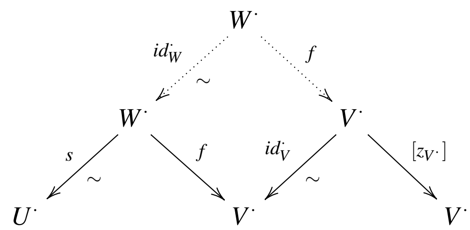
where $s : U^\cdot \longrightarrow W^\cdot$ is a quasiisomorphism and $f : W^\cdot \longrightarrow V^\cdot$ is a morphism in $K^*(\mathscr{A})$. On the other hand, $[[z_{U^\cdot}]]$ and $[[z_{V^\cdot}]]$ are represented by roofs

$$\begin{array}{ccccc} & & U^\cdot & & \\ & \overset{id_{U^\cdot}}{\underset{\sim}{\swarrow}} & & \overset{[z_{U^\cdot}]}{\searrow} & \\ U^\cdot & & & & U^\cdot \end{array}$$

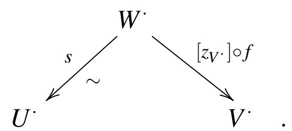
and

$$\begin{array}{ccccc} & & V^\cdot & & \\ & \overset{id_{U^\cdot}}{\underset{\sim}{\swarrow}} & & \overset{[z_{V^\cdot}]}{\searrow} & \\ V^\cdot & & & & V^\cdot \end{array} .$$

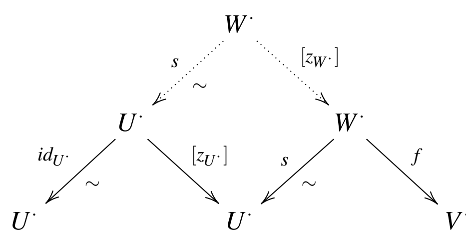
To calculate the composition $[[z_{V^\cdot}]] \circ \varphi$ we consider the composition diagram



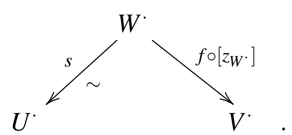
which obviously commutes. This implies that the composition is represented by the roof



Analogously, to calculate $\varphi \circ [[z_{U^\cdot}]]$ we consider the composition diagram



which commutes since $K^*(z)$ is in the center of $K^*(\mathscr{A})$. This implies that the composition is represented by the roof



Since $f \circ [z_{W^\cdot}] = [z_{V^\cdot}] \circ f$, these two roofs are identical and $[[z_{V^\cdot}]] \circ \varphi = \varphi \circ [[z_{V^\cdot}]]$. Hence, the assignment $V^\cdot \longmapsto [[z_{V^\cdot}]]$ defines an element of the t-center $Z_0(D^*(\mathscr{A}))$ of $D^*(\mathscr{A})$ which we denote by $D^*(z)$. Moreover, we have the following result.

Lemma 2.4 *The map $z \longmapsto D^*(z)$ defines an injective morphism of the center $Z(\mathscr{A})$ of $\mathscr{A}$ into the t-center $Z_0(D^*(\mathscr{A}))$ of $D^*(\mathscr{A})$.*

For any $z \in Z(\mathscr{A})$, we have

$$H^p([[z_{V^\cdot}]]) = z_{H^p(V^\cdot)} \text{ for any } V^\cdot \text{ in } D^*(\mathscr{A}) \text{ and any } p \in \mathbb{Z}.$$

Proof. The second statement follows immediately from the construction.

