Vol. 49, No. 4 DUKE MATHEMATICAL JOURNAL © December 1982

ASYMPTOTIC BEHAVIOR OF MATRIX COEFFICIENTS OF ADMISSIBLE REPRESENTATIONS

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Introduction. In this paper we will restate and reprove several old, but largely unpublished results of Harish-Chandra ([11], [12], [13], [22]) regarding the behavior at infinity of matrix coefficients of certain representations of reductive Lie groups. Our methods are rather different from those of Harish-Chandra. Very briefly put, the difference is that we make a coordinate change that allows us to formulate things in terms of systems of complex differential equations and thus apply elegant but elementary results of Deligne [8].

More precisely, let G be a reductive group in what we call the Harish-Chandra class (see Section 1), K a maximal compact subgroup and $\mathfrak{g}$ the complexified Lie algebra of G. Suppose that (π, V) is a smooth representation of G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$, the center of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. We will be concerned with a description of the matrix coefficient $\langle \pi(x)v, \tilde{v}\rangle$ as $x \in G$ tends to infinity, where v is a K-finite vector in V and $\tilde{v}$ a K-finite vector in the dual $\tilde{V}$ of V. If θ is a Cartan involution of G associated to K and A a maximal θ-stable closed vector subgroup of G, then $G = KAK$; so that, because of the K-finiteness assumption one may as well assume $x \in A$. Loosely put, the K-finiteness of v and $\tilde{v}$, together with the assumption that I

Received June 30, 1982. Casselman partially supported by the National Science and Engineering Research Council of Canada and Miličić partially supported by the National Science Foundation and Sonderforschungsbereich "Theoretische Mathematik", Universität Bonn, FRG.

annihilates V, imply that on A the matrix coefficient $\langle \pi(a)v, \tilde{v} \rangle$ satisfies a certain system of differential equations which extend onto the complex torus one gets from A by base field extension from $\mathbf{R}$ to $\mathbf{C}$. The solutions of this system are related to the horizontal sections of a certain holomorphic connection in the sense of [8], and this connection turns out to have regular singularities at infinity, which enables us to prove Harish-Chandra's results with almost no explicit calculations.

Rather than state here the main results (Theorem 5.6 and 6.2) precisely, we will look here at the simplest example: $G = \mathrm{SL}(2, \mathbf{R})$. Suppose π to contain a nontrivial vector fixed by $K = \mathrm{SO}(2)$ and suppose also that the Casimir element acts on V by the scalar λ. Choose $\tilde{v} \neq 0$ fixed by K in $\tilde{V}$. Then the function $\langle \pi(x)v, \tilde{v} \rangle$ may be considered as a function F on the upper half-plane $\mathcal{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ which is an eigenfunction for the non-Euclidean Laplacian. Since V may be written as a sum of eigenvectors with respect to K, one may as well assume that v is an eigenvector with respect to non-Euclidean rotations around i; or that in non-Euclidean polar coordinates, if φ is the angular variable, the function F satisfies $(\partial/\partial\varphi)F = inF$ for some $n \in \mathbf{Z}$. Now, if r = radial (non-Euclidean) distance from i, the Laplacian may be expressed as

$$\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{\tanh r}\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\left(\frac{\partial}{\partial \varphi}\right)^2,$$

so that if $F = F(r, \varphi) = f(r)e^{in\varphi}$, the function f satisfies

$$\frac{d^2 f}{dr^2} + \frac{1}{\tanh r}\frac{df}{dr} - \frac{n^2}{\sinh^2 r} f = \lambda f.$$

This is how Harish-Chandra would express things (see [22, 9.1. Heuristics]). What we do is use not (r, φ) but (y, φ) as our coordinates, where $y = e^r$. The equation above becomes

$$\left[\left(y\frac{d}{dy}\right)^2 - \frac{1+y^2}{1-y^2}\, y\frac{d}{dy} - n^2\frac{y^2}{(1-y^2)^2}\right] f = \lambda f$$

(compare with 3.7). This looks a little more complicated, but in making the coordinate change $y = e^r$, the irregular singularities of the first equation at $\pm\infty$ become regular singularities at $0, \infty$. Thus one may apply the classical theory of Frobenius and—for example—the convergence of Harish-Chandra's series is, in some sense, explained naturally. (We recall that the natural context of differential equations with regular singularities is the theory of complex variables, even though the original equation was considered only for real values of y, and that it is this which lets one prove things painlessly.) Thus, for generic values of λ, one deduces the existence of functions f_1, f_2 holomorphic in the open disc $|y| < 1$ in C such that

$$f(y) = y^{s_1} f_1(y) + y^{s_2} f_2(y)$$

where s_1, s_2 are the roots of the indicial equation

$$s^2 - s = \lambda.$$

(This is true when $s_1 - s_2 \notin \mathbf{Z}$. In general one must introduce some terms involving $\log y$ as well. For example, when $s_1 = s_2 = \frac{1}{2}$ then there exist f_1, f_2 as above with

$$f(y) = y^{1/2} f_1(y) + y^{1/2} \log y\, f_2(y).)$$

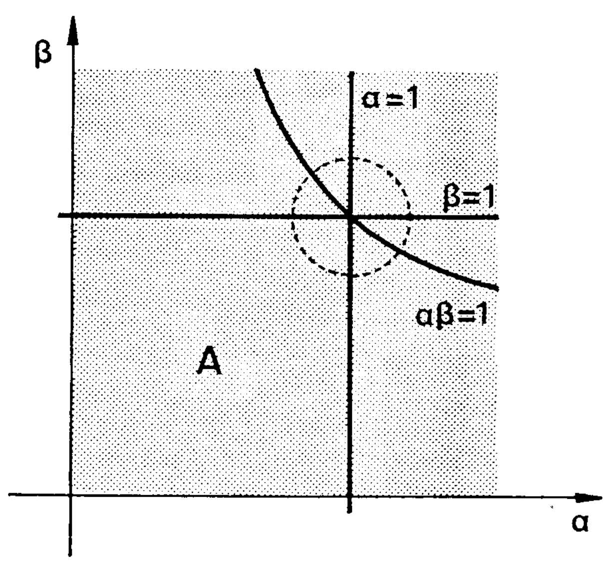
It is precisely this sort of analysis that we will carry out for arbitrary reductive groups. To simplify things slightly, assume G is semi-simple, and let Δ be a choice of simple positive roots—i.e. multiplicative homomorphisms of A into $\mathbf{R}_+^*$. Then one can imbed A into $\mathbf{C}^\Delta$ via the map $a \to (\alpha(a),\ \alpha \in \Delta)$. Modulo certain technicalities, K-finite matrix coefficients, when restricted to A, will satisfy a certain system of differential equations which extend to all of $\mathbf{C}^\Delta$, with singularities on the hyperplanes $\alpha = 0$ ($\alpha \in \Delta$) and the hypersurfaces $\gamma^2 = 1$ (γ any root). For our purposes the crucial point will be that this system has *regular* singularities along the hyperplanes $\alpha = 0$ ($\alpha \in \Delta$), which are, in some sense, the points at infinity on A. (Note that all points at infinity on A are transforms under the Weyl group of these.) This will imply, at least around the points of these hyperplanes which are not on the hypersurfaces $\gamma^2 = 1$ (and in particular around the origin), an expansion analogous to the one found above for $\mathrm{SL}(2, \mathbf{R})$.

Perhaps a picture will help. Let $G = \mathrm{SL}(3, \mathbf{R})$. Then

$$A = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle|\, a_i > 0,\ a_1 a_2 a_3 = 1 \right\}.$$

The simple roots may be chosen as $\alpha = a_1/a_2$, $\beta = a_2/a_3$; and the only other positive root is $\alpha\beta = a_1/a_3$. Thus one may picture A, as well as a neighborhood in $\mathbf{R}^\Delta$, as shown on Fig. 0.1 (the shaded region represents A with all the singularities intersecting A drawn in also). The dotted circle is a neighborhood of the identity element in A. We have indicated it in order to contrast our coordinate system with that of Harish-Chandra, where our coordinate lines are at infinity and the curves $\gamma = 1$ become lines meeting at the origin (as for SL(2, R), the relation between the two is exponentiation).

In this example, the regularity of singularities gives expansions around all points on the coordinate lines except (0, 1) and (1, 0). In fact these points may also be dealt with (as well as their analogues for other groups, of course). This is an important matter: one is able to deduce properties of the matrix coefficients everywhere on G from properties around the origin in $\mathbf{C}^{\Delta}$. For example, suppose that a matrix coefficient $\langle \pi(a)v, \tilde{v} \rangle$ vanishes as $\alpha(a) \to 0$ for $\alpha \in \Delta$ in a neighborhood of the origin. Then in fact it vanishes whenever $\alpha(a) \to 0$ for $\alpha \in \Delta$. One can deal similarly with integrability and growth properties (see Section 7). This part of the theory is called the asymptotics "along the walls" because the hyperplanes $\alpha = 1$ $(\alpha \in \Delta)$ are the walls of the "negative" Weyl chamber $A^- = \{a \in A \mid \alpha(a) < 1 \text{ for } \alpha \in \Delta\}$ in A. Some version of this is due to Harish-Chandra, but has only been written down in a well known but unpublished and rather intricate manuscript [12]. In our context, these results follow (in Section 6) form the earlier ones (in Section 5) by monodromy arguments in $\mathbf{C}^{\Delta}$. These arguments appear to us considerably simpler to follow (although we have not often convinced our colleagues of this simplicity). In particular, we do not need for this part of the argument any special information about the nature of the singularities of our system of differential equations on the root hypersurfaces $\gamma^2 = 1$. In fact, we know these to be regular ([3], [4]), but we will not prove this in this paper, nor shall we refer to it in the body of the paper. Note that for SL(2, R) the equation we write for f clearly has a regular singularity



at $y = 1$; the consequence we wish to point out is that although f has no singularity at $y = 1$, the two functions f_1 and f_2 will in general have singularities at this point. Getting around this possibility in general is something that must be taken into account (see the beginning of Section 6); it forces us to state the behavior of matrix coefficients separately on certain subsets of $\mathbf{C}^{\Delta}$ indexed by subsets of Δ.

If this paper appears long, it is because we have tried to make it as self-contained as possible. In Sections 2 and 3 we restate, in a different language, well known results (see [13], [22, Ch. 9]) on the "radial components" of differential operators in $\mathfrak{U}(\mathfrak{g})$—the analogue of expressing the non-Euclidean Laplacian in the coordinates (y, φ). In Section 4 we show that the matrix coefficients satisfy a system of first order complex differential equations of the type considered in [8]. Our main results, as already mentioned, appear in Sections 5 and 6, and consequences appear in Sections 7 and 8. We include also a rather lengthy appendix expressing results of Deligne [8] in down-to-earth terms.

This paper is an outgrowth of some parts of unpublished manuscripts [4] and [16].

We would like to thank Professor Harish-Chandra for showing to one of us (D.M.) the manuscripts [12] and [13] during his stay at the Institute for Advanced Study in 1975/76.

1. Generalities on reductive groups. Let G be a Lie group with the Lie algebra $\mathfrak{g}_0$. Denote by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$, and by Ad the adjoint representation of G in $\mathfrak{g}$. Let G^0 be the identity component of G and G_1 its commutator subgroup. Let G_c be the group of all inner automorphisms of $\mathfrak{g}$. The group G is said to belong to the *Harish-Chandra class* if

(i) $\mathfrak{g}$ is a reductive Lie algebra,
(ii) $[G : G^0]$ is finite,
(iii) $\mathrm{Ad}(G) \subset G_c$,
(iv) the center of G_1 is finite.

Connected semisimple Lie groups with finite center and groups of real-valued points on Zariski-connected reductive algebraic groups defined over $\mathbf{R}$ belong to this class. What is crucial is a hereditary property: if G belongs to this class than so do Levi components of parabolic subgroups of G. For more about such groups, see §§II.1, II.6 of [20].

In the following we fix, once for all, a group G in the Harish-Chandra class.

All maximal compact subgroups of G are conjugate by the elements of G^0. We fix a maximal compact subgroup K of G. Denote by $\mathfrak{k}_0$ its Lie algebra and by $\mathfrak{k}$ the complexification of $\mathfrak{k}_0$.

Let θ be a Cartan involution of G corresponding to K, i.e. an involutive automorphism of G whose set of fixed points is equal to K. We denote its differential, which is an involutive automorphism of $\mathfrak{g}$, by the same letter.

There exists a G-invariant bilinear form $B: \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbf{R}$ which is symmetric and nondegenerate and has the following properties

(i) $B(\theta X, Y) = B(X, \theta Y)$ for all $X, Y \in \mathfrak{g}_0$,

(ii) the Lie algebra of G_1 is orthogonal to the Lie algebra of the center Z_G of G with respect to B,

(iii) the bilinear map

$$(X, Y) \to -B(X, \theta Y)$$

is a positive definite inner product for $\mathfrak{g}_0$.

This inner product extends uniquely to a Hermitian inner product on $\mathfrak{g}$.

We fix such a bilinear form B in the following. In the case of a semisimple group G we can take for B the Killing form of $\mathfrak{g}_0$.

Let P be a minimal parabolic subgroup of G. Denote by N the nilpotent radical of P and put $L = P \cap \theta(P)$. Then L is the unique θ-stable Levi-component of P, and P is the semidirect product of L and N. The subgroup $M = L \cap K$ is the maximal compact subgroup of L and if we denote by A the maximal θ-stable closed vector subgroup of L, we have the direct product decomposition $L = MA$.

We have the following well-known decompositions

$$G = KAN \qquad \text{(Iwasawa decomposition)}$$

$$G = KAK \qquad \text{(Cartan decomposition)}$$

$$P = MAN \qquad \text{(Langlands decomposition)}$$

of G, respectively P.

Let $\mathfrak{p}_0, \mathfrak{l}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0$ be the Lie algebras of P, L, M, A, N and $\mathfrak{p}, \mathfrak{l}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ their complexifications, respectively.

The group A acts on $\mathfrak{g}$ by the adjoint action. The linear operators $\operatorname{Ad} a$, $a \in A$, are self-adjoint with respect to the Hilbert space structure on $\mathfrak{g}$. Therefore, if we denote for a positive character $\alpha: A \to \mathbf{R}_+^*$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\operatorname{Ad} a)X = \alpha(a)X, \forall a \in A\}$$

we get an orthogonal decomposition

$$\mathfrak{g} = \oplus \mathfrak{g}_\alpha .$$

Obviously, $\mathfrak{g}_1 = \mathfrak{l}$. If $\mathfrak{g}_\alpha \neq \{0\}$ for some $\alpha \neq 1$ we say that α is a *root* of $\mathfrak{g}$ with respect to A. We denote by Σ the set of all roots of $\mathfrak{g}$ with respect to A.

We fix an ordering on Σ so that the set of all positive roots Σ^+ is equal to $\{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \mathfrak{n}\}$. Let Δ be the corresponding set of simple roots.

Let

$$A_{\text{reg}} = \{a \in A \mid \alpha(a) \neq 1, \forall \alpha \in \Sigma\}$$

be the set of regular elements in A. It is the disjoint union of connected Weyl chambers. We put

$$A^{-} = \{a \in A \mid \alpha(a) < 1, \forall \alpha \in \Delta\}$$

for the "negative" Weyl chamber corresponding to P. We have a stronger form of the Cartan decomposition

$$G = K \cdot \mathrm{Cl}(A^{-}) \cdot K.$$

Finally we shall recall some well-known results about the universal enveloping algebras.

Let $\mathfrak{U}(\mathfrak{g}), \mathfrak{U}(\mathfrak{k})$ etc. be the universal enveloping algebras of $\mathfrak{g}, \mathfrak{k}$, etc. respectively, equipped with their canonical filtrations. Let $\mathfrak{Z}(\mathfrak{g}), \mathfrak{Z}(\mathfrak{l})$ be the centers of the universal enveloping algebras $\mathfrak{U}(\mathfrak{g}), \mathfrak{U}(\mathfrak{l})$ respectively, with the induced filtrations.

As it is well-known, the algebras $\mathrm{Gr}\,\mathfrak{U}(\mathfrak{g}), \mathrm{Gr}\,\mathfrak{U}(\mathfrak{l})$ are canonically isomorphic to the symmetric algebras $S(\mathfrak{g}), S(\mathfrak{l})$ of $\mathfrak{g}, \mathfrak{l}$, respectively. Under this isomorphism the algebras $\mathrm{Gr}\,\mathfrak{Z}(\mathfrak{g})$ and $\mathrm{Gr}\,\mathfrak{Z}(\mathfrak{l})$ correspond to the algebras $I(\mathfrak{g}), I(\mathfrak{l})$ of $\mathfrak{g}$-, $\mathfrak{l}$-invariants in the symmetric algebras $S(\mathfrak{g}), S(\mathfrak{l})$. Looking at the adjoint action of A in $\mathfrak{U}(\mathfrak{g})$, it is easy to conclude that

$$\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{Z}(\mathfrak{l}) \oplus \mathfrak{n}\mathfrak{U}(\mathfrak{g}).$$

Let $\sigma : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{l})$ be the projection map with respect to this decomposition. This map σ is an algebra homomorphism compatible with the filtrations on $\mathfrak{Z}(\mathfrak{g})$ and $\mathfrak{Z}(\mathfrak{l})$ (compare [2, Ch. VIII., §6, no. 4]). Also $\mathrm{Gr}\,\sigma : I(\mathfrak{g}) \to I(\mathfrak{l})$ is the restriction of the orthogonal projection of $S(\mathfrak{g})$ onto $S(\mathfrak{l})$ with respect to the natural inner product structure on $S(\mathfrak{g})$ defined by the Hilbert space structure on $\mathfrak{g}$. We include a proof of the following result for the sake of completeness.

PROPOSITION 1.1. *The algebra* $I(\mathfrak{l})$ *is a finitely generated module over* $\mathrm{Gr}\,\sigma(I(\mathfrak{g}))$.

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Hence it is a Cartan subalgebra of $\mathfrak{g}$ too. Denote by W_1, W_2 the Weyl groups of $\mathfrak{g}$, respectively $\mathfrak{l}$, with respect to $\mathfrak{h}$; and by $S(\mathfrak{h})^{W_1}, S(\mathfrak{h})^{W_2}$ the corresponding algebras of invariants. By a result of Chevalley [2, Ch. VIII, §8, no. 3, Corollary 2. of Theorem 1] the orthogonal projection of $S(\mathfrak{g})$ onto $S(\mathfrak{h})$ defines an algebra isomorphism of $I(\mathfrak{g}), I(\mathfrak{l})$ onto $S(\mathfrak{h})^{W_1}, S(\mathfrak{h})^{W_2}$ respectively. Now, [1, Ch. V, §1, no. 9, Theorem 2.] implies our assertion. Q.E.D.

By [1, Ch. III, §2, no. 9, Corollary 1. of Proposition 12.] we have the following consequence.

COROLLARY 1.2. *The algebra* $\mathfrak{Z}(\mathfrak{l})$ *is a finitely generated module over* $\sigma(\mathfrak{Z}(\mathfrak{g}))$.

By the Poincaré–Birkhoff–Witt theorem, the Iwasawa decomposition of the

Lie algebra $\mathfrak{g}$ implies the following decomposition of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$,

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}) \oplus (\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{k}).$$

Let $\chi : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{a})$ be the projection map corresponding to this decomposition.

We have $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$. The projection map of $\mathfrak{l}$ onto $\mathfrak{a}$ induces an algebra homomorphism $\omega : \mathfrak{U}(\mathfrak{l}) \to \mathfrak{U}(\mathfrak{a})$. It is evident that the restriction of χ to $\mathfrak{Z}(\mathfrak{g})$ is equal to $\omega \circ \sigma$, therefore $\chi : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{a})$ is an algebra homomorphism. Obviously we have $\operatorname{Gr}\chi = \operatorname{Gr}\omega \circ \operatorname{Gr}\sigma$, so 1.1 and the fact that $\operatorname{Gr}\omega : I(\mathfrak{l}) \to S(\mathfrak{a})$ is surjective imply the following result.

COROLLARY 1.3. *The algebra $S(\mathfrak{a})$ is a finitely generated module over* $\operatorname{Gr}\chi(I(\mathfrak{g}))$.

2. The infinitesimal Cartan decomposition. In this section we study a decomposition of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ closely related to the Cartan decomposition of the group G. Its full importance cannot be fully appreciated before Section 3 where it will play a crucial role in the study of the action of $\mathfrak{U}(\mathfrak{g})$ on spherical functions on G.

Let $a \in A$. For $X \in \mathfrak{U}(\mathfrak{g})$ we put $X^a = (\operatorname{Ad} a^{-1})X$. Define the trilinear map $B_a : \mathfrak{U}(\mathfrak{a}) \times \mathfrak{U}(\mathfrak{k}) \times \mathfrak{U}(\mathfrak{k}) \to \mathfrak{U}(\mathfrak{g})$ by

$$B_a(H, X, Y) = X^a H Y$$

for $H \in \mathfrak{U}(\mathfrak{a})$, $X, Y \in \mathfrak{U}(\mathfrak{k})$. Obviously for $Z \in \mathfrak{U}(\mathfrak{m})$ we have

$$B_a(H, XZ, Y) = B_a(H, X, ZY).$$

Regarding the first $\mathfrak{U}(\mathfrak{k})$ as a right $\mathfrak{U}(\mathfrak{m})$-module by right multiplication and the second $\mathfrak{U}(\mathfrak{k})$ as a left $\mathfrak{U}(\mathfrak{m})$-module by left multiplication, the map B_a induces a linear map

$$\Gamma_a : \mathfrak{U}(\mathfrak{a}) \otimes \mathfrak{U}(\mathfrak{k}) \otimes_{\mathfrak{U}(\mathfrak{m})} \mathfrak{U}(\mathfrak{k}) \to \mathfrak{U}(\mathfrak{g})$$

such that

$$\Gamma_a(H \otimes X \otimes Y) = X^a H Y$$

for every $H \in \mathfrak{U}(\mathfrak{a})$, $X, Y \in \mathfrak{U}(\mathfrak{k})$.

In the following we put

$$\mathfrak{C} = \mathfrak{U}(\mathfrak{a}) \otimes \mathfrak{U}(\mathfrak{k}) \otimes_{\mathfrak{U}(\mathfrak{m})} \mathfrak{U}(\mathfrak{k}),$$

viewed as a complex linear space.

We have the following infinitesimal version of the Cartan decomposition for the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

THEOREM 2.1. *For $a \in A_{\mathrm{reg}}$, $\Gamma_a : \mathfrak{C} \to \mathfrak{U}(\mathfrak{g})$ is a linear isomorphism.*

To prove the above theorem we need a few preliminary remarks.

LEMMA 2.2. *Let γ be a root, $Z \in \mathfrak{g}_\gamma$ and $a \in A$ such that $\gamma(a) \neq 1$. Then $U = Z + \theta Z \in \mathfrak{k}$ and*

$$Z = \frac{\gamma(a)}{1 - \gamma(a)^2}(U^a - \gamma(a)U).$$

Proof. Since θ acts as inversion on A, $\theta\mathfrak{g}_\gamma = \mathfrak{g}_{\gamma^{-1}}$ for every root γ. Therefore, the relation $U = Z + \theta Z$ implies

$$U^a = \gamma(a)^{-1}Z + \gamma(a)\theta Z,$$

which immediately implies our assertion. Q.E.D.

Let $\mathfrak{q}$ be the image of $(I + \theta): \mathfrak{n} \to \mathfrak{k}$. This is the orthogonal complement to $\mathfrak{m}$ in $\mathfrak{k}$.

