HOMOLOGICAL ALGEBRA OF $(\mathfrak{g}, K)$-MODULES

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1. ALGEBRAIC REPRESENTATIONS

1.1. **Algebraic representations.** Let G be an algebraic group defined over $\mathbb{C}$. Let $\mathcal{R}\text{ep}(G)$ be the category of representations of group G on complex vector spaces.

A representation (π, V) of G is called *algebraic*, if V is a union of finite-dimensional subrepresentations $(V_i, i \in I)$ such that the representations of G on V_i, $i \in I$, are given by algebraic morphisms of G into $\mathrm{GL}(V_i)$.

We denote by $\mathcal{AR}\text{ep}(G)$ the full subcategory of $\mathcal{R}\text{ep}(G)$ consisting of all algebraic representations in $\mathcal{R}\text{ep}(G)$. We have the following result.

1.1.1. **Lemma.** (i) *A subrepresentation of an algebraic representation is algebraic.*
(ii) *A quotient representation of an algebraic representation is algebraic.*
(iii) *The direct sum of two algebraic representations is algebraic.*

Proof. (i) Let (π, V) be an algebraic representation of G. Assume that V is a union of finite-dimensional G-invariant subspaces $(V_i; i \in I)$ such that the representations of G on V_i are given by algebraic morphisms of G into $\mathrm{GL}(V_i)$ for all $i \in I$. Let U be an G-invariant subspace of V. Put $U_i = V_i \cap U$ for $i \in I$. Then U_i are G-invariant subspaces of U. Clearly, U is the union of $(U_i; i \in I)$.

Let $i \in I$. Then U_i is a G-invariant subspace of V_i. Let P_i be the subgroup of $\mathrm{GL}(V_i)$ consisting of all maps leaving U_i invariant. This is a closed subgroup of $\mathrm{GL}(V_i)$. The representation of G on V_i is given by a morphism of G into $\mathrm{GL}(V_i)$ which factors through P_i. Let p_i be the natural restriction morphism from P_i into $\mathrm{GL}(U_i)$. Then the representation of G on U_i is given as a composition of the morphism of G into P_i with the restriction morphism p_i. Therefore, the representation on U_i is algebraic.

(ii) Let (ρ, W) be the representation of G on $W = V/U$. Denote by ϕ the natural projection of V onto W. Let $W_i = \phi(V_i)$, $i \in I$. Clearly, W_i are finite-dimensional G-invariant subspaces of W. Moreover, W is the union of all W_i, $i \in I$. Let q_i be the natural morphisms from P_i into $\mathrm{GL}(W_i)$. The representation of G on W_i is given as a composition of the morphism of G into P_i with the morphism q_i. Therefore, the representation (ρ, W) is algebraic.

(iii) Let (ν, U) and (π, V) be two algebraic representations. Assume that U is a union of finite-dimensional G-invariant subspaces $(U_i; i \in I)$ and V a union of finite-dimensional G-invariant subspaces $(V_j; i \in J)$ such that the representations of G on U_i, $i \in I$, and V_j, $j \in J$ respectively, are given by algebraic morphisms of G into $\mathrm{GL}(U_i)$ and $\mathrm{GL}(V_j)$ respectively. Then, $U \oplus V$ is a union of G-invariant subspaces $(U_i \oplus U_j; (i,j) \in I \times J)$ and the representations of G on $U_i \oplus V_j$ are given by the algebraic morphisms of G into $\mathrm{GL}(U_i) \times \mathrm{GL}(V_j) \subset \mathrm{GL}(U_i \oplus V_j)$. Therefore, $(\nu \oplus \pi, U \oplus V)$ is an algebraic representation. □

This implies that $\mathcal{AR}\mathrm{ep}(G)$ is a abelian subcategory of $\mathcal{R}\mathrm{ep}(G)$.

1.2. **The largest algebraic subrepresentation.** Let (π, V) be a representation of G. A vector $v \in V$ is called *algebraic* if there exists an algebraic subrepresentation of π containing v.

1.2.1. **Lemma.** *The set of all algebraic vectors in V is a G-invariant vector subspace.*

Proof. Let v be an algebraic vector and $\alpha \in \mathbb{C}$. Then there exists a subrepresentation (ν, U) of π which is algebraic and contains vector v. Clearly $\alpha v \in U$. Hence, αv is algebraic.

Let v_1 and v_2 be two algebraic vectors. Then there exist two algebraic subrepresentations (ν_1, U_1) and (ν_2, U_2) of π such that $v_1 \in U_1$ and $u_2 \in U_2$. The sum $U_1 + U_2$ is an G-invariant subspace of V containing $v_1 + v_2$. It is a quotient of the algebraic representation on $U_1 \oplus U_2$. By 1.1.1, the representation on $U_1 + U_2$ is algebraic. Therefore, $v_1 + v_2$ is an algebraic vector.