To prove injectivity, assume that $D^*(z) = 0$ for some $z \in Z(\mathscr{A})$. For an object V in $\mathscr{A}$, denote by $D(V)^{\cdot}$ the complex such that $D(V)^0 = V$ and $D(V)^p = 0$ for $p \neq 0$. By our assumption, we have $[[z_{D(V)^{\cdot}}]] = 0$. This implies that $z_V = H^0([[z_{D(V)^{\cdot}}]]) = 0$. Therefore, $z_V = 0$ for any V in $\mathscr{A}$, i.e., $z = 0$. □

Let z be an element of the t-center $Z_0(D^*(\mathscr{A}))$ of $D^*(\mathscr{A})$. Then

$$H^{p+1}(z_{U^{\cdot}}) = H^p(T(z_{U^{\cdot}})) = H^p(z_{T(U^{\cdot})})$$

for any object $U^{\cdot}$ in $D^*(\mathscr{A})$ and $p \in \mathbb{Z}$. Therefore, $H^0(z_{U^{\cdot}}) = 0$ for all objects $U^{\cdot}$ in $D^*(\mathscr{A})$ is equivalent to $H^p(z_{U^{\cdot}}) = 0$ for all objects $U^{\cdot}$ in $D^*(\mathscr{A})$ and $p \in \mathbb{Z}$. In particular,

$$\begin{aligned} I_0(D^*(\mathscr{A})) &= \{z \in Z_0(D^*(\mathscr{A})) \mid H^p(z_{U^{\cdot}}) = 0 \text{ for all } U^{\cdot} \text{ in } D^*(\mathscr{A}) \text{ and } p \in \mathbb{Z}\} \\ &= \{z \in Z_0(D^*(\mathscr{A})) \mid H^0(z_{U^{\cdot}}) = 0 \text{ for all } U^{\cdot} \text{ in } D^*(\mathscr{A})\} \end{aligned}$$

is an ideal in $Z_0(D^*(\mathscr{A}))$.

On the other hand, let $D : \mathscr{A} \longrightarrow D^*(\mathscr{A})$ be the functor which attaches to each object V in $\mathscr{A}$ the complex $D(V)^{\cdot}$, such that $D(V)^0 = V$ and $D(V)^p = 0$ for all $p \neq 0$. This functor is an isomorphism of $\mathscr{A}$ onto the full additive subcategory of $D^*(\mathscr{A})$ consisting of all complexes $U^{\cdot}$ such that $U^p = 0$ for all $p \neq 0$ [4]. Therefore, we have a natural homomorphism r of $Z(D^*(\mathscr{A}))$ into $Z(\mathscr{A})$ which attaches to an element z of the center of $D^*(\mathscr{A})$ the element of the center of $\mathscr{A}$ given by $V \longmapsto H^0(z_{D(V)})$ for any V in $\mathscr{A}$. In particular, we have a natural homomorphism $r : Z_0(D^*(\mathscr{A})) \longrightarrow Z(\mathscr{A})$.

From 2.4, we see that

$$r(D^*(z))_V = H^0(D^*(z)_{D(V)}) = H^0([[z_{D(V)}]]) = z_V$$

for any z in the center of $\mathscr{A}$ and any V in $\mathscr{A}$. Therefore, we have the following result.

Proposition 2.5 *The natural homomorphism $r : Z_0(D^*(\mathscr{A})) \longrightarrow Z(\mathscr{A})$ is a left inverse of the homomorphism $D^* : Z(\mathscr{A}) \longrightarrow Z_0(D^*(\mathscr{A}))$. In particular, it is surjective.*

Its kernel is the ideal $I_0(D^*(\mathscr{A}))$.

The situation is particularly nice for bounded derived categories.[2]

Proposition 2.6 *The natural homomorphism $r : Z_0(D^b(\mathscr{A})) \longrightarrow Z(\mathscr{A})$ is an isomorphism.*

Proof. We have to prove that $I_0(D^b(\mathscr{A})) = 0$. Let z be an element of $I_0(D^b(\mathscr{A}))$.

Clearly, for any object V in $\mathscr{A}$, we have $z_{D(V)} = 0$. Moreover, since z is in the t-center, $z_{T^p(D(V))} = 0$ for any $p \in \mathbb{Z}$.

[2] I do not know any example where this result fails in unbounded case.