COROLLARY 2.3. *For any $a \in A_{\text{reg}}$*

$$\mathfrak{g} = \mathfrak{q}^a \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Proof. By 2.2 and the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ it follows immediately that $\mathfrak{g}$ is spanned by $\mathfrak{q}^a$, $\mathfrak{a}$ and $\mathfrak{k}$. Now

$$\begin{aligned}\dim \mathfrak{q}^a + \dim \mathfrak{a} + \dim \mathfrak{k} &= \dim \mathfrak{q} + \dim \mathfrak{a} + \dim \mathfrak{k} \\ &\leqslant \dim \mathfrak{n} + \dim \mathfrak{a} + \dim \mathfrak{k} = \dim \mathfrak{g}\end{aligned}$$

which implies $\mathfrak{g} = \mathfrak{q}^a \oplus \mathfrak{a} \oplus \mathfrak{k}$. Q.E.D.

To conclude the proof of Theorem 2.1, apply Poincaré–Birkhoff–Witt.

Denote by $\mathfrak{R}$ the algebra of functions on A_{reg} generated by α, $\alpha \in \Delta$, and $(1 - \gamma^2)^{-1}$, $\gamma \in \Sigma$. For each $a \in A_{\text{reg}}$ there is a unique linear map of $\mathfrak{R} \otimes \mathfrak{C}$ into $\mathfrak{U}(\mathfrak{g})$ which takes $f \otimes X$, $f \in \mathfrak{R}$, $X \in \mathfrak{C}$, into $f(a)\Gamma_a(X)$. We denote this map by Γ_a too.

THEOREM 2.4. *For each $X \in \mathfrak{U}(\mathfrak{g})$ there exists a unique $\Pi(X) \in \mathfrak{R} \otimes \mathfrak{C}$ such that $\Gamma_a(\Pi(X)) = X$ for every $a \in A_{\text{reg}}$.*

Proof. By 2.1 the uniqueness is obvious. We prove the existence of $\Pi(X)$ by the induction in the degree of X. If X is of the degree zero the assertion is obvious.

Let $X \in \mathfrak{U}_{n+1}(\mathfrak{g})$, $n \in \mathbf{Z}_+$. By Poincaré–Birkhoff–Witt and the Iwasawa decomposition of $\mathfrak{g}$ we have the decomposition

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{n}\mathfrak{U}(\mathfrak{g}) \oplus \mathfrak{U}(\mathfrak{a})\mathfrak{U}(\mathfrak{k}).$$

Therefore, there exists $X_0 \in \mathcal{U}(\mathfrak{a})\mathcal{U}(\mathfrak{k})$ such that $X - X_0 \in \mathfrak{n}\mathcal{U}_n(\mathfrak{g})$. It follows immediately that we have to prove the assertion only for the elements of $\mathfrak{n}\mathcal{U}_n(\mathfrak{g})$. In addition, we can suppose that X is of the form ZY, where $Z \in \mathfrak{g}_\gamma$ for $\gamma \in \Sigma^+$ and $Y \in \mathcal{U}_n(\mathfrak{g})$. By 2.2 we have

$$ZY = \frac{\gamma(a)}{1 - \gamma(a)^2}\left(U^a Y - \gamma(a) YU - \gamma(a)[U, Y]\right)$$

for every $a \in A_{\text{reg}}$. Applying the induction assumption to $Y, [U, Y] \in \mathcal{U}_n(\mathfrak{g})$, the assertion follows immediately. Q.E.D.

The adjoint action of M on $\mathcal{U}(\mathfrak{k})$ defines a natural action of M on $\mathcal{R} \otimes \mathcal{C}$ by

$$m \cdot (f \otimes H \otimes X \otimes Y) = f \otimes H \otimes (\operatorname{Ad} m)X \otimes (\operatorname{Ad} m)Y$$

for $m \in M$, $f \in \mathcal{R}$, $H \in \mathcal{U}(\mathfrak{a})$ and $X, Y \in \mathcal{U}(\mathfrak{k})$. If we consider $\mathcal{U}(\mathfrak{g})$ as a M-module under the adjoint action we have the following result.

PROPOSITION 2.5. *The linear map $\Pi : \mathcal{U}(\mathfrak{g}) \to \mathcal{R} \otimes \mathcal{C}$ is a M-module homomorphism.*

Proof. For every $m \in M$, $f \in \mathcal{R}$, $H \in \mathcal{U}(\mathfrak{a})$ and $X, Y \in \mathcal{U}(\mathfrak{k})$ we have

$$\begin{aligned}\Gamma_a(m(f \otimes H \otimes X \otimes Y)) &= f(a)((\operatorname{Ad} m)X)^a H(\operatorname{Ad} m)Y \\ &= (\operatorname{Ad} m)(f(a)X^a HY) = (\operatorname{Ad} m)\Gamma_a(f \otimes H \otimes X \otimes Y),\end{aligned}$$

which immediately implies our assertion. Q.E.D.

The filtration on the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ induces a filtration of $\mathcal{R} \otimes \mathcal{C}$ by

$$(\mathcal{R} \otimes \mathcal{C})_n = \mathcal{R} \otimes \mathcal{U}_n(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{m})} \mathcal{U}(\mathfrak{k})$$

for $n \in \mathbf{Z}$. We call it the $\mathfrak{a}$-*filtration* and the corresponding degree the $\mathfrak{a}$-*degree*. The linear map $\Pi : \mathcal{U}(\mathfrak{g}) \to \mathcal{R} \otimes \mathcal{C}$ is obviously compatible with filtrations on $\mathcal{U}(\mathfrak{g})$ and $\mathcal{R} \otimes \mathcal{C}$.

Denote by $\mathcal{M}$ the ideal in $\mathcal{R}$ generated by the functions $\alpha, \alpha \in \Delta$.

PROPOSITION 2.6. (i) *If $X \in \mathfrak{n}\mathcal{U}_n(\mathfrak{g})$, $n \in \mathbf{Z}_+$, then the $\mathfrak{a}$-degree of $\Pi(X)$ is less than or equal to n,*

(ii) *If $X \in \mathfrak{n}\mathcal{U}(\mathfrak{g})$, then $\Pi(X) \in \mathcal{M} \otimes \mathcal{C}$.*

Proof. We can suppose that $X = ZY$ where $Z \in \mathfrak{g}_\gamma$, $\gamma \in \Sigma^+$, and $Y \in \mathcal{U}_n(\mathfrak{g})$. Then by 2.2 we conclude that

$$X = \frac{\gamma(a)}{1 - \gamma(a)^2}\left(U^a Y - \gamma(a) YU - \gamma(a)[U, Y]\right).$$

That Y and $[U, Y]$ lie in $\mathfrak{U}_n(\mathfrak{g})$ immediately implies (i). The assertion (ii) is also obvious from the above relation. Q.E.D.

Example 2.7. Let $G = \mathrm{SL}(2, \mathbf{R})$ and choose

$$K = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \middle| \varphi \in \mathbf{R} \right\},$$

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbf{R} \right\}.$$

With this, the Cartan involution θ takes an element of $\mathfrak{g}$ to its negative transpose, and the single positive root α is given by

$$\alpha\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2.$$

As a basis of $\mathfrak{g}$ one has the elements

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is generated by the Casimir element

$$C = H^2 - H - Y \cdot \theta Y.$$

By 2.2, for $a \in A_{\mathrm{reg}}$, one has

$$Y = \frac{\alpha(a)}{1 - \alpha(a)^2}(X^a - \alpha(a)X).$$

After a short calculation it follows that

$$C = H^2 - \frac{1 + \alpha(a)^2}{1 - \alpha(a)^2}H + \frac{\alpha(a)^2}{\left(1 - \alpha(a)^2\right)^2}\left((X^a)^2 + X^2\right) - \frac{\alpha(a)\left(1 + \alpha(a)^2\right)}{\left(1 - \alpha(a)^2\right)^2}X^aX,$$

for $a \in A_{\mathrm{reg}}$; which implies

$$\Pi(C) = 1 \otimes H^2 \otimes 1 \otimes 1 - \frac{1+\alpha^2}{1-\alpha^2} \otimes H \otimes 1 \otimes 1$$

$$+ \frac{\alpha^2}{(1-\alpha^2)^2} \otimes 1 \otimes (X^2 \otimes 1 + 1 \otimes X^2) - \frac{\alpha(1+\alpha^2)}{(1-\alpha^2)^2} \otimes 1 \otimes X \otimes X.$$

3. The τ-radial components. Let (τ, E) be a finite-dimensional smooth representation of $K \times K$. A *τ-spherical function* on G is a smooth function $F: G \to E$ such that

$$F(k_1^{-1} x k_2) = \tau(k_1, k_2)^{-1} F(x)$$

for every $x \in G$, $k_1, k_2 \in K$. We denote by $C_\tau^\infty(G)$ the linear space of all τ-spherical functions on G.

For example, τ might be the representation of $K \times K$ on $E = \mathrm{Hom}_{\mathbf{C}}(U_2, U_1)$ arising from a pair of finite-dimensional smooth representations (τ_i, U_i), $i = 1, 2$, of K:

$$\tau(k_1, k_2)(T) = \tau_1(k_1) T \tau_2(k_2^{-1}), \qquad k_1, k_2 \in K,$$

for every $T \in E$. In this case a τ-spherical function F on G satisfies

$$F(k_1 x k_2) = \tau_1(k_1) F(x) \tau_2(k_2)$$

for every $k_1, k_2 \in K$, $x \in G$.

Let E^M be the linear space of M-invariants in E, with M inbedded diagonally into $K \times K$.

Recalling the Cartan decomposition of G we see that a τ-spherical function F is completely determined by its restriction $F | A_{\mathrm{reg}}$ to A_{reg}. Also, it is evident that the restriction $F | A_{\mathrm{reg}}$ is a smooth E^M-valued function on A_{reg}. Therefore, the restriction map $\mathrm{Res}: F \to F | A_{\mathrm{reg}}$ is a linear injection of the space $C_\tau^\infty(G)$ into the space $C^\infty(A_{\mathrm{reg}}; E^M)$ of smooth E^M-valued functions on A_{reg}.

The elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ act as left-invariant differential operators on G. Therefore the element $X \in \mathfrak{U}(\mathfrak{g})$ maps a τ-spherical function F on G into a smooth E-valued function $X \cdot F$ on G. Our main aim in this section is to find an expression for $\mathrm{Res}(X \cdot F)$, $X \in \mathfrak{U}(\mathfrak{g})$, $F \in C_\tau^\infty(G)$, in terms of $\mathrm{Res}\, F$. To accomplish that we shall use the results on the infinitesimal Cartan decomposition from Section 2.

Let $X \to X^t$ be the principal antiautomorphism of $\mathfrak{U}(\mathfrak{g})$, i.e. the anti-automorphism extending the map $X \to -X$ on $\mathfrak{g}$.

Let $\xi_\tau: \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{k}) \to \mathrm{Hom}_{\mathbf{C}}(E^M, E)$ be the linear map defined by

$$\xi_\tau(X \otimes Y)(T) = \tau(X \otimes Y^t)(T)$$

for $X, Y \in \mathfrak{U}(\mathfrak{k})$ and $T \in E^M$. Then

$$\xi_\tau(XZ \otimes Y) = \xi_\tau(X \otimes ZY)$$

for all $Z \in \mathfrak{U}(\mathfrak{m})$ and $X, Y \in \mathfrak{U}(\mathfrak{k})$. Therefore ξ_τ induces a linear map from $\mathfrak{U}(\mathfrak{k}) \otimes_{\mathfrak{U}(\mathfrak{m})} \mathfrak{U}(\mathfrak{k})$ into $\mathrm{Hom}_{\mathbf{C}}(E^M, E)$ which we denote also by ξ_τ.

Further, $\xi_\tau : \mathfrak{U}(\mathfrak{k}) \otimes_{\mathfrak{U}(\mathfrak{m})} \mathfrak{U}(\mathfrak{k}) \to \mathrm{Hom}_{\mathbf{C}}(E^M, E)$ defines a linear map $\eta_\tau = 1 \otimes 1 \otimes \xi_\tau$ from $\mathfrak{R} \otimes \mathfrak{E} = \mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \mathfrak{U}(\mathfrak{k}) \otimes_{\mathfrak{U}(\mathfrak{m})} \mathfrak{U}(\mathfrak{k})$ into $\mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \mathrm{Hom}_{\mathbf{C}}(E^M, E)$.

Finally, we put

$$\Pi_\tau = \eta_\tau \circ \Pi : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \mathrm{Hom}_{\mathbf{C}}(E^M, E).$$

Considering the elements of $\mathfrak{U}(\mathfrak{a})$ as invariant differential operators on A, the elements of $\mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \mathrm{Hom}_{\mathbf{C}}(E^M, E)$ can be viewed as differential operators on A_{reg} mapping smooth E^M-valued functions into smooth E-valued functions by the rule

$$(f \otimes H \otimes S)F = f \cdot H(SF)$$

where $f \in \mathfrak{R}$, $H \in \mathfrak{U}(\mathfrak{a})$, $S \in \mathrm{Hom}_{\mathbf{C}}(E^M, E)$ and $F \in C^\infty(A_{\mathrm{reg}}; E^M)$.

It follows that Π_τ is a linear map attaching to every left-invariant differential operator X on G a differential operator $\Pi_\tau(X)$ on A_{reg}. Because of the following result we call $\Pi_\tau(X)$ the *τ-radial component* of X.

THEOREM 3.1. *For every τ-spherical function F on G and $X \in \mathfrak{U}(\mathfrak{g})$ we have*

$$\mathrm{Res}(X \cdot F) = \Pi_\tau(X) \cdot \mathrm{Res}\, F.$$

Proof. Let $C^\infty(G; E)$ be the space of all smooth E-valued functions on G. Let L and R be the left and right regular representations of G on $C^\infty(G; E)$ respectively. By definition, $R_X F = X \cdot F$ for every $X \in \mathfrak{U}(\mathfrak{g})$ and $F \in C^\infty(G; E)$. Also

$$(R_X F)(1) = (L_{X'} F)(1)$$

for every $X \in \mathfrak{U}(\mathfrak{g})$ and $F \in C^\infty(G; E)$. This immediately implies

$$\begin{aligned}(X^g \cdot F)(g) &= (R_{g^{-1}} R_X R_g F)(g) = (R_X R_g F)(1)\\ &= (L_{X'} R_g F)(1) = (R_g L_{X'} F)(1) = (L_{X'} F)(g)\end{aligned}$$

for all $X \in \mathfrak{U}(\mathfrak{g})$, $g \in G$ and $F \in C^\infty(G; E)$.

If we take now $F \in C_\tau^\infty(G)$, $X, Y \in \mathfrak{U}(\mathfrak{k})$, $H \in \mathfrak{U}(\mathfrak{a})$ and $a \in A$ we get

$$\begin{aligned}(X^a \cdot H \cdot Y \cdot F)(a) &= (H \cdot L_{X'} \cdot Y \cdot F)(a) = \tau(X \otimes Y')(H \cdot F)(a)\\ &= \xi_\tau(X \otimes Y)(H \cdot F)(a).\end{aligned}$$

By 2.4 we have $X = \Gamma_a(\Pi(X))$ for every $a \in A_{\text{reg}}$ and $X \in \mathfrak{U}(\mathfrak{g})$. Therefore, by the above equality we see immediately that

$$(X \cdot F)(a) = (\Gamma_a(\Pi(X))F)(a) = (\Pi_\tau(X) \cdot \operatorname{Res} F)(a)$$

for every $X \in \mathfrak{U}(\mathfrak{g})$, $F \in C_\tau^\infty(G)$ and $a \in A_{\text{reg}}$, proving our assertion. Q.E.D.

Denote by $\mathfrak{U}(\mathfrak{g})^M$ the M-invariants in $\mathfrak{U}(\mathfrak{g})$ with respect to the adjoint action.

PROPOSITION 3.2. *The map* Π_τ *maps* $\mathfrak{U}(\mathfrak{g})^M$ *into* $\mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E^M)$.

Proof. This follows immediately from 2.5. Q.E.D.

Put

$$\mathfrak{D} = \mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E^M).$$

It is easy to check that $\mathfrak{D}$ is a subalgebra of the algebra of all differential operators on A_{reg} mapping $C^\infty(A_{\text{reg}}; E^M)$ into itself.

Let $\mathfrak{U}(\mathfrak{g})^K$ be the algebra of K-invariants in $\mathfrak{U}(\mathfrak{g})$ with respect to the adjoint action. It is obvious that for every $X \in \mathfrak{U}(\mathfrak{g})^K$ and $F \in C_\tau^\infty(G)$ the function $X \cdot F$ is again a τ-spherical function, i.e. $C_\tau^\infty(G)$ has a natural $\mathfrak{U}(\mathfrak{g})^K$-module structure. By 3.2 the map Π_τ maps $\mathfrak{U}(\mathfrak{g})^K$ into $\mathfrak{D}$. Moreover, we have the following result showing that the restriction map is compatible with the natural module structures.

THEOREM 3.3. *The map* $\Pi_\tau : \mathfrak{U}(\mathfrak{g})^K \to \mathfrak{D}$ *is an algebra homomorphism.*

Proof. Let $X, Y \in \mathfrak{U}(\mathfrak{g})^K$ and $F \in C_\tau^\infty(G)$. As we remarked before $Y \cdot F \in C_\tau^\infty(G)$. Therefore

$$\operatorname{Res}(X \cdot Y \cdot F) = \Pi_\tau(X) \cdot \operatorname{Res}(Y \cdot F) = \Pi_\tau(X) \cdot \Pi_\tau(Y) \cdot \operatorname{Res} F,$$

by 3.1. This proves that $\Pi_\tau(X \cdot Y) - \Pi_\tau(X)\Pi_\tau(Y)$ annihilates the restrictions of all τ-spherical functions on A_{reg}.

By the Cartan decomposition the map $(k_1, k_2, a) \to k_1 a k_2^{-1}$ is a differentiable map of $K \times K \times A^-$ onto an open dense submanifold G' of G. If we consider M as diagonally imbedded into $K \times K$, this map induces a diffeomorphism of $[(K \times K)/M] \times A^-$ onto G'.

Let φ be a compactly supported smooth E^M-valued function on A^-. By the above remark, setting

$$F(k_1^{-1} a k_2) = \tau(k_1, k_2)^{-1} \varphi(a)$$

for $k_1, k_2 \in K$ and $a \in A^-$, defines a smooth E-valued function F on G'. Putting F equal to zero outside G' we get a τ-spherical function F on G such that $F \mid A^- = \varphi$.

By the previous discussion, the differential operator $\Pi_\tau(X \cdot Y) - \Pi_\tau(X)\Pi_\tau(Y)$ annihilates φ. This implies that the support of $\Pi_\tau(X \cdot Y) - \Pi_\tau(X)\Pi_\tau(Y)$ is disjoint from A^-. Now the fact that its coefficients are rational functions of roots clearly implies that the differential operator $\Pi_\tau(X \cdot Y) - \Pi_\tau(X)\Pi_\tau(Y)$ is zero on A_{reg}. Q.E.D.

The space $\mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)$ is canonically filtered by the degree of the differential operators. The map $\eta_\tau : \mathfrak{R} \otimes \mathcal{C} \to \mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)$ is evidently compatible with the filtrations. Therefore the map $\Pi_\tau : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)$ as well is compatible with the filtrations. Moreover, we have the following direct consequences of 2.6.

PROPOSITION 3.4. (i) *If* $X \in \mathfrak{n}\mathfrak{U}_n(\mathfrak{g})$, $n \in \mathbf{Z}_+$, *then the degree of* $\Pi_\tau(X)$ *is less than or equal to* n.

(ii) *If* $X \in \mathfrak{n}\mathfrak{U}(\mathfrak{g})$, *then* $\Pi_\tau(X) \in \mathfrak{M} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)$.

It is evident that

$$\operatorname{Gr}\big(\mathfrak{R} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)\big) = \mathfrak{R} \otimes S(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E),$$

where the grading on $\mathfrak{R} \otimes S(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E)$ is inherited from the grading of $S(\mathfrak{a})$; moreover

$$\operatorname{Gr} \mathfrak{D} = \mathfrak{R} \otimes S(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E^M),$$

as a graded algebra. Therefore the map Π_τ defines a map

$$\operatorname{Gr} \Pi_\tau : S(\mathfrak{g}) \to \mathfrak{R} \otimes S(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E).$$

The homomorphism $\Pi_\tau : \mathfrak{U}(\mathfrak{g})^K \to \mathfrak{D}$ being compatible with the filtrations, $\operatorname{Gr} \Pi_\tau : \operatorname{Gr} \mathfrak{U}(\mathfrak{g})^K \to \operatorname{Gr} \mathfrak{D}$ is an algebra homomorphism.

COROLLARY 3.5. *We have*

$$(\operatorname{Gr} \Pi_\tau)X = 1 \otimes (\operatorname{Gr} \chi)X \otimes 1_{E^M}$$

for every $X \in S(\mathfrak{g})$.

Proof. Let $Y \in \mathfrak{U}_n(\mathfrak{g})$. By the definition of χ we have

$$Y - \chi(Y) \in \mathfrak{n}\mathfrak{U}_{n-1}(\mathfrak{g}) + \mathfrak{U}_{n-1}(\mathfrak{g})\mathfrak{k}.$$

From 3.4 we immediately conclude that the degree of $\Pi_\tau(Y - \chi(Y))$ is less than or equal to $n - 1$, which implies our assertion at once. Q.E.D.

Finally $\mathfrak{Z}(\mathfrak{l})$ is contained in $\mathfrak{U}(\mathfrak{g})^M$ which, by 3.2, implies that Π_τ maps $\mathfrak{Z}(\mathfrak{l})$ into $\mathfrak{D}$.

PROPOSITION 3.6. *The map* $\Pi_\tau : \mathfrak{Z}(\mathfrak{l}) \to \mathfrak{D}$ *is an algebra homomorphism.*

Proof. Obviously $\mathfrak{Z}(\mathfrak{l}) = \mathfrak{Z}(\mathfrak{m})\mathfrak{U}(\mathfrak{a})$. If $X \in \mathfrak{Z}(\mathfrak{m})$ and $H \in \mathfrak{U}(\mathfrak{a})$ we have

$$\Pi(X \cdot H) = 1 \otimes H \otimes X \otimes 1$$

and

$$\Pi_\tau(X \cdot H)F = \tau(X \otimes 1)(H \cdot F)$$

for $F \in C^\infty(A_{\text{reg}}; E^M)$. This immedately implies our assertion. Q.E.D.

Example 3.7. The τ-radial components of the Casimir element for $SL(2,\mathbf{R})$. We follow the notation from 2.7. Let

$$\tau_n\begin{pmatrix} \cos\varphi \sin\varphi \\ -\sin\varphi\cos\varphi \end{pmatrix} = e^{in\varphi}, \qquad \varphi \in \mathbf{R},$$

for $n \in \mathbf{Z}$. Then an irreducible smooth representation τ of $K \times K$ has the form

$$\tau(k_1,k_2) = \tau_n(k_1)\tau_m(k_2), \qquad k_1,k_2 \in K,$$

for some $n,m \in \mathbf{Z}$. Therefore, by 2.7, the τ-radial component of the Casimir element C is given by

$$\Pi_\tau(C) = H^2 - \frac{1+\alpha^2}{1-a^2}H - (n^2+m^2)\frac{\alpha^2}{(1-\alpha^2)^2} - nm\frac{\alpha(1+\alpha^2)}{(1-\alpha^2)^2}.$$

If we identify A with $\mathbf{R}_+^*$ via the map $\alpha: A \to R_+^*$, the differential operator H on A corresponds to the differential operator $x(d/dx)$ on $\mathbf{R}_+^*$. Under this identification $\Pi_\tau(C)$ defines a holomorphic differential operator

$$\Pi_\tau(C) = \left(z\frac{d}{dz}\right)^2 - \frac{1+z^2}{1-z^2}z\frac{d}{dz} - (n^2+m^2)\frac{z^2}{(1-z^2)^2} - nm\frac{z(1+z^2)}{(1-z^2)^2}$$

on $\mathbf{C}\setminus\{1,-1\}$. It is easy to check that it has regular singularities at $0, 1, -1$ and ∞.