Let v be an algebraic vector. Then there exists a finite-dimensional G-invariant subspace U of V containing v such that the representation of G on U is given by a morphism of G into $\mathrm{GL}(U)$. By definition, all vectors in U are algebraic. In particular, $\pi(g)v$ is algebraic for $g \in G$. Hence, the set of all algebraic vectors if G-invariant. □

We denote the vector space of all algebraic vectors in V by $V_{[G]}$. The representation of G on $V_{[G]}$ is denoted by $\pi_{[G]}$. This representation is clearly algebraic.

1.2.2. **Lemma.** *Let (ν, U) be a subrepresentation of (π, V). If (ν, U) is algebraic, $U \subset V_{[G]}$.*

Proof. Let $u \in U$. Since ν is algebraic, u is an algebraic vector in V. Hence $u \in V_{[G]}$. □

Therefore, we call $(\pi_{[G]}, V_{[G]})$ the *largest algebraic subrepresentation* of (π, V). Clearly, (π, V) is an algebraic representation if and only if $V_{[G]} = V$.

1.2.3. **Lemma.** *Let (ν, U) and (π, V) be two representations of G and $\phi : U \longrightarrow V$ a morphism of representations. Let $u \in U$ be an algebraic vector. Then $\phi(u)$ is an algebraic vector in V.*

Proof. By restricting ϕ to the largest algebraic subrepresentation of π, we can reduce the proof to the case where (ν, U) is algebraic. By taking a quotient by the kernel of ϕ, by 1.1.1, we can assume that ϕ is injective. In this case, the result is obvious. □

Therefore, the morphism $\phi : U \longrightarrow V$ maps $U_{[G]}$ into $V_{[G]}$ and the restriction of ϕ defines a morphism of $\nu_{[G]}$ into $\pi_{[G]}$. It is straightforward to check that in this way we get an additive functor $\pi \longmapsto \pi_{[G]}$ from $\mathcal{R}\mathrm{ep}(G)$ into $\mathcal{AR}\mathrm{ep}(G)$ which we call the *algebraization* functor.

1.2.4. **Theorem.** *The algebraization functor from $\mathcal{R}\mathrm{ep}(G)$ into $\mathcal{AR}\mathrm{ep}(G)$ is a right adjoint of the forgetful functor from $\mathcal{AR}\mathrm{ep}(G)$ to $\mathcal{R}\mathrm{ep}(G)$.*

Proof. Let (ν, U) be an algebraic representation and (π, V) a representation of G. Then for $\phi : U \longrightarrow V$, it follows that $\operatorname{im}\phi(U) \subset V_{[G]}$ by 1.2.3. Therefore, it follows that

$$\operatorname{Hom}_G(U, V) = \operatorname{Hom}_G(U, V_{[G]}).$$

□

Therefore, we have the following consequence.

1.2.5. **Corollary.** *The algebraization functor from $\mathcal{R}\mathrm{ep}(G)$ into $\mathcal{AR}\mathrm{ep}(G)$ is left exact.*

1.3. **The bifunctor** $\operatorname{Hom}_{\mathbb{C}}(-,-)_{[G]}$**.** Let (ν, U) and (π, V) be two algebraic representations of G. Then the space of all linear maps $\operatorname{Hom}_{\mathbb{C}}(U, V)$ from U into V is a representation of G. The action of G is given by

$$\rho(g)T = \pi(g)T\nu(g^{-1})$$

for $g \in G$ and $T \in \operatorname{Hom}_{\mathbb{C}}(U, V)$.

Therefore, we get an algebraic representation $\operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$. This is clearly a bifunctor, contravariant in the first variable and covariant in the second. Since $\operatorname{Hom}_{\mathbb{C}}(-,-)$ is exact in both variables, $\operatorname{Hom}_{\mathbb{C}}(-,-)_{[G]}$ is a functor from $\mathcal{AR}\mathrm{ep}(G)^{opp} \times \mathcal{AR}\mathrm{ep}(G)$ into $\mathcal{AR}\mathrm{ep}(G)$ which is left exact in both variables.

By differentiating the representations ν and π we get representations of the Lie algebra $\mathfrak{g}$ on U and V respectively. By abuse of notation, we denote them by the same letters. Then we have the natural representation ρ of $\mathfrak{g}$ on $\operatorname{Hom}_{\mathbb{C}}(U, V)$ given by

$$\rho(X)T = \pi(X)T - T\nu(X)$$

for any $X \in \mathfrak{g}$ and $T \in \operatorname{Hom}_{\mathbb{C}}(U, V)$.