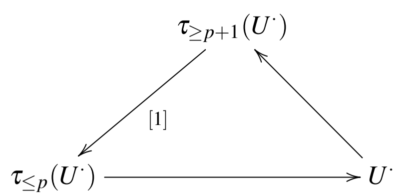
For any object $U^\cdot$ in $D^b(\mathscr{A})$ we put

$$\ell(U^\cdot) = \mathrm{Card}\{p \in \mathbb{Z} \mid H^p(U^\cdot) \neq 0\},$$

and call $\ell(U^\cdot)$ the *cohomological length* of $U^\cdot$.

Now we want to prove that $z_{U^\cdot} = 0$ for all $U^\cdot$ in $D^b(\mathscr{A})$. The proof is by induction in the cohomological length $\ell(U^\cdot)$. If $\ell(U^\cdot) = 0$, $U^\cdot = 0$ and $z_{U^\cdot} = 0$. If $\ell(U^\cdot) = 1$, there exists $p \in \mathbb{Z}$ such that $H^q(U^\cdot) = 0$ for all $q \neq p$. In this case, $U^\cdot$ is isomorphic to the complex which is zero in all degrees $q \neq p$ and in degree p is equal to $H^p(U^\cdot)$, i.e., to $T^{-p}(D(H^p(U^\cdot)))$. Hence, by the above remark, $z_{U^\cdot} = 0$.

Assume now that $\ell(U^\cdot) > 1$. Let $\tau_{\leq p}$ and $\tau_{\geq p}$ be the usual truncation functors [4]. Then, for any $p \in \mathbb{Z}$, we have the truncation distinguished triangle



and by choosing a right $p \in \mathbb{Z}$, we have $\ell(\tau_{\leq p}(U^\cdot)) < \ell(U^\cdot)$ and $\ell(\tau_{\geq p+1}(U^\cdot)) < \ell(U^\cdot)$. Therefore, by the induction assumption, there exists $p \in \mathbb{Z}$ such that $z_{\tau_{\leq p}(U^\cdot)} = 0$ and $z_{\tau_{\geq p+1}(U^\cdot)} = 0$. As we remarked before, this distinguished triangle leads to the long exact sequence

$$\cdots \to \mathrm{Hom}(U^\cdot, \tau_{\leq p}(U^\cdot)) \to \mathrm{Hom}(U^\cdot, U^\cdot) \to \mathrm{Hom}(U^\cdot, \tau_{\geq p+1}(U^\cdot)) \to \ldots$$

of $Z_0(D^b(\mathscr{A}))$-modules. By our construction, z annihilates the first and third module. Therefore, it must annihilate $\mathrm{Hom}(U^\cdot, U^\cdot)$ too. This implies that

$$0 = z(\mathrm{id}_{U^\cdot}) = \mathrm{id}_{U^\cdot} \circ z_{U^\cdot} = z_{U^\cdot}.$$

□

3 Centers and derived functors

3.1 Homogeneous functors

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories. Let R be a commutative ring with identity and $\alpha : R \longrightarrow Z(\mathscr{A})$ and $\beta : R \longrightarrow Z(\mathscr{B})$ ring morphisms of rings with identity.

By 2.4, α and β define ring morphisms $\alpha = D^* \circ \alpha : R \longrightarrow Z_0(D^*(\mathscr{A}))$ and $\beta = D^* \circ \beta : R \longrightarrow Z_0(D^*(\mathscr{B}))$

Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor. We say that F is *R-homogeneous* if for any $r \in R$ we have

$$\beta(r)_{F(V)} = F(\alpha(r)_V) \text{ for any object } V \text{ in } \mathscr{A}.$$

Assume now that F is left exact. Assume that there exists a subcategory $\mathscr{R}$ of $\mathscr{A}$ right adapted to F [4, ch. III, §6, no. 3][3]. Then F has the right derived functor $RF : D^+(\mathscr{A}) \longrightarrow D^+(\mathscr{B})$.