Let F be a τ-spherical function on G annihilated by a nonzero ideal I in $\mathfrak{Z}(\mathfrak{g}) = \mathbf{C}[C]$. The ideal I being generated by a polynomial $P(C)$ in the Casimir element C, this is equivalent to the differential equation $P(C) \cdot F = 0$ on G. Now, by 3.1 and 3.3, it follows that the restriction $\operatorname{Res} F$ of F to $A_{\text{reg}} = \mathbf{R}_+^* \setminus \{1\}$ satisfies

$$P(\Pi_\tau(C)) \cdot \operatorname{Res} F = 0,$$

and, by the above remark, this differential equation has regular singularities at $0, 1, -1$ and ∞. Therefore, using classical results on such equations [7, Ch. 4], we can find the expansion of $\operatorname{Res} F$ on $A^- = (0,1)$.

As we remarked in the Introduction, this classical remark suggests the approach taken in the next sections, to the analysis of τ-spherical functions in general, as a natural one.

4. Differential equations satisfied by spherical functions. The space $C_\tau^\infty(G)$ of all τ-spherical functions on G has a natural $\mathfrak{Z}(\mathfrak{g})$-module structure. We say that a τ-spherical function F is $\mathfrak{Z}(\mathfrak{g})$-*finite* if it generates a finite-dimensional $\mathfrak{Z}(\mathfrak{g})$-invariant subspace of $C_\tau^\infty(G)$. Let $A_\tau(G)$ be the subspace of all $\mathfrak{Z}(\mathfrak{g})$-finite τ-spherical functions on G. A simple argument using the regularity theorem for elliptic operators proves that the elements of $A_\tau(G)$ are in fact real analytic functions on G (see for example [20, p. 134]).

A τ-spherical function F on G is $\mathfrak{Z}(\mathfrak{g})$-finite if and only if its annihilator I in $\mathfrak{Z}(\mathfrak{g})$ is an ideal of finite codimension. For an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ we denote by $A_\tau(G; I)$ the subspace of all $\mathfrak{Z}(\mathfrak{g})$-finite τ-spherical functions on G annihilated by I.

All elements $F \in A_\tau(G; I)$ satisfy the differential equations

$$Z \cdot F = 0, \qquad Z \in I,$$

on G. Therefore, by 3.1, their restrictions to A_{reg} satisfy

$$\Pi_\tau(Z) \cdot \operatorname{Res} F = 0, \qquad Z \in I,$$

on A_{reg}. The main point in this study of $\mathfrak{Z}(\mathfrak{g})$-finite τ-spherical functions on G is that this system of differential equations, because of the results of Section 3, has a number of nice properties.

The following result is crucial for all that follows.

Let I be an ideal in $\mathfrak{Z}(\mathfrak{g})$. We denote by $\mathfrak{D}_I$ the left ideal in $\mathfrak{D}$ generated by $\Pi_\tau(I)$.

THEOREM 4.1. *If the ideal I has finite codimension in $\mathfrak{Z}(\mathfrak{g})$, the $\mathfrak{R}$-module $\mathfrak{D}/\mathfrak{D}_I$ is finitely generated.*

To prove 4.1 we need first a simple lemma.

LEMMA 4.2. *Let I be an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. Then* $\operatorname{Gr}\chi(\operatorname{Gr} I)$ *generates an ideal of finite codimension in $S(\mathfrak{a})$.*

Proof. If we equip I and $\mathfrak{Z}(\mathfrak{g})/I$ with induced and quotient filtrations respectively, the exact sequence

$$0 \to I \to \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{g})/I \to 0$$

gives the exact sequence

$$0 \to \operatorname{Gr} I \to \operatorname{Gr} \mathfrak{Z}(\mathfrak{g}) \to \operatorname{Gr}(\mathfrak{Z}(\mathfrak{g})/I) \to 0$$

by [1, Chapter III, §2, no. 4, Proposition 2]. The space $\mathfrak{Z}(\mathfrak{g})/I$ being finite-dimensional, $\mathrm{Gr}(\mathfrak{Z}(\mathfrak{g})/I)$ is finite-dimensional too, which implies that $\mathrm{Gr}\, I$ is an ideal of finite codimension in $\mathrm{Gr}\,\mathfrak{Z}(\mathfrak{g}) = I(\mathfrak{g})$. By 1.3, $S(\mathfrak{a})$ is finitely generated as a module over $\mathrm{Gr}\,\chi(I(\mathfrak{g}))$, which clearly implies our assertion. Q.E.D.

Now we can prove 4.1. If we equip $\mathfrak{D}_I$ and $\mathfrak{D}/\mathfrak{D}_I$ with the induced and quotient filtrations respectively, by [1, Chapter III, §2, no. 4, Proposition 2] we get an exact sequence

$$0 \to \mathrm{Gr}\,\mathfrak{D}_I \to \mathrm{Gr}\,\mathfrak{D} \to \mathrm{Gr}(\mathfrak{D}/\mathfrak{D}_I) \to 0$$

of $\mathfrak{R}$-modules. By [1, Chapter III, §2, no. 9, Proposition 12] it is enough to show that the $\mathfrak{R}$-module $\mathrm{Gr}\,\mathfrak{D}/\mathrm{Gr}\,\mathfrak{D}_I$ is finitely generated.

Let $\Pi_\tau(I)$ be equipped with the induced filtration. Then $\mathrm{Gr}\,\mathfrak{D}_I$ is the left ideal in $\mathrm{Gr}\,\mathfrak{D}$ generated by $\mathrm{Gr}(\Pi_\tau(I))$. The homomorphism $\Pi_\tau : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{D}$ being compatible with the filtrations, we have

$$(\mathrm{Gr}\,\Pi_\tau)(\mathrm{Gr}\,I) \subset \mathrm{Gr}(\Pi_\tau(I)).$$

Let $\mathfrak{J}$ be the left ideal in $\mathrm{Gr}\,\mathfrak{D}$ generated by $(\mathrm{Gr}\,\Pi_\tau)(\mathrm{Gr}\,I)$. Then obviously $\mathrm{Gr}\,\mathfrak{D}_I$ contains $\mathfrak{J}$. But, by 3.5, we have

$$\mathfrak{J} = \mathfrak{R} \otimes S(\mathfrak{a}) \cdot \mathrm{Gr}\,\chi(\mathrm{Gr}\,I) \otimes \mathrm{Hom}_{\mathbf{C}}(E^M, E^M),$$

and by 4.2 we conclude that $\mathrm{Gr}\,\mathfrak{D}/\mathfrak{J}$ is a finitely generated $\mathfrak{R}$-module. This immediately proves 4.1, $\mathrm{Gr}\,\mathfrak{D}/\mathrm{Gr}\,\mathfrak{D}_I$ being a quotient of $\mathrm{Gr}\,\mathfrak{D}/\mathfrak{J}$.

Let $A_\Delta = \{a \in A \mid \alpha(a) = 1,\ \forall \alpha \in \Delta\}$. Obviously, A_Δ is a maximal θ-stable closed vector subgroup of the center Z_G of G, and

$$A = (A \cap G_1) \times A_\Delta.$$

Let Λ be a finite set of characters $\lambda : A \to \mathbf{R}_+^*$ such that

(i) $\Delta \subset \Lambda$,

(ii) the characters $\lambda \in \Lambda \backslash \Delta$ are trivial on $A \cap G_1$,

(iii) the differentials of $\lambda \in \Lambda$ form a basis of the linear dual of $\mathfrak{a}$

Let $(H_\lambda;\ \lambda \in \Lambda)$ be the corresponding dual basis of $\mathfrak{a}$, i.e. such that

$$d\lambda(H_\mu) = \delta_{\lambda\mu} \qquad \text{if} \quad \lambda, \mu \in \Lambda.$$

Let F be a $\mathfrak{Z}(\mathfrak{g})$-finite τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. By 4.1 there exist $D_1 = 1, D_2, \ldots, D_p \in \mathfrak{D}$ such that their images in $\mathfrak{D}/\mathfrak{D}_I$ generate it as a $\mathfrak{R}$-module. Then there exist functions $g_{\lambda ij} \in \mathfrak{R}$, $\lambda \in \Lambda$, $1 \leqslant i, j \leqslant p$; such that

$$H_\lambda D_i - \sum_{j=1}^{p} g_{\lambda ij} D_j \in \mathfrak{D}_I$$

for every $\lambda \in \Lambda$, $1 \leqslant i \leqslant p$.

By its definition $\mathfrak{D}$, annihilates $\operatorname{Res} F$. Therefore, if we put

$$\Phi = \begin{bmatrix} D_1 \operatorname{Res} F \\ \vdots \\ D_p \operatorname{Res} F \end{bmatrix},$$

and

$$G_\lambda = \begin{bmatrix} g_{\lambda 11} & g_{\lambda 12} & \cdots & g_{\lambda 1p} \\ \vdots & & & \vdots \\ g_{\lambda p1} & g_{\lambda p2} & \cdots & g_{\lambda pp} \end{bmatrix} \otimes 1_{E^M}$$

for $\lambda \in \Lambda$, we see that the real analytic $(E^M)^p$-valued function Φ on A_{reg} satisfies the system of differential equations

$$H_\lambda \Phi = G_\lambda \Phi, \qquad \lambda \in \Lambda.$$

Let $\underline{\lambda}: A \to \mathbf{C}^\Lambda$ be the imbedding of A into $\mathbf{C}^\Lambda$ defined by

$$\underline{\lambda}(a) = (\lambda(a); \lambda \in \Lambda)$$

for every $a \in A$.

The functions from $\mathfrak{R}$ extend to rational functions on $\mathbf{C}^\Lambda$ holomorphic on the complement of the union Y of the hypersurfaces

$$Y_\gamma = \{ z \in \mathbf{C}^\Lambda \mid \gamma(z)^2 = 1 \},$$

for $\gamma \in \Sigma$.

The differential operators H_λ, $\lambda \in \Lambda$, correspond naturally to the holomorphic differential operators $z_\lambda \partial_\lambda$, $\lambda \in \Lambda$, on $\mathbf{C}^\Lambda$. Therefore, the elements of $\mathfrak{D}$ correspond to holomorphic differential operators on the complement of Y in $\mathbf{C}^\Lambda$.

The function F, being real analytic on A, extends to a holomorphic function on an open set Ω in $(\mathbf{C}^*)^\Lambda$ containing A, with values in E^M, which we denote also by F.

Therefore the corresponding function Φ extends to a holomorphic function on $\Omega \setminus Y$ with values in $(E^M)^p$, which satisfies the system of first order differential equations

$$z_\lambda \partial_\lambda \Phi = G_\lambda \Phi, \qquad \lambda \in \Lambda,$$

on $\Omega \setminus Y$, where G_λ, $\lambda \in \Lambda$, are holomorphic matrix valued functions on $\mathbf{C}^\Lambda \setminus Y$.

We shall see in the following sections that the study of the above system of differential equations will enable us to describe the asymptotic behavior of τ-spherical functions inside the negative Weyl chamber A^- (Section 5) and "along the walls" of A^- (Section 6).

5. Asymptotic behavior of τ-spherical functions on A^-. In this section we shall study the asymptotic behavior of a τ-spherical function F from $A_\tau(G;I)$ inside the negative Weyl chamber A^- using the differential equations studied in the last section. By the procedure described there we associate to F a holomorphic function Φ on an open set $\Omega\setminus Y$ in $\mathbf{C}^\Lambda\setminus Y$, with values in $(E^M)^p$, which satisfies the system of first order differential equations

$$z_\lambda \partial_\lambda \Phi = G_\lambda \Phi, \qquad \lambda \in \Lambda,$$

on $\Omega\setminus Y$; where G_λ, $\lambda\in\Lambda$, are holomorphic matrix valued functions on $\mathbf{C}^\Lambda\setminus Y$.

Let D be the unit disc in $\mathbf{C}$ and $D^* = D\setminus\{0\}$. Then obviously

$$(D^\Delta \times \mathbf{C}^{\Lambda\setminus\Delta}) \cap Y = \emptyset,$$

that implies that the functions G_λ, $\lambda\in\Lambda$, are holomorphic on $D^\Delta\times\mathbf{C}^{\Lambda\setminus\Delta}$. It follows that the function Φ satisfies on $\Omega\cap(D^\Delta\times\mathbf{C}^{\Lambda\setminus\Delta})$ the system of differential equations of the type considered in the Appendix. By A.1.2 we know that Φ extends to a multivalued solution of this system on $(D^*)^\Delta\times(\mathbf{C}^*)^{\Lambda\setminus\Delta}$; and by Deligne's result [A.1.6] we know that Φ has the unique canonical form

$$\Phi = \sum \Phi_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}} \log^{\mathbf{m}} z$$

on $(D^*)^\Delta\times(\mathbf{C}^*)^{\Lambda\setminus\Delta}$.

For $\mathbf{s}\in\mathbf{C}^\Lambda$ and $\mathbf{m}\in\mathbf{Z}_+^\Lambda$ we put

$$(\lambda^{\mathbf{s}})(a) = \prod_{\lambda\in\Lambda} \lambda(a)^{s_\lambda}$$

and

$$(\log^{\mathbf{m}}\lambda)(a) = \prod_{\lambda\in\Lambda} (\log\lambda(a))^{m_\lambda}$$

for $a\in A$. Using this notation, the fact that F is the first component of Φ gives us the existence part of the following preliminary form of the main result of this section. The uniqueness follows from A.1.7.

LEMMA 5.1. *There exist*

(i) *a finite set S of mutually integrally inequivalent elements of $\mathbf{C}^\Lambda$, and*

(ii) *for each $\mathbf{s}\in S$ a finite set $F_{\mathbf{s},\mathbf{m}}$, $\mathbf{m}\in\mathbf{Z}_+^\Lambda$, of nontrivial holomorphic E^M-valued functions on $D^\Delta\times\mathbf{C}^{\Lambda\setminus\Delta}$ such that on each of the coordinate hyperplanes at least one of them is not identically zero, such that*

$$F = \sum_{\mathbf{s},\mathbf{m}} F_{\mathbf{s},\mathbf{m}} \lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda$$

on A^-.

This S and the $F_{\mathbf{s},\mathbf{m}}$ are unique.

Let

$$F_{\mathbf{s},\mathbf{m}} = \sum_{\mathbf{k}} c_{\mathbf{s}+\mathbf{k},\mathbf{m}} \lambda^{\mathbf{k}}$$

be the power series expansion of $F_{\mathbf{s},\mathbf{m}}$ on A^-. Then

$$F = \sum_{\mathbf{l},\mathbf{m}} c_{\mathbf{l},\mathbf{m}} \lambda^{\mathbf{l}} \log^{\mathbf{m}} \lambda$$

on A^- and such an expansion is unique. If $c_{\mathbf{l},\mathbf{m}} \neq 0$ for some $\mathbf{m} \in \mathbf{Z}_+^{\Lambda}$ we say that $\mathbf{l}$ is an *exponent* of F.

We say that $\mathbf{t}, \mathbf{s} \in \mathbf{C}^{\Lambda}$ are Δ-integrally equivalent if $\mathbf{t} - \mathbf{s} \in \mathbf{Z}^{\Delta}$. Also we put

$$\mathbf{t} \leqslant_{\Delta} \mathbf{s} \qquad \text{if} \quad \mathbf{s} - \mathbf{t} \in \mathbf{Z}_+^{\Delta} .$$

We call this relation the Δ-*order* on $\mathbf{C}^{\Lambda}$.

The minimal elements of the set of all exponents of F with respect to the Δ-order are called the *leading exponents* of the τ-spherical function F.

If $\mathbf{t} \in \mathbf{C}^{\Lambda}$ is a leading exponent of F we say that the corresponding character $\lambda^{\mathbf{t}}: A \to \mathbf{C}^*$ of A is a *leading character* and

$$F_{\mathbf{t}} = \sum_{\mathbf{m}} c_{\mathbf{t},\mathbf{m}} \lambda^{\mathbf{t}} \log^{\mathbf{m}} \lambda$$

is a *leading term* of the τ-spherical function F.

The fact that the τ-spherical function F is annihilated by the ideal I imposes a severe restriction on the leading terms and leading characters of F. They are consequences of the following theorem.

THEOREM 5.2. *Let $F \in A_{\tau}(G; I)$. Then all leading terms of F are annihilated by $\Pi_{\tau}(\sigma(I))$.*

Firstly we need a simple fact which follows by direct computation. Let $\mathbf{e}_{\mu} \in \mathbf{Z}_+^{\Lambda}$ by such that all its coordinates are zero except the μ-th coordinate which is equal to one.

LEMMA 5.3. *Let $\mu \in \Lambda$, $\mathbf{l} \in \mathbf{C}^{\Lambda}$ and $\mathbf{m} \in \mathbf{Z}_+^{\Lambda}$. Then*

$$H_{\mu} \lambda^{\mathbf{l}} \log^{\mathbf{m}} \lambda = l_{\mu} \lambda^{\mathbf{l}} \log^{\mathbf{m}} \lambda + m_{\mu} \lambda^{\mathbf{l}} \log^{\mathbf{m} - \mathbf{e}_{\mu}} \lambda.$$

To prove 5.2 we observe first that by 3.4 for every $Z \in \mathfrak{Z}(\mathfrak{g})$ we have

$$\Pi_{\tau}(Z) - \Pi_{\tau}(\sigma(Z)) \in \mathfrak{M} \otimes \mathfrak{U}(\mathfrak{a}) \otimes \operatorname{Hom}_{\mathbf{C}}(E^M, E^M).$$

By 5.3 this implies that for $Z \in \mathfrak{Z}(\mathfrak{g})$ we have

$$\Pi_{\tau}(\sigma(Z)) F_{\mathbf{t}} \equiv \Pi_{\tau}(Z) F_{\mathbf{t}}$$

modulo terms involving $\lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda$ where $\mathbf{t} \leqslant_{\Delta} \mathbf{s}$, $\mathbf{s} \neq \mathbf{t}$. Also, the fact that $\mathbf{t}$ is a leading exponent coupled with 5.3 implies that

$$\Pi_\tau(Z)F_{\mathbf{t}} \equiv \Pi_\tau(Z)F$$

modulo terms involving $\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$, $\mathbf{s} \neq \mathbf{t}$. This immediately implies that

$$\Pi_\tau(\sigma(Z))F_{\mathbf{t}} = 0$$

for all $Z \in I$, that proves 5.2.

Let U be a finite-dimensional A-module. For a character $\omega : A \to \mathbf{C}^*$ we denote by U_ω the submodule of U defined by

$$U_\omega = \{ u \in U \,|\, (a - \omega(a))^n u = 0 \text{ for some } n \in \mathbf{N} \text{ and all } a \in A \}.$$

The submodule U_ω is called the *ω-component* of U. If $U_\omega \neq \{0\}$, ω is called an *A-weight* of U. Of course, U is the sum of its various ω-components.

The ideal I being of finite codimension in $\mathfrak{Z}(\mathfrak{g})$, by 1.2 the ideal $\mathfrak{Z}(\mathfrak{l})\sigma(I)$ generated by $\sigma(I)$ in $\mathfrak{Z}(\mathfrak{l})$ is of finite codimension. Therefore, $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$ is a finite-dimensional $\mathfrak{a}$-module, and because A is simply-connected it has a natural structure as A-module.

If ω is an A-weight of $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$ we say that it *lies over I*.

PROPOSITION 5.4. *If $F \in A_\tau(G; I)$, all leading characters of F lie over I*

Proof. Let $\mathbf{t}$ be a leading exponent of F. By 5.2

$$\Pi_\tau(\sigma(I))F_{\mathbf{t}} = 0;$$

and by 3.6 we see that the annihilator J of $F_{\mathbf{t}}$ in $\mathfrak{Z}(\mathfrak{l})$ is an ideal containing $\mathfrak{Z}(\mathfrak{l})\sigma(I)$. Therefore $\mathfrak{Z}(\mathfrak{l})/J$ is a quotient of the finite-dimensional $\mathfrak{a}$-module $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$. Obviously, an A-weight of $\mathfrak{Z}(\mathfrak{l})/J$ is also an A-weight of $\mathfrak{Z}(\mathfrak{l})/\mathfrak{Z}(\mathfrak{l})\sigma(I)$, i.e. it lies over I.

The differential $d\lambda^{\mathbf{t}}$ of the character $\lambda^{\mathbf{t}} : A \to \mathbf{C}^*$ is a $\mathbf{C}$-linear form on $\mathfrak{a}$. By 5.3 we have

$$\left(H - (d\lambda^{\mathbf{t}})(H)\right)^n \lambda^{\mathbf{t}}\log^{\mathbf{m}}\lambda = 0$$

for all $H \in \mathfrak{a}$ and sufficiently large $n \in \mathbf{N}$. This implies that for sufficiently large $n \in \mathbf{N}$

$$\left(H - (d\lambda^{\mathbf{t}})(H)\right)^n F_{\mathbf{t}} = 0,$$

for all $H \in \mathfrak{a}$; i.e.

$$\left(H - (d\lambda^{\mathbf{t}})(H)\right)^n \in J$$

for all $H \in \mathfrak{a}$. This obviously implies that $\lambda^{\mathbf{t}}$ is the only A-weight of $\mathfrak{Z}(\mathfrak{l})/J$, and by the above remark it lies over I. Q.E.D.

Remark 5.5. By 5.4 there are only finitely many possible leading characters for all spherical functions on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. This finite set is independent of the representation τ of $K \times K$.

Now we can formulate the final version of the main result of this section. It is a slight improvement of the result of Harish-Chandra [13], [22, Vol. II, 9.1.1.1].

Put $\underline{\alpha}: A \to \mathbf{C}^{\Delta}$ defined by

$$\underline{\alpha}(a) = (\alpha(a); \alpha \in \Delta).$$

THEOREM 5.6. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. Then there exist*

(i) *a finite set S_Δ of mutually Δ-integrally inequivalent elements of $\mathbf{C}^{\Lambda}$,*

(ii) *for each $\mathbf{s} \in S_\Delta$ a finite set $F^{\Delta}_{\mathbf{s},\mathbf{m}}, \mathbf{m} \in \mathbf{Z}^{\Lambda}_{+}$, of nontrivial holomorphic E^M-valued functions on D^{Δ} such that on each of the coordinate hyperplanes at least one of them is not identically zero, such that*

$$F = \sum_{\mathbf{s},\mathbf{m}} (F^{\Delta}_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on A^{-}.

This S_Δ and the $F^{\Delta}_{\mathbf{s},\mathbf{m}}$ are unique.

Proof. By the definition, all exponents of F are contained in the union

$$\cup(\mathbf{t} + \mathbf{Z}^{\Delta}_{+})$$

where $\mathbf{t}$ varies in the set of all leading exponents of F. By 5.5 there are just finitely many terms in this union, which implies that the holomorphic functions $F_{\mathbf{s},\mathbf{m}}$ appearing in the expression for F in 5.1 depend polynomially on the variables z_λ, $\lambda \in \Lambda \backslash \Delta$; i.e.

$$F_{\mathbf{s},\mathbf{m}} = \sum_{\mathbf{k}} G_{\mathbf{s}+\mathbf{k},\mathbf{m}} z^{\mathbf{k}}$$

where $\mathbf{k} \in \mathbf{Z}^{\Lambda \backslash \Delta}_{+}$, the E^M-valued functions $G_{\mathbf{t},\mathbf{m}}$ are holomorphic on D^{Δ} and the sum is finite. This proves the existence of the above expansion. The uniqueness follows from the uniqueness of 5.1. Q.E.D.

Remark 5.7. There is a simple relation between the leading exponents of F and the elements of S_Δ. To each class of Δ-integrally equivalent leading exponents we associate an element $s \in S_\Delta$, whose coordinates are the minima of the corresponding coordinates of the leading exponents in this class.

Example 5.8. As we remarked in the Introduction, the functions $F^{\Delta}_{\mathbf{s},\mathbf{m}}$, in general, have singularities on the boundary of D^{Δ}, so that the expansion from 5.6 does not hold for regions of A larger than A^{-}. The simplest example of this general phenomenon is in the case of $SL(2, \mathbf{R})$. In the following, we use the notation and results of 2.7 and 3.7.

Assume that F is a nonzero τ-spherical function on $SL(2, \mathbf{R})$ satisfying

$$C \cdot F = \frac{\lambda^2 - 1}{4} F,$$

where $\lambda \in \mathbf{C} \backslash \mathbf{Z}$. Then the restriction of F to A_{reg} satisfies

$$\Pi_\tau(C) \cdot \operatorname{Res} F = \frac{\lambda^2 - 1}{4} \operatorname{Res} F$$

and the indicial equation at 0 is

$$s^2 - s = \frac{\lambda^2 - 1}{4},$$

i.e. possible leading exponents are $\frac{1}{2}(1 + \lambda)$ and $\frac{1}{2}(1 - \lambda)$. Therefore on A^- we have

$$F = F_+ \alpha^{1/2(1+\lambda)} + F_- \alpha^{1/2(1-\lambda)},$$

and a more detailed inspection shows that F_+ and F_- are nonzero. The roots of the indicial equation at 1 are $\pm(n + m)$; therefore F is, up to a constant factor, the only solution of the above equation on A regular at 1. This evidently implies that both F_+ and F_- have singularities at this point.

6. Asymptotic behavior of τ-spherical functions along the walls of A^-. Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. Theorem 5.6 gives an expression for F on $K \cdot A^- \cdot K$ which describes the behavior of $F(x)$ as x passes off to infinity in certain ways. For example, if $a \in A^-$, then it yields a perfectly satisfactory description of the asymptotic behavior of $F(a^t)$ as $t > 0$ goes to infinity. On the contrary, if $a \in A$ is a boundary point of A^-, it doesn't give us any information about the behavior of $F(a^t)$. Roughly speaking, 5.6 describes the behavior of $F(a)$ completely if $a \in A^-$ "stays far away" from the boundary of A^- as it goes to infinity. Unfortunately, for many purposes what is needed is a description of the behavior of $F(x)$ as x goes to infinity more or less arbitrarily; and in such cases 5.6 is clearly insufficient. Therefore, we must give expressions for F on the closure of the Weyl chamber A^-. In a perhaps ideal situation we would be able to replace the open polydisc D^Δ in 5.6 by its closure; this would certainly yield the uniformity we require. However, even in the case of $SL(2, \mathbf{R})$, as we remarked in 5.8, the functions $F^\Delta_{\mathbf{s},\mathbf{m}}$ are not defined in general on $\mathrm{Cl}(A^-)$. What we are forced to do, is to cover $\mathrm{Cl}(A^-)$ by (overlapping) subsets indexed by $\Theta \subseteq \Delta$ and give for each Θ a grouping of the terms $F^\Delta_{\mathbf{s},\mathbf{m}} \lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda$ with the property that the sum of terms in a group is defined on the corresponding subset. In some sense, by this grouping procedure we cancel out the singularities of $F^\Delta_{\mathbf{s},\mathbf{m}}$ along the wall of A^- corresponding to Θ. What we will have, then, is not a single expansion good on all of $\mathrm{Cl}(A^-)$, but on each of the elements of the cover a different expansion. However, all of these expansions, as it will be obvious from the construction, are completely determined by the expansion inside A^- given by 5.6, which corresponds to $\Theta = \emptyset$. For SL(2, R), for example, the cover has only two elements: one is A^- itself and the other is $\mathrm{Cl}(A^-)$. On A^- there is no grouping of terms necessary, but on $\mathrm{Cl}(A^-)$ it is only the sum of all the $F^{\Delta}_{\mathbf{s},\mathbf{m}}\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$ which we know to be defined.

Let Θ be a subset of the set of simple roots Δ. Put

$$A_{\Theta}^- = \{a \in A \mid \alpha(a) = 1, \alpha \in \Theta; \alpha(a) < 1, \alpha \in \Delta \backslash \Theta\}.$$

We call A_{Θ}^- the *wall* of A^- determined by Θ. We have

$$A_{\emptyset}^- = A^-;$$

the walls A_{Θ}^-, $\Theta \subseteq \Delta$, are mutually disjoint; and

$$\mathrm{Cl}(A^-) = \bigcup_{\Theta \subseteq \Delta} A_{\Theta}^- .$$

Now we can define the cover of $\mathrm{Cl}(A^-)$ we alluded to above. We put

$$A^-(\Theta) = \{a \in A \mid \alpha(a) \leqslant 1, \alpha \in \Theta; \alpha(a) < 1, a \in \Delta \backslash \Theta\}.$$

Evidently $A^-(\Theta)$ is a neighborhood of the wall A_{Θ}^- in $\mathrm{Cl}(A^-)$, and

$$A^-(\Theta) = \bigcup_{\Psi \subseteq \Theta} A_{\Psi}^- ,$$

in particular

$$A^-(\Delta) = \mathrm{Cl}(A^-)$$

Now we shall describe the grouping procedure. First, let's recall the statement of 5.6. If F is a τ-spherical function on G annihilated by I, there exist

(i) a finite set S_{Δ} of mutually Δ-integrally inequivalent elements of $\mathbf{C}^{\Lambda}$;

(ii) for each $\mathbf{s} \in S_{\Delta}$ a finite set $F^{\Delta}_{\mathbf{s},\mathbf{m}}$, $\mathbf{m} \in \mathbf{Z}^{\Lambda}_{+}$, of nontrivial holomorphic functions on D^{Δ}, such that on each of the coordinate hyperplanes at least one of them is not identically zero, with

$$F = \sum_{\mathbf{s},\mathbf{m}} (F^{\Delta}_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on A^-; and S_{Δ} and $F^{\Delta}_{\mathbf{s},\mathbf{m}}$ are unique.

In the following, we shall consider $\mathbf{C}^{\Lambda\backslash\Theta}$ as naturally imbedded into $\mathbf{C}^{\Lambda}$. Generalizing the definitions from the last section we say that $\mathbf{t}, \mathbf{s} \in \mathbf{C}^{\Lambda\backslash\Theta}$ are $(\Delta\backslash\Theta)$-*integrally equivalent* if $\mathbf{t} - \mathbf{s} \in \mathbf{Z}^{\Lambda\backslash\Theta}$, and we put

$$\mathbf{t} \leqslant_{\Delta\backslash\Theta} \mathbf{s} \qquad \text{if} \quad \mathbf{s} - \mathbf{t} \in \mathbf{Z}^{\Delta\backslash\Theta}_{+};$$

we call this relation the $(\Delta\backslash\Theta)$-*order* on $\mathbf{C}^{\Lambda\backslash\Theta}$. Let $\mathrm{pr}_{\Lambda\backslash\Theta}$ be the projection map from $\mathbf{C}^{\Lambda}$ onto $\mathbf{C}^{\Lambda\backslash\Theta}$.

The set $\mathrm{pr}_{\Delta\setminus\Theta}(S_\Delta)$ splits into a finite number of classes of $(\Delta\setminus\Theta)$-integrally equivalent elements. To each equivalence class we associate an element $\mathbf{s} \in \mathbf{C}^{\Delta\setminus\Theta}$ whose coordinates are the minima of the corresponding coordinates of the elements in the equivalence class. We denote the set of all such $\mathbf{s} \in \mathbf{C}^{\Delta\setminus\Theta}$ by $S_{\Delta\setminus\Theta}$. Evidently, the elements of $S_{\Delta\setminus\Theta}$ are mutually $(\Delta\setminus\Theta)$-integrally inequivalent.

Let $\mathbf{s} \in S_{\Delta\setminus\Theta}$ and $\mathbf{m} \in \mathbf{Z}_+^{\Delta\setminus\Theta}$. We put

$$F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta} = \sum F_{\mathbf{t},\mathbf{m}+\mathbf{n}}^{\Delta} z^{\mathbf{t}-\mathbf{s}} \log^{\mathbf{n}} z$$

where the sum is taken over all $\mathbf{t} \in S_\Delta$ such that $\mathrm{pr}_{\Delta\setminus\Theta}(\mathbf{t})$ is $(\Delta\setminus\Theta)$-integrally equivalent to $\mathbf{s}$ and $\mathbf{n} \in \mathbf{Z}_+^{\Theta}$. Evidently $\mathbf{t} - \mathbf{s} \in \mathbf{Z}_+^{\Delta\setminus\Theta} \times \mathbf{C}^{\Theta}$; hence this function is well defined on $D^{\Delta\setminus\Theta} \times (0,1)^{\Theta}$ and it extends to a holomorphic function on any simply connected open set in $D^{\Delta\setminus\Theta} \times (D^*)^{\Theta}$ containing $D^{\Delta\setminus\Theta} \times (0,1)^{\Theta}$.

From A.1.7 and the corresponding statement for the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta}$, it follows immediately that for each $\mathbf{s} \in S_{\Delta\setminus\Theta}$ and any coordinate hyperplane in $\mathbf{C}^{\Delta}$ corresponding to an element of $\Delta\setminus\Theta$, there exists $\mathbf{m} \in \mathbf{Z}_+^{\Delta\setminus\Theta}$ such that $F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta}$ is not identically zero on this coordinate hyperplane.

Also, by the construction, we have

$$F = \sum_{\mathbf{s},\mathbf{m}} (F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta} \circ \underline{\alpha}) \lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda,$$

where $\mathbf{s} \in S_{\Delta\setminus\Theta}$ and $\mathbf{m} \in \mathbf{Z}_+^{\Delta\setminus\Theta}$, on A^-.

To summarize this discussion, we have the following version of the expansion from 5.6 "relative to Θ":

There exist

(i) a finite set $S_{\Delta\setminus\Theta}$ of mutually $(\Delta\setminus\Theta)$-integrally inequivalent elements of $\mathbf{C}^{\Delta\setminus\Theta}$;

(ii) for each $\mathbf{s} \in S_{\Delta\setminus\Theta}$ a finite set $F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta}$, $\mathbf{m} \in \mathbf{Z}_+^{\Delta\setminus\Theta}$, of nontrivial holomorphic functions on a neighborhood of $D^{\Delta\setminus\Theta} \times (0,1)^{\Theta}$ in $D^{\Delta\setminus\Theta} \times (D^*)^{\Theta}$, such that on each of the coordinate hyperplanes corresponding to an element of $\Delta\setminus\Theta$ at least one of them is not identically zero, with

$$F = \sum (F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta} \circ \underline{\alpha}) \lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda$$

on A^-.

If $\Theta = \emptyset$ this expansion is exactly the same as in 5.6, but in the other cases the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta}$ can be holomorphically continued to a larger region giving us control over F on $\mathrm{Cl}(A^-)$.

More specifically, we have the following crucial result.

Lemma 6.1. *There exists a domain $C(\Theta)$ in $D^{\Delta\setminus\Theta} \times (\mathbf{C}^*)^{\Theta}$ containing $D^{\Delta\setminus\Theta} \times (0,1]^{\Theta}$ such that the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta}$ extend to holomorphic functions on $C(\Theta)$.*

We postpone the proof of 6.1 for a moment. Recallying the definition of the elements of our cover $A^-(\Theta)$, $\Theta \subsetneq \Delta$, of $\mathrm{Cl}(A^-)$, we see that 6.1 immediately implies the following crucial result, which gives us control over the behavior of F on all of $\mathrm{Cl}(A^-)$.

THEOREM 6.2. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. For any set $\Theta \subseteq \Delta$ we have*

$$F = \sum (F^{\Delta\setminus\Theta}_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on $A^-(\Theta)$.

It remains to prove 6.1. It is nontrivial only in the case when $\Theta \neq \emptyset$. Therefore, in the following, we fix a proper subset Θ of Δ.

We denote by Σ_Θ the subset of Σ consisting of all roots γ which are products of elements of Θ, i.e. of roots γ such that $A_\Theta^- \subset Y_\gamma$.

Let $0 < \epsilon < 1$ and put

$$X_1 = \{z \in \mathbf{C}^{\Delta\setminus\Theta} \,|\, |z_\alpha| < \epsilon, \alpha \in \Delta\setminus\Theta\}.$$

Evidently $X_1 \times (0,1]^\Theta$ intersects Y_γ if and only if $\gamma \in \Sigma_\Theta$. Therefore, there exists the largest positive number δ such that, if we put

$$X_2 = \{z \in \mathbf{C}^\Theta \,|\, 0 < \operatorname{Re} z_\alpha < 1 + \delta, |\operatorname{Im} z_\alpha| < \delta, \alpha \in \Theta\}$$

and

$$X(\Theta,\epsilon) = X_1 \times X_2 \subset \mathbf{C}^\Delta,$$

the following condition holds: for each $\gamma \in \Sigma$, $Y_\gamma \cap X(\Theta,\epsilon) \neq \emptyset$ implies $\gamma \in \Sigma_\Theta$ (see Figure 6.1).

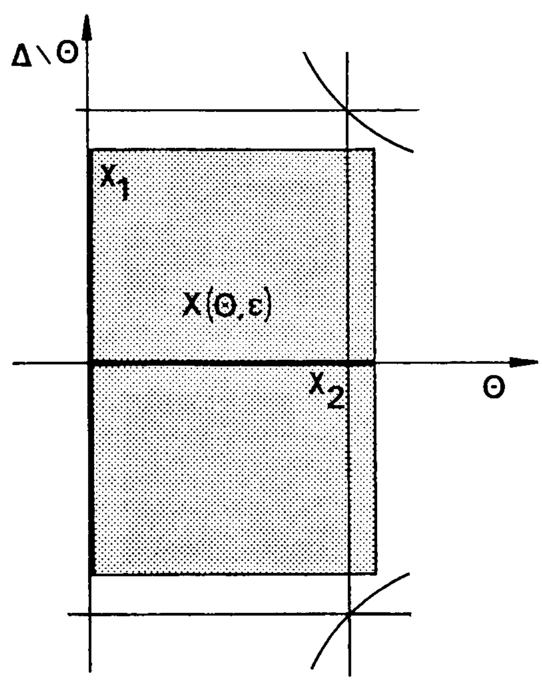


FIGURE 6.1

It is obvious that the union of all $X(\Theta,\epsilon)$, $0<\epsilon<1$, is a domain in $D^{\Delta\backslash\Theta}\times(\mathbf{C}^*)^{\Theta}$ contianing $D^{\Delta\backslash\Theta}\times(0,1]^{\Theta}$. Therefore, to prove 6.1 it is enough to show that the functions $F^{\Delta\backslash\Theta}_{\mathbf{s},\mathbf{m}}$ extend to holomorphic functions on $X = X(\Theta,\epsilon)$. The proof of this statement consists of two steps.

In the first step of the proof we show that the τ-spherical function F extends to a multivalued holomorphic function on the complement of coordinate hyperplanes in X. The proof of this assertion is based on a careful study of the system of differential equations for F from Section 4 and monodromy transformations around its singularities.

Let Y_1 be the union of the intersections of X_1 with the coordinate hyperplanes, i.e.

$$Y_1 = \{z\in X_1 \mid z_\lambda = 0 \text{ for some } \lambda\in\Lambda\backslash\Theta\};$$

Y_2 the union of the intersections of X_2 with the root hypersurfaces Y_γ for $\gamma\in\Sigma_\Theta$, i.e.

$$Y_2 = \{z\in X_2 \mid \gamma(z)^2 = 1 \text{ for some } \gamma\in\Sigma_\Theta\}$$

and

$$X_1^* = X_1\backslash Y_1, \qquad X_2^* = X_2\backslash Y_2.$$

Finally we put $X^* = X_1^*\times X_2^*$. We fix a base point $x_0 = (x_1,x_2)\in X^*\cap A^-$ of X^*.

We can realize the universal covering space $\tilde{X}_1^*$ of X_1^* as

$$\tilde{X}_1^* = \{x\in\mathbf{C}^{\Lambda\backslash\Theta} \mid \operatorname{Re} x_\alpha < \log\epsilon,\ \alpha\in\Delta\backslash\Theta\},$$

the covering map $p_1:\tilde{X}_1^*\to X_1^*$ being the ordinary exponential map. We fix a base point $\tilde{x}_1$ of $\tilde{X}_1^*$ as the point above x_1 with real coordinates.

Let $\tilde{X}_2^*$ be the universal covering space of X_2 and $\tilde{x}_2$ its base point above x_2. Then

$$\tilde{X}^* = \tilde{X}_1^*\times\tilde{X}_2^*$$

is the universal covering space of X^* with the base point $\tilde{x}_0 = (\tilde{x}_1,\tilde{x}_2)$.

Now we invoke the results and notation of Section 4. The restriction of our τ-spherical function F on A is real analytic, so it extends to a holomorphic function on an open set Ω in $(\mathbf{C}^*)^{\Lambda}$ containing A, which we denote by F too. The corresponding function Φ extends to a holomorphic function on $\Omega\backslash Y$ which satisfies there the system

$$z_\lambda\partial_\lambda\Phi = G_\lambda\Phi, \qquad \lambda\in\Lambda,$$

of first order differential equations, where G_λ, $\lambda\in\Lambda$, are holomorphic matrix valued functions on $\mathbf{C}^{\Lambda}\backslash Y$. In particular, G_λ, $\lambda\in\Lambda$, are holomorphic on X^*. Therefore, the function Φ satisfies this system on $X^*\cap\Omega$.

By shrinking Ω if necessary, we can assume that $X \cap \Omega$ is connected. Then $X^* \cap \Omega$, as a complement of a proper analytic set in $X \cap \Omega$, is connected too. Let U be the connected component of its inverse in $\tilde{X}^*$ containing the base point $\tilde{x}_0$. By A.1.2 the pull-back of Φ to U extends to a multivalued solution of our system on $\tilde{X}^*$. This implies, in particular, that the pullback of F to U extends to a holomorphic function in $\tilde{X}^*$. Abusing notation slightly, we denote it by F too.

The critical point, which enables the whole argument to work, is that

$$\pi_1(X^*, x_0) = \pi_1(X_1^*, x_1) \times \pi_1(X_2^*, x_2)$$

as the direct product of groups. Roughly speaking, the loops around coordinate hyperplanes corresponding to $\Lambda \backslash \Theta$ and around the singularities Y_γ, $\gamma \in \Sigma_\Theta$, commute.

The structure of Y_2 implies immediately that each element of $\pi_1(X_2^*, x_2)$ is represented by a loop lying in $(\{x_1\} \times X_2^*) \cap \Omega$. Therefore the monodromies with respect to the elements of $\pi_1(X_2^*, x_2)$ act trivially on F, and F can be viewed as a holomorphic function on $\tilde{X}_1^* \times X_2^*$.

Now we can consider $\tilde{X}_1^* \times X_2^*$ as imbedded in $\tilde{X}_1^* \times X_2$, which is in a natural way the universal covering space of $X_1^* \times X_2$. Let V be the connected component of the inverse of $(X_1^* \times X_2) \cap \Omega$ in $\tilde{X}_1^* \times X_2$ containing the base point $(\tilde{x}_1, x_2)$. Then F extends to a holomorphic function on $(\tilde{X}_1^* \times X_2^*) \cup V$.

The complement of $\tilde{X}_1^* \times X_2^*$ in $\tilde{X}_1^* \times X_2$ is just $\tilde{X}_1^* \times Y_2$. Obviously each connected component of the set of points where $\tilde{X}_1^* \times Y_2$ is nonsingular and of codimension one in $\tilde{X}_1^* \times X_2$ intersects V. Therefore, by A.1.8, the function F extends to a holomorphic E^M-valued function on $\tilde{X}_1^* \times X_2$. This concludes the first part of the proof.

In the second part of the proof we show that the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta \backslash \Theta}$ extend to holomorphic functions on X.

First, we decompose $X_1 = X_1' \times \mathbf{C}^{\Lambda \backslash \Delta}$, where

$$X_1' = \{ z \in \mathbf{C}^{\Delta \backslash \Theta} \mid |z_\alpha| < \epsilon, \alpha \in \Delta \backslash \Theta \}.$$

We put $X' = X_1' \times X_2$. The intersection of X' with D^Δ, viewed as a subset of $\mathbf{C}^\Delta$, is equal to

$$X' \cap D^\Delta = X_1' \times (D^\Theta \cap X_2).$$

It is obvious from the previous discussion that the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta \backslash \Theta}$ extend to holomorphic functions on $X' \cap D^\Delta$, and the function F is given by

$$F = \sum_{\mathbf{s},\mathbf{m}} F_{\mathbf{s},\mathbf{m}}^{\Delta \backslash \Theta} z^{\mathbf{s}} \log^{\mathbf{m}} z$$

on $\tilde{X}_1^* \times (D^\Theta \cap X_2)$. Now, it is easy to rearrange this expression to

$$F = \sum_{\mathbf{t},\mathbf{m}} G_{\mathbf{t},\mathbf{m}} z^{\mathbf{t}} \log^{\mathbf{m}} z$$

where $\mathbf{t} \in \mathbf{C}^{\Lambda\setminus\Theta}$ are mutually integrally inequivalent and

$$G_{\mathbf{t},\mathbf{m}} = \sum_{\mathbf{s}} F^{\Delta\setminus\Theta}_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}-\mathbf{t}},$$

where $\mathbf{s} - \mathbf{t} \in \mathbf{Z}_+^{\Lambda\setminus\Delta}$. By A.1.7 the functions $G_{\mathbf{t},\mathbf{m}}$ extend to holomorphic functions on X. This evidently implies that the functions $F^{\Delta\setminus\Theta}_{\mathbf{s},\mathbf{m}}$ extend to holomorphic functions on X, which concludes the proof of 6.1.

7. Leading characters and growth estimates on the group. By combining the results of Sections 5 and 6 we are now able to describe various aspects of the asymptotic behavior of τ-spherical functions on the group G solely in terms of their leading characters.

We equip E with inner product such that τ is unitary. In this case obviously

$$\|F(k_1ak_2)\| = \|F(a)\|, \qquad k_1, k_2 \in K, \quad a \in A.$$

Therefore, by the Cartan decomposition, the growth of F is completely determined by the behavior of $\|F\|$ on $\mathrm{Cl}(A^-)$.

Before formulating our results we need some notation. We define an ordering relation on *positive* characters of A:

$$\chi_1 \leqslant \chi_2 \quad \text{if} \quad \chi_1(a) \leqslant \chi_2(a) \quad \text{for all} \quad a \in A^-.$$

Obviously $\chi_1 \leqslant \chi_2$ implies $\chi_1|A_\Delta = \chi_2|A_\Delta$. Also we put

$$\chi_1 < \chi_2 \quad \text{if} \quad \chi_1(a) < \chi_2(a) \quad \text{for all} \quad a \in \mathrm{Cl}(A^-)\setminus A_\Delta.$$

THEOREM 7.1. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ and ω a positive character of A. Then the following conditions are equivalent*:

(i) *for every leading character ν of F we have*

$$|\nu| \leqslant \omega;$$

(ii) *there exist $M \geqslant 0$ and $m \geqslant 0$ such that*

$$\|F(a)\| \leqslant M\omega(a)(1 + \|\log a\|)^m$$

for all $a \in \mathrm{Cl}(A^-)$.