Theorem 3.1 *The functor $RF : D^+(\mathscr{A}) \longrightarrow D^+(\mathscr{B})$ is R-homogeneous.*

Proof. Let $V^\cdot$ be a complex in $D^+(\mathscr{A})$. Since $\mathscr{R}$ is right adapted to F, there exists a bounded from below complex $R^\cdot$ consisting of objects in $\mathscr{R}$ and a quasiisomorphism $q : V^\cdot \longrightarrow R^\cdot$. Let z be an element of the center of $\mathscr{A}$. Then we have the commutative diagram

$$\begin{array}{ccc} V^\cdot & \xrightarrow{\ q\ } & R^\cdot \\ {\scriptstyle [[z_{V^\cdot}]]}\big\downarrow & & \big\downarrow{\scriptstyle [[z_{R^\cdot}]]} \\ V^\cdot & \xrightarrow[\ q\ ]{} & R^\cdot \end{array} .$$

By applying the functor RF to it, we get the diagram

$$\begin{array}{ccc} RF(V^\cdot) & \xrightarrow{RF(q)} & F(R^\cdot) \\ {\scriptstyle RF([[z_{V^\cdot}]])}\big\downarrow & & \big\downarrow{\scriptstyle [[F(z)_{F(R^\cdot)}]]} \\ RF(V^\cdot) & \xrightarrow[RF(q)]{} & F(R^\cdot) \end{array} .$$

If $r \in R$, $\alpha(r)$ is in the center of $\mathscr{A}$ and the above diagram implies that

$$\begin{array}{ccc} RF(V^\cdot) & \xrightarrow{RF(q)} & F(R^\cdot) \\ {\scriptstyle RF([[\alpha(r)_{V^\cdot}]])}\big\downarrow & & \big\downarrow{\scriptstyle [[\beta(r)_{F(R^\cdot)}]]} \\ RF(V^\cdot) & \xrightarrow[RF(q)]{} & F(R^\cdot) \end{array}$$

is commutative. Moreover, $\beta(r)$ is in the center of $\mathscr{B}$, hence we also have

$$\begin{array}{ccc} RF(V^\cdot) & \xrightarrow{RF(q)} & F(R^\cdot) \\ {\scriptstyle [[\beta(r)_{RF(V^\cdot)}]]}\big\downarrow & & \big\downarrow{\scriptstyle [[\beta(r)_{F(R^\cdot)}]]} \\ RF(V^\cdot) & \xrightarrow[RF(q)]{} & F(R^\cdot) \end{array} .$$

Hence, we conclude that $RF([[\alpha(r)_{V^\cdot}]]) = [[\beta(r)_{RF(V^\cdot)}]]$, i.e., RF is R-homogeneous. □

[3] I would prefer a proof of the next theorem which doesn't use the construction of the derived functor, but its universal property. Unfortunately, I do not know such argument.

Let V be an object in $\mathscr{A}$. Then

$$\beta(r)_{RF(D(V))} = RF(\alpha(r)_{D(V)}) \text{ for any } r \in R.$$

By taking cohomology, we get

$$\beta(r)_{R^pF(V)} = H^p(\beta(r)_{RF(D(V))}) = R^pF(\alpha(r)_V) \text{ for any } r \in R \text{ and } p \in \mathbb{Z}_+.$$

Therefore, we have the following consequence.

Corollary 3.2 *The functors $R^pF : \mathscr{A} \longrightarrow \mathscr{B}$ are R-homogeneous.*

We leave to the reader the formulation and proofs of the analogous results for a right exact functor F and its left derived functor $LF : D^-(\mathscr{A}) \longrightarrow D^-(\mathscr{B})$.

3.2 Special cases

Now we are going to illustrate how 1.1 and 1.2 follow from the above discussion.

First, we prove a well-known result about the center of the category of modules. This is not necessary for our applications, but puts the constructions in a proper perspective.

Let A be a ring with identity and Z its center. Let $\mathscr{M}(A)$ be the category of A-modules. Any element z in Z determines an endomorphism z_U of an A-module U. Clearly, the assignment $U \longmapsto z_U$ defines an element of the center $Z(\mathscr{M}(A))$ of $\mathscr{M}(A)$. Therefore, we have a natural homomorphism $i : Z \longrightarrow Z(\mathscr{M}(A))$ of rings.