Proof. Let $\mathbf{t} \in \mathbf{R}^\Lambda$ be such that $\omega = \lambda^{\mathbf{t}}$. Then the condition (i) is equivalent to

$$\operatorname{Re} s_\alpha \geqslant t_\alpha, \qquad \alpha \in \Delta,$$

$$\operatorname{Re} s_\lambda = t_\lambda, \qquad \lambda \in \Lambda\setminus\Delta,$$

for all leading exponents $\mathbf{s}$ of F, or by 5.7 to

$$\operatorname{Re} S_\Delta \subseteq \mathbf{t} + \mathbf{R}_+^\Delta .$$

Therefore if (i) holds we have

$$\operatorname{Re} S_{\Delta\setminus\Theta} \subset \operatorname{pr}_{\Delta\setminus\Theta}(\mathbf{t}) + \mathbf{R}_{+}^{\Delta\setminus\Theta}$$

for all $\Theta \subset \Delta$.

We fix $0 < \epsilon < 1$ and put

$$A_\epsilon^-(\Theta) = \{a \in A \mid \epsilon \leqslant \alpha(a) \leqslant 1, \alpha \in \Theta, \alpha(a) < \epsilon, \alpha \in \Delta\setminus\Theta\},$$

(Fig. 7.1), then obviously

$$\operatorname{Cl}(A^-) = \bigcup_{\Theta \subset \Delta} A_\epsilon^-(\Theta).$$

Also, by 6.2, for $\Theta \subset \Delta$,

$$F = \sum (F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on $A_\epsilon^-(\Theta)$, and by 6.1 the functions $F_{\mathbf{s},\mathbf{m}}^{\Delta\setminus\Theta} \circ \underline{\alpha}$ are bounded on $A_\epsilon^-(\Theta)$. This implies that there exist $M_\Theta \geqslant 0$ and $m_\Theta \geqslant 0$ such that

$$\|F(a)\| \leqslant M_\Theta \omega(a)(1 + \|\log a\|)^{m_\Theta}$$

for all $a \in A_\epsilon^-(\Theta)$. This clearly implies (ii).

Suppose (ii) holds. Let $\alpha \in \Delta$. Fix points $p \in (0,1)^{\Delta\setminus\{\alpha\}}$ and $q \in (\mathbf{R}_+^*)^{\Lambda\setminus\Delta}$. Let

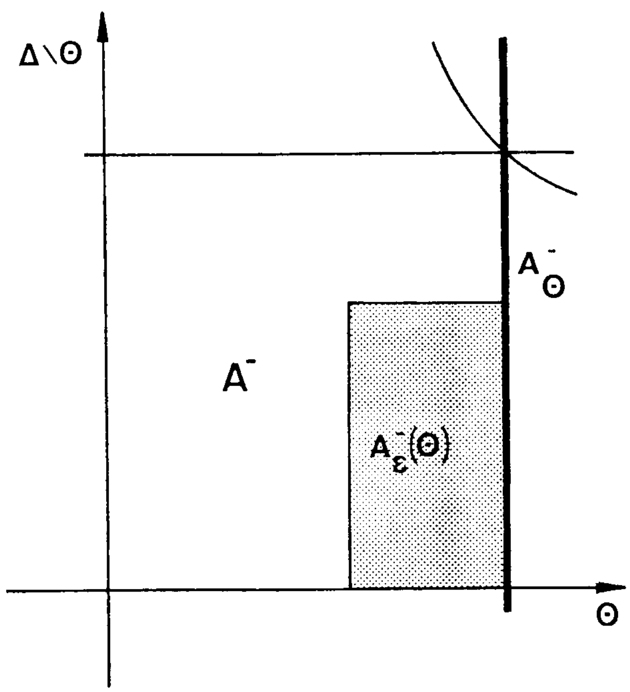


FIGURE 7.1

$r \to a_r$ be a map from $(0,1)$ into A^- such that

$$\alpha(a_r) = r, \qquad \beta(a_r) = p_\beta \qquad \text{for} \quad \beta \in \Delta \backslash \{\alpha\},$$
$$\lambda(a_r) = q_\lambda \qquad \text{for} \quad \lambda \in \Lambda \backslash \Delta.$$

Then by (ii) we have, for some $M_\alpha(p,q) \geqslant 0$,

$$\|F(a_r)\| \leqslant M_\alpha(p,q) r^{t_\alpha}(1 + \|\log r\|)^m$$

for all $r \in (0,1)$.

Also, by the discussion from the beginning of Section 6, for $\Theta = \Delta \backslash \{\alpha\}$, we know that on A^- we have

$$F = \sum \left(F^{\{\alpha\}}_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha}\right)\lambda^{\mathbf{s}} \log^{\mathbf{m}} \lambda$$

where $\mathbf{s} \in S_{\{\alpha\}} \subset \mathbf{C}^{\{\alpha\}} \times \mathbf{C}^{\Lambda\backslash\Delta}$ and $\mathbf{m} \in \mathbf{Z}_+^{\{\alpha\}} \times \mathbf{Z}_+^{\Lambda\backslash\Delta}$. We put

$$\mathbf{s} = (s_\alpha, \mathbf{s}') \quad \text{and} \quad \mathbf{m} = (m_\alpha, \mathbf{m}'),$$

where $s_\alpha \in \mathbf{C}^{\{\alpha\}}$, $\mathbf{s}' \in \mathbf{C}^{\Lambda\backslash\Delta}$, $m_\alpha \in \mathbf{Z}_+^{\{\alpha\}}$ and $\mathbf{m}' \in \mathbf{Z}_+^{\Lambda\backslash\Delta}$. Then

$$F(a_r) = \sum F^{\{\alpha\}}_{\mathbf{s},\mathbf{m}}(p,r) q^{\mathbf{s}'} \log^{\mathbf{m}'} q \; r^{s_\alpha} \log^{m_\alpha} r$$

for all $r \in (0,1)$. By A.2.1, if

$$\sum_{\mathbf{s}',\mathbf{m}'} F^{\{\alpha\}}_{((s_\alpha,\mathbf{s}'),(m_\alpha,\mathbf{m}'))}(p,0) q^{\mathbf{s}'} \log^{\mathbf{m}} q \neq 0,$$

we must have $\operatorname{Re} s_\alpha \geqslant t_\alpha$.

By the properties of $F^{\{\alpha\}}_{\mathbf{s},\mathbf{m}}$ for each s_α and $\mathbf{s}'$ there exists $\mathbf{m} = (m_\alpha, \mathbf{m}')$ such that $F^{\{\alpha\}}_{\mathbf{s},\mathbf{m}}(p,0) \neq 0$ for some $p \in (0,1)^{\Delta\backslash\{\alpha\}}$. By the linear independence of the functions $q \to q^{\mathbf{s}'} \log^{\mathbf{m}'} q$ there exist $q \in (\mathbf{R}_+^*)^{\Lambda\backslash\Delta}$ such that the above expression is different from zero. This implies that $\operatorname{Re} s_\alpha \geqslant t_\alpha$. The relation of $S_{\{\alpha\}}$ with S_Δ now implies that $\operatorname{Re} s_\alpha \geqslant t_\alpha$, $\alpha \in \Delta$, for each $\mathbf{s} \in S_\Delta$.

Take now $\mu \in \Lambda \backslash \Delta$. Fix a point $p \in (0,1)^\Delta$ and $q \in (\mathbf{R}_+^*)^{\Lambda\backslash(\Delta\cup\{\mu\})}$. Let $r \to a_r$ be a map from $\mathbf{R}_+^*$ into A^- such that

$$\alpha(a_r) = p_\alpha \qquad \text{for} \quad \alpha \in \Delta, \quad \mu(a_r) = r,$$
$$\lambda(a_r) = q_\lambda \qquad \text{for} \quad \lambda \in \Lambda \backslash (\Delta \cup \{\mu\}).$$

Then by (ii) we have, for some $M_\mu(p,q) \geqslant 0$,

$$\|F(a_r)\| \leqslant M_\mu(p,q) r^{t_\mu}(1 + \|\log r\|)^m$$

for all $r \in (\mathbf{R}_+^*)$.

Also, by the discussion from the beginning of Section 6, for $\Theta = \Delta$ we know that on A^- we have

$$F = \sum (F^{\varnothing}_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

where $\mathbf{s} \in S_\varnothing \subset \mathbf{C}^{\Lambda\setminus\Delta}$ and $\mathbf{m} \in \mathbf{Z}_+^{\Lambda\setminus\Delta}$. We put

$$\mathbf{s} = (s_\mu, \mathbf{s}') \quad \text{and} \quad \mathbf{m} = (m_\mu, \mathbf{m}'),$$

where $s_\mu \in \mathbf{C}^{\{\mu\}}$, $\mathbf{s}' \in \mathbf{C}^{\Lambda\setminus(\Delta\cup\{\mu\})}$, $m_\mu \in \mathbf{Z}_+^{\{\mu\}}$ and $\mathbf{m}' \in \mathbf{Z}_+^{\Lambda\setminus(\Delta\cup\{\mu\})}$. Then

$$F(a_r) = \sum F^{\varnothing}_{\mathbf{s},\mathbf{m}}(p) q^{\mathbf{s}'}\log^{\mathbf{m}'}q \, r^{s_\mu}\log^{m_\mu}r,$$

for $r \in \mathbf{R}_+^*$.

Obviously for each $\mathbf{s} \in S_\varnothing$ there exists $\mathbf{m}$ such that $F^{\varnothing}_{\mathbf{s},\mathbf{m}}(p) \neq 0$ for some $p \in (0,1)^\Delta$. Now by the linear independence of the functions $q \to q^{\mathbf{s}'}\log^{\mathbf{m}'}q$, for each s_μ we can find m_μ and $q \in (\mathbf{R}_+^*)^{\Lambda\setminus(\Delta\cup\{\mu\})}$ such that

$$\sum_{\mathbf{s}',\mathbf{m}'} F^{\varnothing}_{((s_\mu,\mathbf{s}'),(m_\mu,\mathbf{m}'))}(p) q^{\mathbf{s}'}\log^{\mathbf{m}'}q \neq 0.$$

By A.2.1 this implies $\operatorname{Re} s_\mu = t_\mu$ for all $\mu \in \Lambda\setminus\Delta$. The relation of $S_\varnothing$ with S_Δ now implies that

$$Re\, s_\mu = t_\mu, \qquad \mu \in \Lambda\setminus\Delta$$

for all $\mathbf{s} \in S_\Delta$. Q.E.D.

Let δ be the positive character of A defined by

$$\delta(a) = \det(\operatorname{Ad}(a) \mid \mathfrak{n}), \qquad a \in A.$$

Put $m(\alpha) = \dim \mathfrak{g}_\alpha$ for $\alpha \in \Sigma$. Then

$$\delta(a) = \prod_{\alpha\in\Sigma^+} \alpha(a)^{m(\alpha)}, \qquad a \in A.$$

Following Harish-Chandra we say that F is *tempered* if there exist $M \geqslant 0$ and $m \geqslant 0$ such that

$$\|F(a)\| \leqslant M\delta^{1/2}(a)(1 + \|\log a\|)^m$$

for $a \in A^-$. By 7.1 we have the following result.

COROLLARY 7.2. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. The function F is tempered if and only if*

$$|\nu| \leqslant \delta^{1/2}$$

for every leading character ν of F.

Let Z_G be the center of the group G. Then Z_G is the direct product of its maximal compact subgroup $K \cap Z_G$ and A_Δ.

Let F be a τ-spherical function on G. The character ζ of Z_G is called the *central character* of F if

$$F(zx) = \zeta(z)F(x), \qquad x \in G, \quad z \in Z_G$$

LEMMA 7.3. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ with the central character ζ. Then the expansion of F in A^- has the form*

$$F = \sum (F^\Delta_{\mathbf{s},\mathbf{m}} \circ \underline{\alpha})\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

where the restrictions of $\lambda^{\mathbf{s}}$, $\mathbf{s} \in S_\Delta$, to A_Δ are equal to $\zeta \mid A_\Delta$, and $\mathbf{m} \in \mathbf{Z}^\Delta_+$.

Proof. This follows immediately from 5.6 and the linear independence of the functions $\lambda^{\mathbf{t}}\log^{\mathbf{n}}\lambda$, $\mathbf{t} \in \mathbf{C}^{\Lambda\setminus\Delta}$, $\mathbf{n} \in \mathbf{Z}^{\Lambda\setminus\Delta}_+$. Q.E.D.

We say that a function f on A with values in a normed linear space *vanishes at infinity* in A^- if for every $\eta > 0$ there exists ϵ, $0 < \epsilon < 1$, such that $\delta(a) < \epsilon$ implies $\|f(a)\| < \eta$ for $a \in A^-$.

THEOREM 7.4. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ with the central character ζ. Let ω be a positive character of A. Then the following conditions are equivalent.*

(i) *for every leading character ν of F we have*

$$|\nu| < \omega,$$

(ii) *the function $\omega^{-1}F$ vanishes at infinity in A^-.*

Proof. Suppose that (i) holds. Let $\mathbf{t} \in \mathbf{R}^\Lambda$ be such that $\omega = \lambda^{\mathbf{t}}$ holds. Then the condition (i) is equivalent to

$$\operatorname{Re} s_\alpha > t_\alpha, \qquad \alpha \in \Delta,$$

$$\operatorname{Re} s_\lambda = t_\lambda, \qquad \lambda \in \Lambda\setminus\Delta,$$

for all leading exponents $\mathbf{s}$ of F. This implies that there exists a positive character ω_1 of A such that

$$|\nu| \leqslant \omega_1 \quad \text{and} \quad \omega_1 < \omega.$$

Now we can find $\eta > 0$ such that

$$\frac{\omega_1}{\omega} \leqslant \delta^\eta.$$

By 7.1 this implies that

$$\omega(a)^{-1} \cdot \|F(a)\| \leqslant M\delta^\eta(a)(1 + \|\log a\|)^m, \qquad a \in A^-,$$

for some $M \geqslant 0$ and $m \geqslant 0$. Now by 7.3 we know that

$$|\zeta|\,|A_\Delta = |\nu|\,|A_\Delta = \omega|A_\Delta.$$

Therefore, the function $\omega^{-1} \cdot \|F\|$ is constant on A_Δ-cosets in A. Therefore we can restrict ourselves to looking at it on $A \cap G_1$. In this case we can find $c > 0$ such that

$$\|\log a\| \leqslant c|\log \delta(a)|, \qquad a \in A^- \cap G_1.$$

This easily implies that $\omega^{-1} \cdot F$ vanishes at infinity in A^-.

Suppose (ii) holds. Let ϵ, $0 < \epsilon < 1$, be such that $a \in A^-$ and $\delta(a) < \epsilon$ imply

$$\omega^{-1}(a) \cdot \|F(a)\| < 1.$$

Then for $a' \in A_\Delta$, we have

$$|\zeta|(a')\omega^{-1}(a')\omega^{-1}(a)\|F(a)\| = \omega^{-1}(a'a) \cdot \|F(a'a)\| < 1$$

because $\delta(a') = 1$. Hence it follows that $a' \to |\zeta|(a')\omega^{-1}(a')$ is a bounded positive character of A_Δ, i.e. equal to 1. This, by 7.3, implies that $|\nu|\,|A = \omega|A_\Delta$ for every leading exponent ν of F.

Now we use the notation from the proof of the implication (ii)$\Rightarrow$(i) in 7.1. Fix $\alpha \in \Delta$, $p \in (0,1)^{\Delta\setminus\{\alpha\}}$ and $q \in (\mathbf{R}_+^*)^{\Lambda\setminus\Delta}$ and define the map $r \to a_r$ as there. Then we have

$$\lim_{r\to 0} r^{t_\alpha}\|f(a_r)\| = 0.$$

As in the above mentioned proof, using also 7.3, we have

$$\lim_{r\to 0} r^{t_\alpha}\sum F_{\mathbf{s},\mathbf{m}}^{\{\alpha\}}(p,r)r^{s_\alpha}\log^{m_\alpha} r = 0.$$

By A.2.1, $F_{\mathbf{s},\mathbf{m}}^{\{\alpha\}}(p,0) \neq 0$ implies that $\operatorname{Re} s_\alpha > t_\alpha$ for $\mathbf{s} \in S_{\{\alpha\}}$. By the relation of $S_{\{\alpha\}}$ with S_Δ it follows that

$$\operatorname{Re} s_\alpha > t_\alpha, \qquad \alpha \in \Delta,$$

for all $\mathbf{s} \in S_\Delta$. The above discussion implies also

$$\operatorname{Re} s_\mu = t_\mu, \qquad \mu \in \Lambda\setminus\Delta.$$

By 5.7, (i) follows immediately. Q.E.D.

Let F be a τ-spherical function on G with *unitary* central character ζ. Then the function $x \to \|F(x)\|$ is constant on Z_G-cosets of G, i.e. we can consider it as a function on G/Z_G.

Let $p \in [1, +\infty)$. We say that the τ-spherical function F is *p-integrable modulo center* if the function $x \to \|F(x)\|^p$ is integrable as a function on G/Z_G.

THEOREM 7.5. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$, with a unitary central character ζ, and $p \in [1, +\infty)$. Then the following conditions are equivalent,*

(i) *for every leading character ν of F we have*

$$|\nu| < \delta^{1/p},$$

(ii) *F is p-integrable modulo center.*

Proof. By 7.3, restrictions of leading characters to A_Δ are unitary. Therefore

$$|\nu| \,|\, A_\Delta = 1,$$

for all leading characters ν. By [20, II. 1.3, Prop. 11] there exists a closed subgroup ${}^0G$ of G of Harish-Chandra's class such that G is the direct product of ${}^0G$ and A_Δ. The center of ${}^0G$ is compact, therefore the above theorem is equivalent to the corresponding statement for $F \,|\, {}^0G$, where p-integrability modulo center in (ii) is replaced with p-integrability. Therefore, without any loss of generality we can assume that $A_\Delta = \{1\}$.

Suppose (i) holds. Then we can find a positive character ω of A such that

$$|\nu| \leqslant \omega \quad \text{and} \quad \omega < \delta^{1/p}$$

for every leading character ν of F. Therefore, by 7.1, there exist $M \geqslant 0$, $m \geqslant 0$ such that

$$\|F(a)\| \leqslant M\omega(a)(1 + |\log \delta(a)|)^m$$

for $a \in \mathrm{Cl}(A^-)$.

Now we have to recall the well-known integral formula connected with the Cartan decomposition

$$\int_G f(x)\,dx = \int_{K \times \mathrm{Cl}(A^-) \times K} f(k_1 a k_2) D(a)\, dk_1\, da\, dk_2,$$

where dx, dk and da are Haar measures on G, K and A respectively, and

$$D(a) = \prod_{\alpha \in \Sigma^+} \left(\alpha(a)^{-1} - \alpha(a)\right)^{m(\alpha)}$$

for $a \in A$. This implies that $\|F\|^p$ is integrable on G if

$$\int_{\mathrm{Cl}(A^-)} \|F(a)\|^p D(a)\, da < +\infty.$$

Obviously

$$D(a) \leqslant \delta(a)^{-1}, \qquad a \in \mathrm{Cl}(A^-);$$

therefore

$$\int_{\mathrm{Cl}(A^-)} \|F(a)\|^p D(a)\,da$$
$$\leqslant M \int_{\mathrm{Cl}(A^-)} \omega(a)^p \delta(a)^{-1}(1 + |\log \delta(a)|)^m\,da < \infty,$$

because of $\omega^p \delta^{-1} < 1$.

Suppose now that (ii) holds. Then by the above integral formula

$$\int_{\mathrm{Cl}(A^-)} \|F(a)\|^p D(a)\,da < +\infty.$$

Fix ϵ, $0 < \epsilon < 1$. There exists c, $0 < c < 1$, such that

$$D(a) \geqslant c\delta(a)^{-1}, \qquad a \in A_\epsilon^-(\varnothing).$$

Therefore the above fact implies that

$$\int_{A_\epsilon^-(\varnothing)} \|F(a)\|^p \delta(a)^{-1}\,da < +\infty.$$

Now we use the notation from the proof of the implication (ii)$\Rightarrow$(i) in 7.1. We fix $\alpha \in \Delta$ and $q \in (0,1)^{\Delta \setminus \{\alpha\}}$. We define the map $r \to a_r$ from $(0,1)$ into A^- as there.

By Fubini's theorem the above relation implies

$$\int_0^\epsilon \|F(a_r)\|^p r^{-m(\alpha)} \frac{dr}{r} < +\infty$$

for almost all $q \in (0,\epsilon)^{\Delta \setminus \{\alpha\}}$.

Now we can represent F as

$$F = \sum \left(F_{s,m}^{\{\alpha\}} \circ \underline{\alpha}\right) \alpha^s \log^m \alpha$$

where $s \in S_{\{\alpha\}} \subset \mathbf{C}$ and $m \in \mathbf{Z}_+$. Then it follows that

$$\int_0^\epsilon \left\| \sum F_{s,m}^{\{\alpha\}}(q,r) r^{s-(m(\alpha)/p)} \log^m r \right\|^p \frac{dr}{r} < +\infty$$

for almost all $q \in (0,\epsilon)^{\Delta \setminus \{\alpha\}}$. This obviously implies that all matrix coefficients of the function

$$r \to \sum F_{s,m}^{\{\alpha\}}(q,r) r^{s-(m(\alpha)/p)} \log^m r$$

are in $L^p((0,\epsilon], dr/r)$ for almost all q. By A.2.1 we now have

$$\operatorname{Re} s - \frac{m(\alpha)}{p} > 0$$

for $s \in S_{(\alpha)}$. This implies that for $\mathbf{s} \in S_\Delta$ we have

$$\operatorname{Re} s_\alpha - \frac{m(\alpha)}{p} > 0$$

for all $\alpha \in \Delta$, or $|\nu|\delta^{-1/p} < 1$ for all leading characters ν of F, which implies (i). Q.E.D.

Comparing 7.4 and 7.5 we see immediately the following result.

COROLLARY 7.6. *Let F be a τ-spherical function on G annihilated by an ideal I of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ with a unitary central character ζ, and $p \in [1, +\infty)$. Then the following conditions are equivalent*:

(i) *F is p-integrable modulo center*;

(ii) *$\delta^{-1/p} \cdot F$ vanishes at infinity in A^-.*

Remark 7.7. In the case $A_\Delta = \{1\}$ and $p = 2$ we see that a τ-spherical function F on G is square-integrable if and only if $\delta^{-1/2} \cdot F$ vanishes at infinity in A^-. In Harish-Chandra's terminology this means that F is "rapidly decreasing" on G. This is one of the crucial results of his theory of discrete series.

8. Admissible representations and their matrix coefficients. Now we want to apply the results about the asymptotic behavior of spherical functions to representation theory. We restrict ourselves, in this paper, to very modest applications culminating in the proof of the subrepresentation theorem (compare [6]).

An *admissible representation* (π, V) *of* $(\mathfrak{g}, K)$ consists of a pair of representations of $\mathfrak{g}$ and K simultaneously on V such that:

(A_1) the representation of K is an algebraic direct sum of irreducible finite-dimensional smooth representations, each isomorphism class occurring with finite multiplicity;

(A_2) the representation of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ coincides with the differential of the representation of K;

(A_3) for any $X \in \mathfrak{U}(\mathfrak{g})$ and $k \in K$,

$$\pi(\operatorname{Ad} k(X)) = \pi(k)\pi(X)\pi(k^{-1}).$$

Remark 8.1. Let (π, V) be a $(\mathfrak{g}, K)$-bimodule satisfying (A_3). A vector $v \in V$ is *K-finite* if its K-orbit spans a finite-dimensional linear subspace of V. Let V_0 be the linear space of all K-finite vectors in V. Then V_0 is obviously a K-submodule. We claim that it is also a $\mathfrak{g}$-module. If $v \in V_0$ we have

$$\pi(k)\pi(X)v = \pi(\operatorname{Ad} k(X))\pi(k)v, \qquad k \in K, \quad X \in \mathfrak{g},$$

by (A_3); this obviously implies that $\pi(X)v$ is K-finite, proving that V_0 is $\mathfrak{g}$-invariant.

Now let (π, V) be an admissible representation of $(\mathfrak{g}, K)$. Let V^* be the algebraic dual of V. It is a $(\mathfrak{g}, K)$-bimodule with respect to the contragredient action which obviously satisfies (A_3). Applying 8.1 we get a pair of representations $(\tilde{\pi}, \tilde{V})$ of $\mathfrak{g}$ and K on the space $\tilde{V}$ of all K-finite linear forms of V. It is easy to check that $(\tilde{\pi}, \tilde{V})$ satisfies (A_1) and (A_2), i.e. it is an admissible representation of $(\mathfrak{g}, K)$—the *contragredient representation* for (π, V).

There exist natural $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-bimodule structures on $\tilde{V} \otimes V$ and $C^\infty(G)$. For $X_1, X_2 \in \mathfrak{g}$, $k_1, k_2 \in K$, we put

$$(X_1, X_2)(\tilde{v} \otimes v) = \tilde{\pi}(X_1)\tilde{v} \otimes v + \tilde{v} \otimes \pi(X_2)v$$

$$(k_1, k_2)(\tilde{v} \otimes v) = \tilde{\pi}(k_1)\tilde{v} \otimes \pi(k_2)v$$

for $\tilde{v} \in \tilde{V}$, $v \in V$; and

$$(X_1, X_2)f = L_{X_1}f + R_{X_2}f$$

$$(k_1, k_2)f = L_{k_1}R_{k_2}f$$

for $f \in C^\infty(G)$.

A *matrix coefficient map* $c : \tilde{V} \otimes V \to C^\infty(G)$ for (π, V) is a linear map such that

(MC_1) c is a $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-bimodule morphism;

(MC_2) for any $v \in V$ and $\tilde{v} \in \tilde{V}$,

$$c(\tilde{v} \otimes v)(1) = \langle v, \tilde{v} \rangle.$$

The function $c_{v,\tilde{v}} = c(\tilde{v} \otimes v)$ is called a *matrix coefficient* of $v \in V$ and $\tilde{v} \in \tilde{V}$.

Remark 8.2. Let (π, X) be an admissible representation of the group G on complete locally convex space X. Then the representations of $\mathfrak{g}$ and K on the linear space of all K-finite vectors in X define an admissible representation of $(\mathfrak{g}, K)$. The map c defined by

$$c_{v,\tilde{v}}(x) = \langle \pi(x)v, \tilde{v} \rangle$$

for K-finite vectors $v \in X$ and K-finite linear forms $\tilde{v}$ on X is a matrix coefficient map. This explains the above definition.

The next result relates matrix coefficient maps to spherical functions.

Let (π, V) be an admissible representation and $c : \tilde{V} \otimes V \to C^\infty(G)$ a matrix coefficient map for (π, V). Suppose that (τ, E) is a finite-dimensional smooth representation of $K \times K$ such that its contragredient $(\tilde{\tau}, \tilde{E})$ is a $K \times K$-submodule of $\tilde{V} \otimes V$. Let $i : \tilde{E} \to \tilde{V} \otimes V$ be the canonical imbedding; then $c \circ i : \tilde{E} \to C^\infty(G)$ is a $K \times K$-module morphism.

LEMMA 8.3. *There exists a unique τ-spherical function F such that*

$$[(c \circ i)(w)](x) = \langle F(x), w \rangle, \qquad x \in G,$$

for all $w \in \tilde{E}$.

Proof. For any $x \in G$ there exists a unique $F(x) \in E$ such that the above relation holds for all $w \in \tilde{E}$. It is evident that the function $F: G \to E$ is smooth. Also, for $k_1, k_2 \in K$ and $x \in G$, we have

$$\begin{aligned}\langle F(k_1^{-1} x k_2), w \rangle &= [(c \circ i)(w)](k_1^{-1} x k_2) \\ &= [(c \circ i)(\tilde{\tau}(k_1, k_2) w)](x) \\ &= \langle \tau(k_1, k_2)^{-1} F(x), w \rangle\end{aligned}$$

for all $w \in E$. Therefore, F is a τ-spherical function. Q.E.D.

We say that F is a *spherical function associated to the matrix coefficient map c for (π, V).*

A vector $v \in V$ is $\mathfrak{Z}(\mathfrak{g})$-*finite* if it generates a finite-dimensional $\mathfrak{Z}(\mathfrak{g})$-invariant subspace of V. Because $\mathfrak{Z}(\mathfrak{g})$ commutes with K, it takes every K-isotopic component of (π, V) into itself, so that by the definition of admissibility it follows:

PROPOSITION 8.4. *Every vector in V is $\mathfrak{Z}(\mathfrak{g})$-finite.*

Therefore the annihilator in $\mathfrak{Z}(\mathfrak{g})$ of a finite-dimensional linear subspace of V is an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$.

Let F be a τ-spherical function associated to the matrix coefficient map c for (π, V). Then there exists a finite-dimensional subspace U of V such that $\tilde{E} \subset \tilde{V} \otimes U$. Let I be the annihilator in $\mathfrak{Z}(\mathfrak{g})$ of the linear space U. It follows that for $Z \in I$, we have

$$Z \cdot c_{v, \tilde{v}} = c(\tilde{v} \otimes \pi(Z) v) = 0$$

for all $v \in U$ and $\tilde{v} \in \tilde{V}$. This implies, in particular, that for all $Z \in I$, we have

$$\langle (Z \cdot F)(x), w \rangle = (Z[(c \circ i)(w)])(x) = 0, \qquad x \in G,$$

for all $w \in \tilde{E}$. Therefore, the ideal I annihilates F; i.e. $F \in A_\tau(G; I)$.

By the above discussion, it follows that any spherical function F associated to a matrix coefficient map for an admissible representation of $(\mathfrak{g}, K)$ is annihilated by an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. This fact, combined with the remarks at the beginning of Section 4, proves that all such spherical functions F are analytic. Therefore, we have:

PROPOSITION 8.5. *Let c be a matrix coefficient map for an admissible representation (π, V) of $(\mathfrak{g}, K)$. Then for any $w \in \tilde{V} \otimes V$ the function $c(w)$ is analytic on G.*

This result has the following important consequence.

PROPOSITION 8.6. *Let (π, V) be an admissible representation of $(\mathfrak{g}, K)$. Then there exists at most one matrix coefficient map c for (π, V).*

Proof. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. By 8.5, the function $c(\tilde{v} \otimes v)$ is analytic on G. Therefore there exists a neighborhood U of zero in $\mathfrak{g}_0$ such that $c(\tilde{v} \otimes v)$ can be represented by its Taylor series, i.e.

$$\begin{aligned} c(\tilde{v} \otimes v)(\exp X) &= \sum_{n=0}^{\infty} \frac{1}{n!} (X^n \cdot c(\tilde{v} \otimes v))(1) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} c(\tilde{v} \otimes \pi(X^n) v)(1) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \pi(X^n) v, \tilde{v} \rangle \end{aligned}$$

for $X \in U$. Therefore the matrix coefficient of $v \in V$ and $\tilde{v} \in \tilde{V}$ is uniquely determined in a neighborhood of identity in G. Applying 8.5 again we see that it is unique on the identity component G^0 of G. Finally, the fact that $G = G^0 \cdot K$ [20, II.1, Theorem 14] implies our assertion. Q.E.D.

Finally, we have

THEOREM 8.7. *Every admissible representation (π, V) of $(\mathfrak{g}, K)$ has a unique matrix coefficient map.*

Therefore we can always talk about *the* matrix coefficients of an admissible representation.

The existence of a matrix coefficient map for irreducible admissible representations of connected semi-simple groups follows from 8.2 and Lepowsky's version [15] of Harish-Chandra's subquotient theorem. The general result was proved, a number of years ago, by the first author using results on systems of differential equations with regular singularities [3], [4]. As we remarked in the Introduction, the differential equations for spherical functions have regular singularities along the root hypersurfaces in A; in particular at the identity. One of the main results of [3], in analogy with the classical situation [7, Ch. 4], states that every "formal" solution at a regular singularity converges. Therefore, roughly speaking, formal "Taylor series" for matrix coefficients at the identity given by the action of $\mathfrak{U}(\mathfrak{g})$ converge. It was observed by several people independently (D. Vogan pointed out this to us), that it is possible to deduce the existence in general from the above special case, using later results in this section.

This argument, although not as elegant as the first one, is very elementary. We include it, for the convenience of the reader, at the end of this section.

Now we shall, using 8.3, relate the matrix coefficients of admissible representations to spherical functions and study their asymptotic behavior along the negative Weyl chamber A^-.

Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then 8.4, combined with the fact that $\mathfrak{Z}(\mathfrak{g})$ commutes with $\mathfrak{g}$ and K, implies that $I = \ker(\pi \mid \mathfrak{Z}(\mathfrak{g}))$ is an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$. Also, it follows from previous discussion that all spherical functions associated to (π, V) are annihilated by I.

Let S_I be the set of all $\mathbf{s} \in \mathbf{C}^\Lambda$ such that the character $\lambda^\mathbf{s}$ lies over I. Then by 5.4 and 5.6 we have the following result.

THEOREM 8.8. *There exist unique linear forms* $c_{\mathbf{s},\mathbf{m}}: \tilde{V} \otimes V \to \mathbf{C}$ *for* $\mathbf{s} \in S_I + \mathbf{Z}_+^\Delta$, $\mathbf{m} \in \mathbf{Z}_+^\Lambda$, *such that for any* $w \in \tilde{V} \otimes V$ *we have*

$$c(w) = \sum c_{\mathbf{s},\mathbf{m}}(w)\lambda^\mathbf{s}\log^\mathbf{m}\lambda$$

on A^-.

We say that $\mathbf{s} \in \mathbf{C}^\Lambda$ is an *exponent* of $v \in V$ if there exists $\mathbf{m} \in \mathbf{Z}_+^\Lambda$ such that $c_{\mathbf{s},\mathbf{m}} \mid \tilde{V} \otimes v \neq 0$. We denote by $\operatorname{Exp}_v(\pi)$ the set of all exponents of v.

We say that $\mathbf{s} \in \mathbf{C}^\Lambda$ is an *exponent* of (π, V) if there exists $\mathbf{m} \in \mathbf{Z}_+^\Lambda$ such that $c_{\mathbf{s},\mathbf{m}} \neq 0$. We denote by $\operatorname{Exp}(\pi)$ the set of all exponents of (π, V).

The following inclusions are obvious

$$\operatorname{Exp}_v(\pi) \subset \operatorname{Exp}(\pi) \subset S_I + \mathbf{Z}_+^\Delta$$

for any $v \in V$.

We denote by $\operatorname{Exp}^0(\pi)$ the set of all minimal elements of $\operatorname{Exp}(\pi)$ with respect to Δ-order. The elements of $\operatorname{Exp}^0(\pi)$ are the *leading exponents* of (π, V). The corresponding characters of A are the *leading characters* of (π, V).

THEOREM 8.9. *Let* (π, V) *be a finitely generated admissible representation of* $(\mathfrak{g}, K)$ *and* $I = \ker(\pi \mid \mathfrak{Z}(\mathfrak{g}))$. *Then the leading characters of* (π, V) *lie over* I.

Proof. Let $\mu = \lambda^\mathbf{s}$, $\mathbf{s} \in \mathbf{C}^\Lambda$, be a leading character of (π, V). Then there exists $\mathbf{m} \in \mathbf{Z}_+^\Lambda$ such that $c_{\mathbf{s},\mathbf{m}} \neq 0$. Therefore we can find $v \in V$ and $\tilde{v} \in \tilde{V}$ such that $c_{\mathbf{s},\mathbf{m}}(\tilde{v} \otimes v) \neq 0$. Now 8.3 implies that there exists a τ-spherical function F associated to (π, V) such that μ is its leading character. The assertion follows from the fact that $F \in \mathcal{A}_\tau(G; I)$ and 5.4. Q.E.D.

COROLLARY 8.10. *The set of all leading characters of a finitely generated admissible representation is finite.*

Therefore to each finitely generated admissible representation (π, V) we associate a finite set of its leading characters which, by the results of Section 7, determines the qualitative asymptotic behavior of its matrix coefficients.

Although the transition is purely formal, for the sake of completeness, we shall reformulate some results of Section 7 in more representation theoretic terms.

Firstly, 7.1 implies

THEOREM 8.11. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$ and ω a positive character of A. Then the following conditions are equivalent:*

(i) *for every leading character ν of (π, V) we have*

$$|\nu| \leqslant \omega;$$

(ii) *for any $v \in V$ and $\tilde{v} \in \tilde{V}$ there exist $M \geqslant 0$ and $m \geqslant 0$ such that*

$$|c_{v,\tilde{v}}(a)| \leqslant M\omega(a)(1 + \|\log a\|)^m$$

for all $a \in \mathrm{Cl}(A^-)$.

We say that a finitely generated admissible representation (π, V) is *tempered* if for any $v \in V$ and $\tilde{v} \in \tilde{V}$ there exist $M \geqslant 0$ and $m \geqslant 0$ such that

$$|c_{v,\tilde{v}}(a)| \leqslant M\delta^{1/2}(a)(1 + \|\log a\|)^m$$

for $a \in A^-$. Then we have the following direct consequence of 8.11.

COROLLARY 8.12. *A finitely generated admissible representation (π, V) is tempered if and only if all its leading characters ν satisfy*

$$|\nu| \leqslant \delta^{1/2}.$$

As we remarked before, the center Z_G of G is the direct product of its maximal compact subgroup $K \cap Z_G$ and A_Δ. Because Z_G commutes with K, its Lie algebra and $K \cap Z_G$ take every K-isotopic component of (π, V) into itself. Therefore, by the simply-connectedness of A_Δ, the representations of $\mathfrak{g}$ and K on V determine a representation of Z_G on V. If Z_G acts on V by a character $\zeta : Z_G \to \mathbf{C}^*$ we call ζ the *central character* of (π, V).

Now 7.4 implies the following result.

THEOREM 8.13. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$ with a central character. Let ω be a positive character of A. Then the following conditions are equivalent*

(i) *for every leading character ν of (π, V) we have*

$$|\nu| < \omega;$$

(ii) *for any $v \in V$ and $\tilde{v} \in \tilde{V}$ the function $\omega^{-1} \cdot c_{v,\tilde{v}}$ vanishes at infinity in A^-.*

Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$ with a *unitary* central character. The functions $x \to |c_{v,\tilde{v}}(x)|$, $v \in V$, $\tilde{v} \in \tilde{V}$, are constant on Z_G-cosets of G; i.e. we can consider them as functions on G/Z_G.

Let $p \in [1, +\infty)$. We say that (π, V) is *p-integrable modulo center* if the functions $x \to |c_{v,\tilde{v}}(x)|^p$, $v \in V$, $\tilde{v} \in \tilde{V}$, are integrable as functions on G/Z_G.

Now 7.5 implies the following result.

THEOREM 8.14. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$ with a unitary central character and $p \in [1, +\infty)$. Then the following conditions are equivalent*

(i) *for every leading character ν of (π, V) we have*

$$|\nu| < \delta^{1/p};$$

(ii) *(π, V) is p-integrable modulo center.*

Finally 8.13 and 8.14 imply

COROLLARY 8.15. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$ with a unitary central character and $p \in [1, +\infty)$. Then the following conditions are equivalent*

(i) *(π, V) is p-integrable modulo center,*

(ii) *for any $v \in V$ and $\tilde{v} \in \tilde{V}$ the functions $\delta^{-1/p} \cdot c_{v,\tilde{v}}$ vanish at infinity in A^-.*

As we remarked in 7.7, in a special case, this is one of the crucial results of Harish-Chandra on the discrete series representation.

Now we want to study some more algebraic consequences of the expansions of matrix coefficients.

Let (π, V) be a finitely generated admissible representations of $(\mathfrak{g}, K)$. Let

$$L_k^+ = \left\{ \mathbf{m} \in \mathbf{Z}_+^{\Delta} \,\middle|\, \sum_{\alpha \in \Delta} m_\alpha \geqslant k \right\},$$

then obviously

$$\operatorname{Exp}(\pi) \subset \operatorname{Exp}^0(\pi) + L_0^+ .$$

Put

$$V_{(k)} = \left\{ v \in V \,|\, \operatorname{Exp}_v(\pi) \subset \operatorname{Exp}^0(\pi) + L_k^+ \right\}.$$

Then it is evident that $(V_{(k)};\, k \in \mathbf{Z}_+)$ is a decreasing linear space filtration of V. Moreover we have the following result.

LEMMA 8.16. (i) *The decreasing filtration $(V_{(k)};\, k \in \mathbf{Z}_+)$ is a $(\mathfrak{p}, M)$-bimodule filtration;*

(ii) *For any $k \in \mathbf{Z}_+$, we have*

$$\pi(X) V_{(k)} \subset V_{(k+1)}, \qquad X \in \mathfrak{n}.$$

Proof. For $X \in \mathfrak{g}$ and $a \in A$ we have

$$c_{\pi(X)v,\tilde{v}}(a) = c(\tilde{v} \otimes \pi(X)v)(a) = R_X c(\tilde{v} \otimes v)(a) = -L_{(\operatorname{Ad} a)(X)} c(\tilde{v} \otimes v)(a).$$

If $X \in \mathfrak{l}$ we have

$$c_{\pi(x)v,\tilde{v}}(a) = -c_{v,\tilde{\pi}(X)\tilde{v}}(a), \qquad a \in A;$$

which implies that

$$\mathrm{Exp}_{\pi(X)v}(\pi) \subset \mathrm{Exp}_v(\pi)$$

and therefore $\pi(X)V_{(k)} \subset V_{(k)}$.

If $X \in \mathfrak{g}_\gamma$, $\gamma \in \Sigma^+$, we have

$$c_{\pi(X)v,\tilde{v}}(a) = -\gamma(a)c_{v,\tilde{\pi}(X)\tilde{v}}(a), \qquad a \in A;$$

which implies that, if $\gamma = \alpha^{\mathbf{m}}$, $\mathbf{m} \in \mathbf{Z}_+^\Delta$, we have

$$\mathrm{Exp}_{\pi(X)v}(\pi) \subset \mathrm{Exp}_v(\pi) + \mathbf{m}.$$

Therefore, $\pi(X)V_{(k)} \subset V_{(k+1)}$, which proves (ii).

Finally, for $m \in M$, we have

$$\begin{aligned} c_{\pi(m)v,\tilde{v}}(a) &= c(\tilde{v} \otimes \pi(m)v)(a) = R_m c(\tilde{v} \otimes v)(a) \\ &= L_{m^{-1}} c(\tilde{v} \otimes v)(a) = c\big(\tilde{\pi}(m^{-1})\tilde{v} \otimes v\big)(a) \\ &= c_{v,\tilde{\pi}(m^{-1})\tilde{v}}(a), \qquad a \in A; \end{aligned}$$

which implies

$$\mathrm{Exp}_{\pi(m)v}(\pi) \subset \mathrm{Exp}_v(\pi)$$

and $\pi(m)V_{(k)} \subset V_{(k)}$. Q.E.D.

The filtration $(V_{(k)};\ k \in \mathbf{Z}_+)$ is called the *asymptotic filtration* of (π, V).

THEOREM 8.17. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then the asymptotic filtration is Hausdorff.*

Proof. The assertion means that

$$\bigcap_{k=0}^{\infty} V_{(k)} = \{0\}.$$

Let $v \in \bigcap_{k=0}^{\infty} V_{(k)}$. Then by the definition of $V_{(k)}$ the set of exponents of v is empty. This obviously implies that $c_{v,\tilde{v}} = 0$ for all $\tilde{v} \in \tilde{V}$ and therefore $v = 0$. Q.E.D.

It is possible to define a similar decreasing $(\mathfrak{p}, M)$-bimodule filtration $(\mathfrak{n}^k V;$

$k \in \mathbf{Z}_+$) of (π, V), while $\mathfrak{n}^k V$ is the linear span of vectors

$$\pi(X_1)\pi(X_2) \ldots \pi(X_k)v, \qquad X_1, X_2, \ldots, X_k \in \mathfrak{n}. \qquad v \in V$$

[5]. This filtration is called the $\mathfrak{n}$*-adic filtration* of (π, V).

The following result follows immediately from 8.16 (ii) by induction.

THEOREM 8.18. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then for any $k \in \mathbf{Z}_+$, we have*

$$\mathfrak{n}^k V \subset V_{(k)} .$$

Therefore the $\mathfrak{n}$-adic filtration is finer than the asymptotic filtration.

From 8.17 and 8.18 it follows

COROLLARY 8.19. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then the $\mathfrak{n}$-adic filtration of (π, V) is Hausdorff.*

From [6, 2.4] we know that the homology group $H_0(\mathfrak{n}, V)$ of a finitely generated admissible representation (π, V) is a finite-dimensional L-module. Now 8.18 implies the following non-vanishing result.

COROLLARY 8.20. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then $H_0(\mathfrak{n}, V)$ is a nontrivial finite-dimensional L-module.*

Let (ω, U) be a finite-dimensional smooth representation of P. Let $\mathrm{Ind}(\omega \mid P, G)$ be the space of all smooth functions $f\colon G \to U$ such that

(i) f is right K-finite,

(ii) $f(px) = \omega(p)f(x)$ for all $p \in P$, $x \in G$.

Then the right regular actions of $\mathfrak{g}$ and K define on $\mathrm{Ind}(\omega \mid P, G)$ the structure of an admissible representation of $(\mathfrak{g}, K)$ [6]. It is called the representation *induced* from (ω, U). For irreducible representations (ω, U) of P, which are evidently trivial on N, the representations $\mathrm{Ind}(\omega \mid P, G)$ are called the *principal series representations*.

Now, as in [6], 8.20 implies the following "subrepresentation" theorem which strengthens Harish-Chandra's "subquotient" theorem.

THEOREM 8.21. *Let (π, V) be an irreducible admissible representation of $(\mathfrak{g}, K)$. Then there exists an irreducible finite-dimensional smooth representation (ω, U) of P such that (π, V) may be imbedded into $\mathrm{Ind}(\omega \mid P, G)$.*

In fact we can extract more information about $H_0(\mathfrak{n}, V)$ from the previous discussion. This gives us more information about the imbeddings of irreducible admissible representations into the principal series.

THEOREM 8.22. *Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then the leading characters of (π, V) are A-weights of $H_0(\mathfrak{n}, V)$.*

Proof. For an A-weight μ of $H_0(\mathfrak{n}, V)$, let m_μ be the dimension of the

μ-component of $H_0(\mathfrak{n}, V)$. Let

$$P(H) = \prod_{\mu} (H - (d\mu)(H))^{m_\mu} \in \mathfrak{U}(\mathfrak{a})$$

for $H \in \mathfrak{a}$. The $P(H)$ annihilates $H_0(\mathfrak{n}, V)$ for all $H \in \mathfrak{a}$. Therefore

$$P(H)V \subset \mathfrak{n}V$$

and by 8.18

$$P(H)V \subset V_{(1)}$$

for all $H \in \mathfrak{a}$.