Lemma 3.3 *The morphism $i : Z \longrightarrow Z(\mathscr{M}(A))$ is an isomorphism.*

Proof. If we consider A as an A-module for the left multiplication, we see that $i(z)_A$ is the multiplication by z for any $z \in Z$. Therefore, $i(z)_A(1) = z$ and $i : Z \longrightarrow Z(\mathscr{M}(A))$ is injective.

Let ζ be an element of the center of $\mathscr{A}$. Then ζ_A is an endomorphism of A considered as A-module for left multiplication. Let $z = \zeta_A(1)$. Then

$$\zeta_A(a) = a\zeta_A(1) = az$$

for any $a \in A$. Moreover, any $b \in A$ defines an endomorphism φ_b of A given by $\varphi_b(a) = ab$ for all $a \in A$. Since we must have $\zeta_A \circ \varphi_b = \varphi_b \circ \zeta_A$, it follows that

$$bz = (\zeta_A \circ \varphi_b)(1) = (\varphi_b \circ \zeta_A)(1) = zb.$$

Since $b \in A$ is arbitrary, z must be in the center Z of A.

Let M be an arbitrary A-module and $m \in M$. Then m determines a module morphism $\psi_m : A \longrightarrow M$ given by $\psi_m(a) = am$ for any $a \in A$. Therefore,

$$\zeta_M(m) = (\zeta_M \circ \psi_m)(1) = (\psi_m \circ \zeta_A)(1) = zm = i(z)_M m.$$

Hence $\zeta = i(z)$, and i is surjective. □

Now we return to the notation from the first section. By 3.3, the center of the category $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ is isomorphic to $\mathscr{Z}(\mathfrak{g})$.

First we discuss 1.1. The functor $F = \operatorname{Hom}_{\mathscr{U}(\mathfrak{g})}(U, -)$ is a functor from the category $\mathscr{U}(\mathfrak{g})$ into the category of $\mathscr{Z}(\mathfrak{g})$-modules. If we define α as the natural morphism of $\mathscr{Z}(\mathfrak{g})$ into the center of $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ and β as multiplication by $\chi_U(z)$, F is clearly $\mathscr{Z}(\mathfrak{g})$-homogeneous. This implies that the functors R^pF are $\mathscr{Z}(\mathfrak{g})$-homogeneous. Hence, for any V in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ we have $R^pF(z_V) = \chi_U(z)$ for all $p \in \mathbb{Z}_+$. In, particular, if $z \in \ker \chi_V$ we have

$$0 = R^pF(0) = R^pF(z_V) = \chi_U(z).$$

This clearly contradicts $\chi_U \neq \chi_V$ if $\operatorname{Ext}^p_{\mathscr{U}(\mathfrak{g})}(U,V) \neq 0$ for some $p \in \mathbb{Z}$.

No we discuss 1.2. The functor $F = H_0(\mathfrak{n}, -)$ is a functor from the category $\mathscr{U}(\mathfrak{g})$ into the category of $\mathscr{Z}(\mathfrak{g})$-modules. If we define α as the natural morphism of $\mathscr{Z}(\mathfrak{g})$ into the center of $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$ and β as the composition of the Harish-Chandra homomorphism with the natural morphism of $\mathscr{U}(\mathfrak{h})$ into the center of $\mathscr{M}(\mathscr{U}(\mathfrak{h}))$, F is clearly $\mathscr{Z}(\mathfrak{g})$-homogeneous. This implies that the functors L_pF are $\mathscr{Z}(\mathfrak{g})$-homogeneous. Hence for any V in $\mathscr{M}(\mathscr{U}(\mathfrak{g}))$, we have $L_pF(z_V) = \gamma(z)_{L_pF(V)}$ for all $p \in \mathbb{Z}_+$. In, particular, if z annihilates V, $\gamma(z)$ annihilates $L_pF(V) = H_p(\mathfrak{n}, V)$ for all $p \in \mathbb{Z}$.

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