Let $\mathbf{t}$ be a leading exponent of (π, V). Then there exist $v \in V$ and $\tilde{v} \in \tilde{V}$ such that

$$c_{v,\tilde{v}} = \sum c_{\mathbf{s},\mathbf{m}}(\tilde{v} \otimes v)\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on A^- and $c_{\mathbf{t},\mathbf{m}}(\tilde{v} \otimes v) \neq 0$ for some $\mathbf{m} \in \mathbf{Z}_+^\Lambda$.

By the previous remark, for $H \in \mathfrak{a}$, we have

$$P(H)c_{v,\tilde{v}} = c_{P(H)v,\tilde{v}} = \sum c_{\mathbf{s},\mathbf{m}}(\tilde{v} \otimes P(H)v)\lambda^{\mathbf{s}}\log^{\mathbf{m}}\lambda$$

on A^- and $c_{\mathbf{t},\mathbf{m}}(\tilde{v} \otimes P(H)v) = 0$ for all $\mathbf{m} \in \mathbf{Z}_+^\Lambda$. Therefore, by 5.3, we easily see that $\lambda^{\mathbf{t}}$ must be an A-weight of $H_0(\mathfrak{n}, V)$. Q.E.D.

It remains to prove 8.7.

First we remark that if an admissible representation (π, V) of $(\mathfrak{g}, K)$ has the matrix coefficient map so do its sub- and quotient-representations. Therefore, the existence of matrix coefficient maps being evident for induced representations $\mathrm{Ind}(\omega \mid P, G)$ by 8.2, the "subquotient" theorem [15] implies their existence for irreducible admissible representations of connected semi-simple groups. This, in turn, implies that 8.20 holds in this situation.

Now, assume that G is from the Harish-Chandra class. The commutator subgroup G_1 of its identity component G^0 is a connected semi-simple Lie group. Then $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is the complexified Lie algebra and $K_1 = K \cap G_1$ a maximal compact subgroup of G_1. Also, it is evident that $\mathfrak{n} \subset \mathfrak{g}_1$, and it is the complexified Lie algebra of the nilpotent radical N of the minimal parabolic subgroup $P_1 = P \cap G_1$ of G_1.

Let (π, V) be a finitely generated admissible representation of $(\mathfrak{g}, K)$. Then there exists a finite-dimensional subspace U which is a sum of K-isotypic components of V and generates V as a $\mathfrak{g}$-module. Evidently, it also generates V as a $\mathfrak{g}_1$-module. Therefore, viewed as a $(\mathfrak{g}_1, K_1)$-module, (π, V) is a finitely generated admissible representation. Applying the above remark to its irreducible quotient implies, in turn, that $H_0(\mathfrak{n}, V)$ is non-zero. *This gives us a proof of* 8.20 *without using* 8.7.

Now, we shall prove 8.7 for finitely generated admissible representations using 8.20. This follows immediately from the following variant of 8.21.

PROPOSITION 8.23. *Let* (π, V) *be a finitely generated admissible representation of* $(\mathfrak{g}, K)$. *Then there exists a finite-dimensional smooth representation* (ω, U) *of* P *such that* (π, V) *may be imbedded into* $\operatorname{Ind}(\omega \mid P, G)$.

Proof. Let $k \in \mathbf{N}$. By 8.20 and [6, 2.3] we know that $V/\mathfrak{n}^k V$ is a nontrivial finite-dimensional $(\mathfrak{p}, M)$-module. The group AN being simply-connected, it is in fact a finite-dimensional P-module which we denote by (ω_k, U_k) in the following. By the Frobenius reciprocity theorem [6, 3.1], the quotient map of V onto U_k defines a $(\mathfrak{g}, K)$-module homomorphism φ_k of V into $\operatorname{Ind}(\omega_k \mid P, G)$. Let W_k be the kernel of φ_k. Then, by the definition, W_k is contained in $\mathfrak{n}^k V$.

Let $I = \ker(\pi \mid \mathfrak{Z}(\mathfrak{g}))$. Evidently, any A-weight of $H_0(\mathfrak{n}, V)$ must lie over I. As before, let S_I be the set of all $\mathbf{s} \in \mathbf{C}^{\Delta}$ such that the character $\lambda^{\mathbf{s}}$ lies over I. Then there exists a $k_0 \in \mathbf{N}$ such that for any two Δ-integrally equivalent $\mathbf{s}, \mathbf{t} \in S_I$ we have

$$\sum_{\alpha \in \Delta} |s_\alpha - t_\alpha| \leqslant k_0 .$$

If we denote by $T^k(\mathfrak{n})$ the kth tensor power of $\mathfrak{n}$ considered as a P-module under the adjoint action, we have the natural surjective $(\mathfrak{p}, M)$-module homomorphism

$$T^k(\mathfrak{n}) \otimes V \to \mathfrak{n}^k V.$$

It evidently induces a surjective P-module homomorphism

$$T^k(\mathfrak{n}) \otimes H_0(\mathfrak{n}, V) \to \mathfrak{n}^k V / \mathfrak{n}^{k+1} V.$$

Therefore, an A-weight of $\mathfrak{n}^k V/\mathfrak{n}^{k+1} V$ is also an A-weight of $T^k(\mathfrak{n}) \otimes H_0(\mathfrak{n}, V)$. Our choice of k_0 now implies that, for $k \geqslant k_0$, none of the A-weights of $\mathfrak{n}^k V/\mathfrak{n}^{k+1} V$ lies over I.

Let W be a subrepresentation of (π, V) contained in $\mathfrak{n}^k V$ for some $k \geqslant k_0$. Because I annihilates W, all A-weights of $H_0(\mathfrak{n}, W)$ lie over I. The inclusion map of W into $\mathfrak{n}^k V$ induces a P-module homomorphism of $H_0(\mathfrak{n}, W)$ into $\mathfrak{n}^k V/\mathfrak{n}^{k+1} V$. By the above remark, our choice of k_0 implies that this homomorphism is zero. Therefore W is contained in $\mathfrak{n}^{k+1} V$. By induction, it follows that

$$W \subset \bigcap_{k=0}^{\infty} \mathfrak{n}^k V.$$

This implies that $H_0(\mathfrak{n}, W)$ is zero, by [6, 2.3] and the Artin–Rees Lemma for $\mathfrak{U}(\mathfrak{n})$ ([18], see also [19] for a simple and elegant argument in the case we need). By 8.20, we now see that W must be zero.

Putting now all the pieces together, we conclude that, for $k \geqslant k_0$, the representation (π, V) may be imbedded into $\operatorname{Ind}(\omega_k \mid P, G)$. Q.E.D.

It remains to prove 8.7 in the general case. Let (π, V) be an admissible representation of $(\mathfrak{g}, K)$. Let $\mathfrak{F}$ be the family of all subrepresentations of (π, V) which have matrix coefficient maps, ordered by inclusion. By 8.23, $\mathfrak{F}$ is non-empty. Also, as it is easy to see from 8.6, it satisfies the conditions of the Zorn Lemma. Let (ρ, W) be a maximal element in $\mathfrak{F}$. Assume that it is different from (π, V). Then we can find a finitely generated subrepresentation (ν, U) of (π, V) such that U is not contained in W. By 8.23, the direct sum $(\nu \oplus \rho, U \oplus W)$ has the matrix coefficent map. Therefore, the representation on the invariant subspace $U + W$ of V, being a quotient of $(\nu \oplus \rho, U \oplus W)$, has a matrix coefficient map. This obviously contradicts the maximality of (ρ, W).

This finally ends the proof of 8.7.

Epilogue 8.24. As we have seen, to each finitely generated admissible representation (π, V) of $(\mathfrak{g}, K)$ we associate the following data:

(i) the set of leading characters, which are purely analytic in nature and determine the asymptotic behavior of its matrix coefficients;

(ii) the set of A-weights of $H_0(\mathfrak{n}, V)$, which are purely algebraic in nature and, by the Frobenius reciprocity theorem [6, 3.2], are related to $(\mathfrak{g}, K)$-morphisms with the principal series representations.

By 8.22 there is a close relationship between these two sets of data associated to (π, V). The complete connection between them is cleared up by [17, Theorem II. 2.1] which states that the leading characters are the *minimal* A-weights of $H_0(\mathfrak{n}, V)$ with respect to the Δ-order.

If we extend formally the proof of this result, considering whole filtrations instead of "top" graded pieces $V/V_{(1)}$ and $H_0(\mathfrak{n}, V) = V/\mathfrak{n}V$, we get that, although in general different, the asymptotic and $\mathfrak{n}$-adic filtrations define the same topology on V (this is equivalent to an unpublished result of H. Hecht and W. Schmid). This gives the ultimate connection between the analysis of asymptotic behavior of matrix coefficients and the algebra of admissible representations.

APPENDIX

1. Systems with simple singularities. For the convenience of the reader we collect in this appendix, with complete proofs, a few mostly well-known technical results we need in the main text.

The main result is an elementary theorem found in Deligne [8] on first order systems of holomorphic partial differential equations with regular singularities. This result, whose simple proof we reproduce below, clarifies greatly the results of Harish-Chandra on differential equations ([12], [22, Vol. II, Appendix]).

Let X be a connected complex manifold and x_0 a base point of X. Let $(\tilde{X}, \tilde{x}_0)$ be the universal covering space of X with base point $\tilde{x}_0$ considered as a complex manifold. We denote by $p: \tilde{X} \to X$ the corresponding covering projection.

The homotopy group $\pi_1(X, x_0)$ acts on $\tilde{X}$ by covering transformations. For $\gamma \in \pi_1(X, x_0)$ let $T_\gamma: \tilde{X} \to \tilde{X}$ be the corresponding covering transformation. The map $\gamma \to T_\gamma$ is a homomorphism of $\pi_1(X, x_0)$ into the group of all holomorphic diffeomorphisms of $\tilde{X}$ and

$$p \circ T_\gamma = p \qquad \text{for all} \quad \gamma \in \pi_1(X, x_0).$$

Let W be a finite-dimensional complex linear space and $\mathcal{O}_X(W)$ and $\mathcal{O}_{\tilde{X}}(W)$ the sheaves of germs of W-valued holomorphic functions on X and $\tilde{X}$, respectively.

The above defined action T of $\pi_1(X, x_0)$ on $\tilde{X}$ induces a representation of $\pi_1(X, x_0)$ on the linear space $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(W))$ of all W-valued holomorphic functions on $\tilde{X}$ by

$$T_\gamma^*(f) = f \circ T_{\gamma^{-1}}, \qquad (\gamma \in \pi_1(X, x_0)).$$

We call T_γ^* the *monodromy transformation* corresponding to γ.

The sheaf $\mathcal{O}_{\tilde{X}}(W)$ is of course just $p^*\mathcal{O}_X(W)$. If $\mathcal{F}$ is a subsheaf of $\mathcal{O}_X(W)$, then the inverse image sheaf $p^*\mathcal{F}$ is a subsheaf of $\mathcal{O}_{\tilde{X}}(W)$. We shall call the global sections of $p^*\mathcal{F}$ the *multivalued sections* of $\mathcal{F}$ on X.

Suppose now that X is a domain of $\mathbf{C}^n$. Let $E = \operatorname{End}_{\mathbf{C}}(W)$ and let $F_1, F_2, \ldots, F_n$ be E-valued holomorphic functions on X. We consider the system of holomorphic partial differential equations

$$\partial_i \Phi = F_i \Phi, \qquad i = 1, 2, \ldots, n, \tag{1}$$

on X. Local solutions of this system determine a subsheaf $\mathcal{S}$ of $\mathcal{O}_X(W)$. Multivalued sections of $\mathcal{S}$ are called *multivalued solutions* of the system (1).

Let $\mathcal{S}_{x_0}$ be the stalk of germs of solutions of the system (1) at the point x_0. The subset

$$W_0 = \{ \varphi(x_0) \mid \varphi \in \mathcal{S}_{x_0} \},$$

is a linear subspace of W.

Lemma A.1.1. *The map $\varphi \to \varphi(x_0)$ is a linear isomorphism of $\mathcal{S}_{x_0}$ onto W_0.*

Proof. The map $\varphi \to \varphi(x_0)$ is linear and surjective by the definition. Suppose that $\varphi \in \mathcal{S}_{x_0}$ is such that $\varphi(x_0) = 0$. Then there exists an open neighborhood U of x_0 in X and $\Phi \in \Gamma(U, \mathcal{S})$, such that φ is its germ at x_0. We may assume that $U = \{ x \in \mathbf{C}^n \mid |x_i - x_{0,i}| < \epsilon, \ 0 \leqslant i \leqslant n \}$ for some $\epsilon > 0$. For $x \in U$ we define a differentiable function $\Phi_x : [0,1] \to W$ by

$$\Phi_x(t) = \Phi(x_0 + t(x - x_0)).$$

Then

$$\Phi_x'(t) = \sum_{i=1}^{n} (x_i - x_{0,i}) \cdot (\partial_i \Phi)(x_0 + t(x - x_0))$$

$$= \left[\sum_{i=1}^{n} (x_i - x_{0,i}) \cdot F_i(x_0 + t(x - x_0)) \right] \Phi_x(t)$$

for $t \in [0,1]$, and $\Phi_x(0) = 0$. By the classical result on first order equations we have $\Phi_x = 0$. This implies $\Phi(x) = \Phi_x(1) = 0$. Therefore $\Phi = 0$ and finally $\varphi = 0$. Q.E.D.

The next result roughly states that every local solution of the system (1) extends to a global *multivalued* solution

THEOREM A.1.2. *The map* $\Phi \to \Phi(\tilde{x}_0)$ *of the space* $\Gamma(\tilde{X}, p^*\mathbb{S})$ *of all multivalued solutions of the system* (1) *into* W_0 *is a linear isomorphism.*

We firstly consider the corresponding local problem. Let D be the unit disc with center at 0 in $\mathbf{C}$.

LEMMA A.1.3. *Let* $X = D^n$ *and* U *a domain in* X. *Let* $\Phi \in \Gamma(U, \mathbb{S})$. *Then* Φ *extends to an element of* $\Gamma(X, \mathbb{S})$.

Proof. We may assume that $U = U_1 \times U_2 \times \cdots \times U_n$ where U_i is a domain in D for all $1 \leqslant i \leqslant n$. By induction in k we prove the following statement:
(E_k) There exists $\Phi_k \in \Gamma(D \times \cdots \times D \times U_k \times \cdots \times U_n, \mathbb{S})$ such that $\Phi_k \mid U = \Phi$.
This assertion is true trivially for $k = 1$, so say $k > 1$. The function Φ_k satisfies the differential equation

$$\partial_k \Phi_k = F_k \Phi_k$$

on $D \times D \times \cdots \times D \times U_k \times \cdots \times U_n$. Considering all variables z_i, $i \neq k$, as parameters and using the classical theorem on first order systems of ordinary differential equations depending on parameters [7], we see that Φ_k extends to a holomorphic function on $D \times \cdots \times D \times U_{k+1} \times \cdots \times U_n$. We denote this function by Φ_{k+1}; it obviously satisfies our system (1). This proves the induction step. The statement (E_{n+1}) is exactly the statement of the above lemma. Q.E.D.

Now we can prove A.1.2. The map $\Phi \to \Phi(\tilde{x}_0)$ is injective by A.1.1. Put

$$T = \{(x,\varphi) \mid \varphi \in \mathbb{S}_x,\, x \in X\}.$$

Let $\pi : T \to X$ be the projection defined by $\pi(x,\varphi) = x$ for $\varphi \in \mathbb{S}_x$, $x \in X$.

For an open subset U in X and a solution $\Phi \in \Gamma(U, \mathbb{S})$ we define a subset $S(U,\Phi)$ of T by

$$S(U,\Phi) = \{(y,\varphi) \mid y \in U,\, \varphi \text{ the germ of } \Phi \text{ at } y\}.$$

Let $\mathfrak{B}$ be the family of all such $S(U,\Phi)$. It is easy to see that $\mathfrak{B}$ is the basis of a topology on T. In the following we consider T to be endowed with this topology.

It is clear that $\pi : T \to X$ is continuous. We claim that it is a covering projection. Let $x \in X$. Let $\epsilon > 0$ be such that $U = \{y \in \mathbf{C}^n \mid |y_i - x_i| < \epsilon,\ 1 \leqslant i \leqslant n\}$ is contained in X. Let $y \in U$ and $\varphi \in \mathbb{S}_y$. Then by A.1.3 there exists $\Phi \in \Gamma(U, \mathbb{S})$ such that φ is its germ. This implies that $\pi^{-1}(U)$ is the disjoint union of open sets $S(U,\Phi)$, $\Phi \in \Gamma(U,\mathbb{S})$. The map π induces a homeomorphism of $S(U,\Phi)$ onto U for every $\Phi \in \Gamma(U,\mathbb{S})$. Therefore π is a covering projection.

Let $w \in W_0$. There exists $\varphi \in \mathbb{S}_{x_0}$ such that $\varphi(x_0) = w$. Let T_1 be the connected component of T containing (x_0, φ) and $\pi_1 = \pi \mid T_1$. Then $\pi_1 : T_1 \to X$ is a covering projection and T_1 is a connected covering space of X. By the universal property of $\tilde{X}$ there exists a covering map $p_1 : \tilde{X} \to T_1$ such that $p = \pi_1 \circ p_1$ and $p_1(\tilde{x}_0) = (x_0, \varphi)$.

Let $e : T \to W$ be the evaluation map $e(y, \psi) = \psi(y)$. Then $\Phi = e \circ p_1$ is a multivalued solution of the system (1) and

$$\Phi(\tilde{x}_0) = e(p_1(\tilde{x}_0)) = e(x_0, \varphi) = w. \quad \text{Q.E.D.}$$

Let $E_0 = \mathrm{Hom}_{\mathbf{C}}(W_0, W)$. By A.1.2 there exists a E_0-valued holomorphic function S on $\tilde{X}$ such that

(i) $S(\tilde{x}_0)$ is the natural injection of W_0 into W,

(ii) for every $w \in W_0$ the function $x \to S(x)w$ on $\tilde{X}$ is a multivalued solution of the system (1).

The function S is called the *fundamental matrix* of the system (1).

The space $\Gamma(\tilde{X}, p^*\mathbb{S})$ of all multivalued solutions of the system (1) is invariant under the action of all monodromy tranformations T_γ^*, $\gamma \in \pi_1(X, x_0)$. By A.1.2 for each $\gamma \in \pi_1(X, x_0)$ there exists a unique linear map $M_\gamma \in \mathrm{End}_{\mathbf{C}}(W_0)$ such that

$$T_\gamma^* S = S \circ M_\gamma.$$

It is easy to check that $\gamma \to M_\gamma$ is a representation of $\pi_1(X, x_0)$ on W_0. We call this representation M of $\pi_1(X, x_0)$ the *monodromy* of the system (1).

The system (1) is called *integrable* if $W_0 = W$. By the Frobenius theorem [9, 10.9] this is equivalent to

$$\partial_i F_j - F_i F_j = \partial_j F_i - F_j F_i$$

for all $i, j = 1, 2, \ldots, n$.

Now we consider a special case of the system (1). Let $D^* = D \setminus \{0\}$.

Let $X = D^n$, $X^* = (D^*)^k \times D^{n-k}$ and $Y = X \setminus X^* = \{x \in X \mid x_i = 0 \text{ for some } 1 \leqslant i \leqslant k\}$ where $1 \leqslant k \leqslant n$. We study the system

$$\partial_i \Phi = F_i \Phi, \qquad i = 1, 2, \ldots, n, \tag{2}$$

where F_i, $1 \leqslant i \leqslant n$, are holomorphic E-valued functions on X^*.

We denote by $H = \{z \in \mathbf{C} \mid \mathrm{Re}\, z < 0\}$ the left half-plane in $\mathbf{C}$. Then we can identify the universal covering space $\tilde{X}^*$ of X^* with $H^k \times D^{n-k}$ via the covering projection $p : \tilde{X}^* \to X^*$ given by

$$p(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_k}, x_{k+1}, \ldots, x_n)$$

for $x \in \tilde{X}^*$.

The homotopy group $\pi_1(X^*, x_0)$ is isomorphic to $\mathbf{Z}^k$. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be its generators corresponding to the counter-clockwise loops around the coordinate hyperplanes $\{z_j = 0\}$, $1 \leqslant j \leqslant k$, in X^*. Let $T_j = T_{\gamma_j}$ be the covering transformation corresponding to γ_j, $1 \leqslant j \leqslant k$. Then

$$T_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_j + 2\pi i, x_{j+1}, \ldots, x_n)$$

for $x \in \tilde{X}^*$.

Let M be the monodromy of the system (2). We put $M_j = M_{\gamma_j}$ for $1 \leqslant j \leqslant k$. It is easy to see that we can find a commuting family of linear maps $R_1, R_2, \ldots, R_k \in \operatorname{End}_{\mathbf{C}}(W_0)$ such that

$$M_j = \exp(-2\pi i R_j), \qquad 1 \leqslant j \leqslant k.$$

Then the function

$$(x_1, \ldots, x_n) \to S(x_1, \ldots, x_n)\exp(-(x_1 R_1 + \cdots + x_k R_k))$$

on $\tilde{X}^*$ is invariant under all monodromy transformations T^*_γ, $\gamma \in \pi_1(X^*, x_0)$. Therefore it defines an E_0-valued holomorphic function P on X^*. Hence the fundamental matrix S has the form

$$S(x_1, \ldots, x_n) = P(p(x_1, \ldots, x_n))\exp(x_1 R_1 + \cdots + x_k R_k)$$

for $x \in \tilde{X}^*$.

If we denote by $z^{\mathbf{R}}$ the multivalued function

$$(x_1, \ldots, x_n) \to \exp(x_1 R_1 + \cdots + x_k R_k)$$

on X^* we can write formally that

$$S(z) = P(z) z^{\mathbf{R}}. \tag{3}$$

Now we shall extend a classical result [7, Ch. 4, Theorem 2.1] on differential equations with regular singularities. We study the following system of first order differential equations

$$\begin{aligned} z_j \partial_j \Phi &= F_j \Phi, \qquad 1 \leqslant j \leqslant k, \\ \partial_j \Phi &= F_j \Phi, \qquad k+1 \leqslant j \leqslant n, \end{aligned} \tag{4}$$

on X^*, where $F_1, F_2, \ldots, F_n$ are holomorphic E-valued functions on X. We say that this system has *simple singularities* along Y. It is obvious that it defines a system of type (2); therefore by the previous discussion its fundamental matrix has the form $S = P \cdot z^{\mathbf{R}}$.

The systems with simple singularities are generalizations of the so-called systems with "singularities of the first kind" in the one-dimensional case. In this case, by the classical result we mentioned, all solutions have "moderate growth" near the singularity, i.e. the singularity is regular.

The result of Deligne [8] we mentioned at the beginning of the appendix generalizes this statement, i.e. we have:

THEOREM A.1.4. *The fundamental matrix of the system* (4) *has the form*

$$S(z) = P(z)z^{\mathbf{R}}$$

where P *is a holomorphic* E_0*-valued function on* X *and* $R_1, \ldots, R_k$ *is a commuting family of linear maps from* $\mathrm{End}_{\mathbf{C}}(W_0)$.

If our system (4) is integrable the proof of the above statement, as remarked by Deligne, reduces easily to the corresponding classical result for the one-dimensional case. But here it seems more suitable to repeat the classical argument to prove the moderate growth of P in (3) when z approaches Y [7, ibid.], which extends easily to our situation. (A similar argument was given by N. Wallach in [21].) Let $0 < \rho < 1$. We put $X_\rho = \{z \in X \mid |z_i| \leqslant \rho\}$ and $X_\rho^* = X_\rho \cap X^*$. We claim that there exists $\mathbf{m} \in \mathbf{Z}_+^k$ such that the function

$$z \to z^{\mathbf{m}} P(z)$$

is bounded on X_ρ^*.

We put

$$T = \{x \in \tilde{X}^* \mid \mathrm{Re}\, x_i \leqslant \log\rho,\ 0 \leqslant \mathrm{Im}\, x_i \leqslant 2\pi,$$
$$1 \leqslant i \leqslant k, \text{ and } |x_i| \leqslant \rho,\ k+1 \leqslant i \leqslant n\}$$

and

$$T_0 = \{x \in \tilde{X}^* \mid \mathrm{Re}\, x_i = \log\rho,\ 0 \leqslant \mathrm{Im}\, x_i \leqslant 2\pi,$$
$$1 \leqslant i \leqslant k, \text{ and } |x_i| \leqslant \rho,\ k+1 \leqslant i \leqslant n\}.$$

Let $\|\cdot\|$ be a differentiable norm on W (i.e. such that it is a differentiable function on $W \setminus \{0\}$). If Φ is a nontrivial multivalued solution of (4) on X^*, the function $\|\Phi\|$ is a positive differentiable function on $\tilde{X}^*$ considered as an open subset of $\mathbf{R}^{2n}$. Let $t_i = \mathrm{Re}\, x_i$. Then for $1 \leqslant i \leqslant k$ we have

$$\left| \frac{\partial}{\partial t_i} \|\Phi\| \right| \leqslant \left\| \frac{\partial \Phi}{\partial x_i} \right\| = \left\| z_i \frac{\partial \Phi}{\partial z_i} \right\| \leqslant C_0 \|\Phi\|$$

on T, where $C_0 \geqslant 0$ satisfies

$$\|F_i(z)\| \leqslant C_0$$

for $1 \leqslant i \leqslant k$ and $z \in X_\rho$. Therefore

$$\left| \frac{\partial}{\partial t_i} \log\|\Phi\| \right| \leqslant C_0$$

on T. By the compactness of T_0 there exists $C_1 \geqslant 0$ such that $\log\|\Phi\| \leqslant C_1$ on T_0.

It follows that

$$\log\|\Phi(x)\| \leqslant C_0 \cdot \sum_{1 \leqslant i \leqslant k} |\mathrm{Re}\, x_i| + C_1$$

for $x \in T$. Therefore there exists $C_2, C_3 \geqslant 0$ such that

$$\|S(x)\| \leqslant C_2 e^{C_3(|\mathrm{Re}\, x_1| + |\mathrm{Re}\, x_2| + \cdots + |\mathrm{Re}\, x_k|)}$$

for $x \in T$. Also we can find $C_4, C_5 \geqslant 0$ such that

$$\|e^{-x_1 R_1 - \cdots - x_k R_k}\| \leqslant e^{|x_1|\,\|R_1\| + \cdots + |x_k|\,\|R_k\|}$$

$$\leqslant C_4 e^{C_5(|\mathrm{Re}\, x_1| + |\mathrm{Re}\, x_2| + \cdots + |\mathrm{Re}\, x_k|)}$$

for $x \in T$. This yields

$$\|P(p(x))\| \leqslant M e^{m(|\mathrm{Re}\, x_1| + \cdots + |\mathrm{Re}\, x_k|)}, \qquad x \in T;$$

for sufficiently large M and $m \in \mathbf{Z}_+$. Our earlier claim follows from the obvious fact that $p(T) = X_\rho^*$.

From the Riemann extension theorem [10] and the Hartogs extension theorem (see A.1.8 below) it follows that the function

$$z \to z^{\mathbf{m}} P(z)$$

entends to a holomorphic function on X. In fact, in our case we need only the following very elementary result.

LEMMA A.1.5. *Let f be a holomorphic function on $(D^*)^k \times D^{n-k}$ bounded in a neighborhood of the origin. Then it extends to a holomorphic function on D^n.*

Proof. We prove the statement by induction in k. For $k = 0$ there is nothing to prove.

Assume that the assertion holds for $k - 1$. Let f be a holomorphic function on $(D^*)^k \times D^{n-k}$ bounded on $\{z \in \mathbf{C}^n \mid 0 < |z_i| \leqslant \rho,\ 1 \leqslant i \leqslant k,\ |z_i| \leqslant \rho,\ k+1 \leqslant i \leqslant n\}$. We can expand f into the Laurent series

$$f(z) = \sum_{\mathbf{m} \in \mathbf{Z}^k \times \mathbf{Z}_+^{n-k}} c_{\mathbf{m}} z^{\mathbf{m}}.$$

If we fix $z_1, \ldots, z_{k-1}$ so that $0 < |z_i| \leqslant \rho$ for $1 \leqslant i \leqslant k-1$ and $z_{k+1}, \ldots, z_n$ so that $|z_i| \leqslant \rho$ for $k+1 \leqslant i \leqslant n$ then

$$g(z) = \sum_{m_k} \Big(\sum c_{\mathbf{m}} z_1^{m_1} \ldots z_{k-1}^{m_{k-1}} z_{k+1}^{m_{k+1}} \ldots z_n^{m_n} \Big) z^{m_k}$$

is a holomorphic function on D^* bounded near the origin. Therefore it extends to a holomorphic function on D. This easily implies $c_{\mathbf{m}} = 0$ for $m_k < 0$. Therefore f extends to $(D^*)^{k-1} \times D^{n-k+1}$, and by the induction assumption to D^n. Q.E.D.

Now we shall reformulate A.1.4 to describe the form of the solutions of the system (4).

We say that the elements $\mathbf{s}, \mathbf{r} \in \mathbf{C}^k$ are *integrally equivalent* if $\mathbf{s} - \mathbf{r} \in \mathbf{Z}^k$.

For $\mathbf{s} \in \mathbf{C}^k$ and $\mathbf{m} \in \mathbf{Z}_+^k$ we denote by $z^{\mathbf{s}}\log^{\mathbf{m}} z$ the multivalued function

$$(x_1, \ldots, x_n) \to \exp(s_1 x_1 + \cdots + s_k x_k) x_1^{m_1} \ldots x_k^{m_k},$$

on X^*.

It is easy to see that the matrix coefficients of the multivalued function $z^{\mathbf{R}}$ are linear combinations of functions $z^{\mathbf{s}}\log^{\mathbf{m}} z$ with $\mathbf{s} \in \mathbf{C}^k$, $\mathbf{m} \in \mathbf{Z}_+^k$. Therefore A.1.4 implies that for a multivalued solution Φ of the sytem (4) there exist

(i) a finite subset S of $\mathbf{C}^k$ such that no elements of S are integrally equivalent;

(ii) For each $\mathbf{s} \in S$ a finite family of nontrivial W-valued holomorphic functions $\Phi_{\mathbf{s},\mathbf{m}}$, $(\mathbf{m} \in \mathbf{Z}_+^k)$, on D^n such that on each coordinate hyperplane $Y_j = \{z \in \mathbf{C}^n \mid z_j = 0\}$, $1 \leqslant j \leqslant k$, at least one of them is not identically zero, with

$$\Phi = \sum \Phi_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}} \log^{\mathbf{m}} z$$

on X^*. We call the above representation of Φ a *canonical form* of the solution Φ.

PROPOSITION A.1.6. *Every multivalued solution Φ of the system* (4) *has a unique canonical form.*

We have remarked already that a canonical form always exists. The uniqueness follows from a weak form of the next result. In its strongest form it is critical in the understanding of the "asymptotics along the walls" (see Section 6).

LEMMA A.1.7. *Let U be an open subset of X intersecting $\{0\} \times D^{n-k}$. Suppose that Φ is a multivalued W-valued holomorphic function on X^* and assume that there exist*

(i) *a finite set S of integrally inequivalent elements of $\mathbf{C}^k$,*

(ii) *for each $\mathbf{s} \in S$ a finite family of W-valued holomorphic functions $\Phi_{\mathbf{s},\mathbf{m}}$, $(\mathbf{m} \in \mathbf{Z}_+^k)$, on U with*

$$\Phi = \sum \Phi_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}} \log^{\mathbf{m}} z$$

on $p^{-1}(U)$.

Then

(a) *the functions $\Phi_{\mathbf{s},\mathbf{m}}$ extend to holomorphic functions on D^n,*

(b) *the above formula holds on whole X^* and it determines the functions $\Phi_{\mathbf{s},\mathbf{m}}$ uniquely.*

To prove the above statement we study the action of covering transformations on Φ and use the following simple version of the Hartogs extension theorem.

Let X be an open subset of $\mathbf{C}^n$ and Y a proper closed analytic subset of X. Let Y_{ns} be the subset of Y where Y is nonsingular and of codimension one. The complement of Y_{ns} in Y is a closed analytic subset of codimension $\geqslant 2$ in X [10, p. 115]. Let $X^* = X \setminus Y$.

LEMMA A.1.8. *Let U be an open set of X which intersects every connected component of Y_{ns} and f a holomorphic function on $X^* \cup U$. Then f extends to a holomorphic function on X.*

First we prove a special, nearly trivial case of the above result.

LEMMA A.1.9. *Let $X = D \times Y$ where Y is a domain in $\mathbf{C}^{n-1}$, $X^* = D^* \times Y$ and U a nonempty open subset of X intersecting $\{0\} \times Y$. If f is a holomorphic function on $X^* \cup U$ it extends to a holomorphic function on X.*

Proof. The function f can be expanded in the Laurent series

$$f(z, y) = \sum_{k=-\infty}^{\infty} a_k(y) z^k, \qquad z \in D^*, \quad y \in Y,$$

on $D^* \times Y$. The coefficients a_k, $k \in \mathbf{Z}$, are holomorphic functions on Y; if γ is a positively oriented loop around the origin in D^* we have

$$a_k(y) = \frac{1}{2\pi i} \int_\gamma f(z, y) z^{-k-1}\, dz, \qquad y \in Y,$$

for every $k \in \mathbf{Z}$. Obviously, $a_k(y) = 0$ for all $y \in Y$ such that $(0, y) \in U$ and $k < 0$. The region Y being connected, it follows that $a_k = 0$ for $k < 0$. Q.E.D.

Now we can prove A.1.8. Let V be the set of all $y \in Y_{\mathrm{ns}}$ such that there exists an open neighborhood U_1 of y in X such that $f | U_1 \cap X^*$ extends to a holomorphic function on U_1. It is obvious that V is open in Y_{ns} and that it contains $U \cap Y_{\mathrm{ns}}$. By A.1.9 it is also closed in Y_{ns}. Our assumption therefore implies that $V = Y_{\mathrm{ns}}$. Hence f extends to a holomorphic function on $X^* \cup Y_{\mathrm{ns}}$. Since the complement of Y_{ns} in Y has codimension $\geqslant 2$ in X, the classical Riemann extension theorem [10, I.C.8] implies the assertion of A.1.8.

Now we can prove A.1.7.

We put

$$|\mathbf{m}| = m_1 + m_2 + \cdots + m_k,$$

$$\mathbf{m}! = m_1!\, m_2! \ldots m_k!$$

for $\mathbf{m} \in \mathbf{Z}_+^k$;

$$\mathbf{m} \geqslant \mathbf{n} \quad \text{if} \quad m_j \geqslant n_j, \quad 1 \leqslant j \leqslant k,$$

for $\mathbf{m}, \mathbf{n} \in \mathbf{Z}_+^k$; and

$$\mathbf{ms} = m_1 s_1 + m_2 s_2 + \cdots + m_k s_k$$

for $\mathbf{s} \in \mathbf{C}^k$, $\mathbf{m} \in \mathbf{Z}_+^k$. Also we denote by

$$\mathbf{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

the jth standard basis vector in $\mathbf{C}^k$.

For $\mathbf{s} \in \mathbf{C}^k$ and $\mathbf{m} \in \mathbf{Z}_+^k$ the covering transformation T_j corresponding to the loop γ_j, $1 \leqslant j \leqslant k$, acts so that

$$\left(T_j^* - e^{-2\pi i s_j}\right) z^{\mathbf{s}} \log^{\mathbf{m}} z \equiv -2\pi i m_j e^{-2\pi i s_j} z^{\mathbf{s}} \log^{\mathbf{m}-\mathbf{e}_j} z$$

modulo terms involving $z^{\mathbf{s}} \log^{\mathbf{k}} z$, $\mathbf{k} \leqslant \mathbf{m} - 2\mathbf{e}_j$. This immediately implies that

$$\left[\prod_{j=1}^{k} \left(T_j^* - e^{-2\pi i s_j}\right)^{m_j}\right] z^{\mathbf{s}} \log^{\mathbf{m}} z = (-2\pi i)^{|\mathbf{m}|} \mathbf{m}!\, e^{-2\pi i \mathbf{m} \mathbf{s}} z^{\mathbf{s}} \tag{5}$$

and

$$\left[\prod_{j=1}^{k} \left(T_j^* - e^{-2\pi i s_j}\right)^{m_j}\right] z^{\mathbf{s}} \log^{\mathbf{n}} z = 0 \tag{6}$$

if $\mathbf{n}$ is not greater or equal to $\mathbf{m}$.

For $\mathbf{s} \in S$ we denote by $M(\mathbf{s})$ the set of $\mathbf{m} \in \mathbf{Z}_+^k$ such that the term $\Phi_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}} \log^{\mathbf{m}} z$ appears in the expression for Φ on $p^{-1}(U)$.

To prove A.1.7 we use the induction in $\operatorname{Card} S$.

Firstly we assume that $\operatorname{Card} S = 1$. To prove the assertion in this case we use the induction in $\operatorname{Card} M(\mathbf{s})$. If $\operatorname{Card} M(\mathbf{s}) = 1$ the function $\Phi_{\mathbf{s},\mathbf{m}}$ is obviously unique and extends to a holomorphic function on $X^* \cup U$ given by $\Phi z^{-\mathbf{s}} \log^{-\mathbf{m}} z$. By A.1.8, $\Phi_{\mathbf{s},\mathbf{m}}$ extends to a holomorphic function on all of X.

Suppose that $\operatorname{Card} M(\mathbf{s}) > 1$. We can choose a maximal element $\mathbf{m}_0 \in M(\mathbf{s})$ By (5) and (6) we have on $p^{-1}(U)$

$$\left[\prod_{j=1}^{k} \left(T_j^* - e^{-2\pi i s_j}\right)^{m_{0j}}\right] \Phi = (-2\pi i)^{|\mathbf{m}_0|} \mathbf{m}_0!\, e^{-2\pi i \mathbf{m}_0 \mathbf{s}} \Phi_{\mathbf{s},\mathbf{m}_0} z^{\mathbf{s}},$$

and by applying the above conclusion we see that $\Phi_{\mathbf{s},\mathbf{m}_0}$ is unique and extends to a holomorphic function on all of X. Put

$$\Psi = \Phi - \Phi_{\mathbf{s},\mathbf{m}_0} z^{\mathbf{s}} \log^{\mathbf{m}_0} z,$$

then we can apply the induction hypothesis to Ψ. Therefore the assertion holds in the case $\operatorname{Card} S = 1$.

Suppose now that the assertion holds when $\operatorname{Card} S \leqslant p$. We prove that it then holds when $\operatorname{Card} S \leqslant p + 1$ by the induction in the number of terms in

$$\Phi = \sum \Phi_{\mathbf{s},\mathbf{m}} z^{\mathbf{s}} \log^{\mathbf{m}} z.$$

Fix two different $\mathbf{s}, \mathbf{t} \in S$. Then there exists $1 \leqslant j \leqslant k$ such that $s_j - t_j \notin \mathbf{Z}$. By (6) there exists $r \in \mathbf{Z}_+$ such that

$$\left(T_j^* - e^{-2\pi i t_j}\right)^r z^{\mathbf{t}} \log^{\mathbf{m}} z = 0$$

for all $\mathbf{m} \in M(\mathbf{t})$. Therefore the induction hypothesis applies to $(T_j^* - e^{-2\pi i t_j})^r \Phi$.

Evidently

$$\left(T_j^* - e^{-2\pi i t_j}\right) z^{\mathbf{s}} \log^{\mathbf{m}} z \equiv \left(e^{-2\pi i s_j} - e^{-2\pi i t_j}\right) z^{\mathbf{s}} \log^{\mathbf{m}} z$$

modulo terms involving $z^{\mathbf{s}} \log^{\mathbf{k}} z$, $\mathbf{k} \leqslant \mathbf{m} - \mathbf{e}_j$. Therefore if $\mathbf{m}_0$ is a maximal element of $M(\mathbf{s})$, the coefficient of $z^{\mathbf{s}} \log^{\mathbf{m}_0} z$ in $(T_j^* - e^{-2\pi i t_j})^r \Phi$ is equal to $(e^{-2\pi i s_j} - e^{-2\pi i t_j})^r \Phi_{\mathbf{s},\mathbf{m}_0}$. It follows that $\Phi_{\mathbf{s},\mathbf{m}_0}$ is unique and extends to a holomorphic function on all of X.

Now we can put

$$\Psi = \Phi - \Phi_{\mathbf{s},\mathbf{m}_0} z^{\mathbf{s}} \log^{\mathbf{m}_0} z$$

and apply the induction hypothesis again. This proves A.1.7.

2. A technical result. In this section we prove a technical result which is needed critically in Section 7. It is almost self-evident and must be well known, but we do not know a suitable reference.

Let $\Phi_{s,m}$, $s \in \mathbf{C}$, $m \in \mathbf{Z}_+$, be a finite family of smooth functions on $[0, 1)$ such that $\Phi_{s,m}(0) \neq 0$. Put

$$\Phi(x) = \sum_{s,m} \Phi_{s,m} x^s \log^m x$$

for $x \in (0, 1)$. Let $0 < \eta < 1$.

PROPOSITION A.2.1. (i) *Suppose there exist $l \in \mathbf{R}$ and $q \geqslant 0$ such that for some $C > 0$ we have*

$$|\Phi(x)| \leqslant C x^l (1 + |\log x|)^q, \qquad x \in (0, \eta].$$

Then we have

$$\operatorname{Re} s \geqslant l$$

for all s, and if $\Phi_{s,m}$ appears in the above formula for s such that $\operatorname{Re} s = l$ we have $m \leqslant q$.

(ii) *If there exists $l \in \mathbf{R}$ such that*

$$\lim_{x \to 0} x^{-l} \Phi(x) = 0$$

we have $\operatorname{Re} s > l$ for all s.

(iii) *If Φ is an element of $L_p((0, \eta], dx/x)$ for some $1 \leqslant p < +\infty$, then $\operatorname{Re} s > 0$ for all s.*

By Taylor's theorem it is enough to prove the above statement in the case when $\Phi_{s,m}$ are polynomials. We may assume that $\eta = 1$, and we rewrite Φ in the form

$$\Phi(x) = \sum P_s(\log x)x^s, \qquad (s \in S \subset \mathbf{C}),$$

where P_s are nonzero polynomials.

We start with a rather pretty lemma due to Harish-Chandra [22, A.3.2]. For the sake of completeness we reproduce its proof here.

LEMMA A.2.2. *Suppose that every $s \in S$ is purely imaginary and $c_s \in \mathbf{C}$. Then*

$$\limsup_{x \to 0} \left| \sum c_s x^s \right| \geqslant \left[\sum |c_s|^2 \right]^{1/2}.$$

Proof. Put $f(x) = \sum c_s x^s$, $x \in (0, 1]$. Then by direct computation we have

$$\lim_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \int_\epsilon^1 |f(x)|^2 \frac{dx}{x} = \sum |c_s|^2,$$

and if we put $M = \limsup_{x \to 0} |f(x)|$ it follows immediately that

$$\lim_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \int_\epsilon^1 |f(x)|^2 \frac{dx}{x} \leqslant M^2,$$

which implies our assertion. Q.E.D.

Now we can prove A.2.1 (i). By dividing with x^l we may assume at the beginning that $l = 0$. Let s_0 be an exponent in

$$\Phi(x) = \sum P_s(\log x)x^s$$

with the smallest real part. Suppose $t = \operatorname{Re} s_0 < 0$. Then

$$\lim_{x \to 0} x^{-t}\Phi(x) = 0.$$

Therefore

$$0 = \lim_{x \to 0} \left| \sum_{\operatorname{Re} s = t} P_s(\log x)x^{s-t} \right|.$$

Let $m_0 = \max\{\deg P_s \mid \operatorname{Re} s = t\}$ and c_{s,m_0} be the coefficient in P_s of the m_0th power, then

$$0 = \lim_{x \to 0} \log^{-m_0} x \left| \sum_{\operatorname{Re} s = t} P_s(\log x)x^{s-t} \right|$$

$$= \lim_{x \to 0} \left| \sum_{\operatorname{Re} s = t} c_{s,m_0} x^{s-t} \right|.$$

By applying A.2.2 we see that $c_{s,m_0} = 0$ for all s, contradicting the choice of m_0. Therefore $\operatorname{Re} s_0 \geqslant 0$.

Suppose $\operatorname{Re} s_0 = 0$. Then obviously we have for some $C_1 \geqslant 0$

$$\left|\sum_{\operatorname{Re} s=0} P_s(\log x)x^s\right| \leqslant C_1 \cdot (1 + |\log x|)^q$$

for $x \in (0, 1]$. Let m_0 and c_{s,m_0} be as above. Then, if $m_0 > q$, we have

$$\begin{aligned} 0 &= \lim_{x\to 0} \log^{-m_0} x\left|\sum_{\operatorname{Re} s=0} P_s(\log x)x^s\right| \\ &= \lim_{x\to 0}\left|\sum_{\operatorname{Re} s=0} c_{s,m_0}x^s\right|, \end{aligned}$$

and by A.2.2 this implies $c_{s,m_0} = 0$ for all s with $\operatorname{Re} s = 0$, contradicting the choice of m_0. Therefore $m_0 \leqslant q$, which proves (i).

To prove (ii) we remark first that by (i) we have $\operatorname{Re} s \geqslant l$. and $\operatorname{Re} s = l$ implies $\deg P_s = 0$. Therefore

$$0 = \lim_{x\to 0} x^{-l}|\Phi(x)| = \lim_{x\to 0}\left|\sum_{\operatorname{Re} s=l} P_s x^{s-l}\right|$$

what by A.2.2 implies that $P_s = 0$ for $\operatorname{Re} s = l$. Hence $\operatorname{Re} s > l$, which proves (ii).

It remains to prove (iii). We start with the observation that if $s \neq 0$ and P is a polynomial, then

$$\int_y^1 P(\log x)x^s \frac{dx}{x} = Q(\log y)y^s$$

where Q is again a polynomial. Let s_0 be as before. Suppose $\operatorname{Re} s_0 \leqslant 0$. Then

$$\Psi(y) = \int_y^1 \Phi(x)x^{-s_0}\frac{dx}{x}$$

has, by the above remark, the form

$$\Psi(x) = Q_{s_0}(\log x) + \sum_{s\neq s_0} Q_s(\log x)x^{s-s_0}$$

where $\deg Q_{s_0} = \deg P_{s_0} + 1$, i.e. Q_{s_0} is certainly not a constant.

Let $1 < q \leqslant +\infty$ be such that $(1/p) + (1/q) = 1$. By the Hölder inequality we have

$$|\Psi(y)| \leqslant \int_y^1 |\Phi(x)|x^{-\operatorname{Re} s_0}\frac{dx}{x} \leqslant \int_y^1 |\Phi(x)|\frac{dx}{x} \leqslant \|\Phi\|_p \cdot |\log y|^{1/q}$$

for all $y \in (0, 1]$. By (i) this implies that $\deg Q_{s_0} = 0$, which is impossible. Therefore $\operatorname{Re} s_0 > 0$.

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