By differentiating the algebraic representation of G on $\operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$ we get a representation of the Lie algebra $\mathfrak{g}$ on $\operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$.

The next result implies that these two actions of $\mathfrak{g}$ on $\operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$ coincide.

1.3.1. **Proposition.** *The differential of the algebraic representation $\rho_{[G]}$ of G on $\operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$ is given by*

$$\rho_{[G]}(X)T = \pi(X)T - T\nu(X)$$

for any $X \in \mathfrak{g}$ and $T \in \operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$.

Proof. Let $u \in U$ and $T \in \operatorname{Hom}_{\mathbb{C}}(U, V)_{[G]}$. consider the function $\omega : G \longrightarrow V$ given by $\omega(g) = \pi(g)(T(\nu(g^{-1})u))$ for $g \in G$. This function is regular by our assumption. It is a composition of the diagonal imbedding $G \longrightarrow G \times G$ with the map from $G \times G$ into V given by $(g, h) \longmapsto \pi(g)(T(\nu(h^{-1})u))$. Both maps are regular, hence by the chain rule we get that

$$\rho_{[G]}(X)u = d\omega(X) = \pi(X)Tu - T\nu(X)u$$

for any $X \in \mathfrak{g}$. □

Assume now that G is a reductive algebraic group. Then the category $\mathcal{AR}\mathrm{ep}(G)$ is semisimple.

1.3.2. **Lemma.** *Let G be a reductive algebraic group. Then the functor $\operatorname{Hom}_{\mathbb{C}}(-,-)_{[G]}$ from $\mathcal{AR}\mathrm{ep}(G)^{opp} \times \mathcal{AR}\mathrm{ep}(G)$ into $\mathcal{AR}\mathrm{ep}(G)$ is exact in both variables.*

Proof. First, we prove the exactness in the first variable. Let

$$0 \longrightarrow U' \xrightarrow{i} U \longrightarrow U'' \longrightarrow 0$$

be an exact sequence in $\mathcal{AR}\mathrm{ep}(G)$. since G is reductive, it splits. Let $P : U \longrightarrow U'$ be the splitting morphism, i.e., $P \circ i = 1_{U'}$. Clearly, we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U'', V) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U', V) \longrightarrow 0$$

of representations of G. The morphism $S \longmapsto S \circ P$ defines a splitting of this short exact sequence.

Also,

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U'', V)_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U', V)_{[G]}$$

is exact. Let $A \in \mathrm{Hom}_{\mathbb{C}}(U', V)_{[G]}$. Then $A \circ P$ is in $\mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]}$ and $A \circ P \circ i = A$, i.e., A is in the image of the morphism $\mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U', V)_{[G]}$. Therefore, the functor $\mathrm{Hom}_{\mathbb{C}}(-, V)_{[G]}$ is exact.

Now, we prove the exactness in the second variable. Let

$$0 \longrightarrow V' \longrightarrow V \xrightarrow{p} V'' \longrightarrow 0$$

be an exact sequence in $\mathcal{AR}\mathrm{ep}(G)$. since G is reductive, it splits. Let $Q : V'' \longrightarrow V$ be the splitting morphism, i.e., $p \circ Q = 1_{V''}$. Clearly, we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V'') \longrightarrow 0$$

of representations of G. The morphism $T \longmapsto Q \circ T$ defines a splitting of this short exact sequence.

Also,

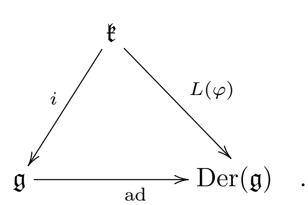
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V')_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V'')_{[G]}$$

is exact. Let $B \in \mathrm{Hom}_{\mathbb{C}}(U, V'')_{[G]}$. Then $Q \circ B$ is in $\mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]}$ and $p \circ Q \circ B = B$, i.e., B is in the image of the morphism $\mathrm{Hom}_{\mathbb{C}}(U, V)_{[G]} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V'')_{[G]}$. Therefore, the functor $\mathrm{Hom}_{\mathbb{C}}(U, -)_{[G]}$ is exact. □

2. Category of $(\mathfrak{g}, K)$-modules

2.1. **Harish-Chandra pairs.** Let $\mathfrak{g}$ be a complex Lie algebra. Let $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of all derivations of $\mathfrak{g}$. Denote by $\mathrm{Aut}(\mathfrak{g})$ the automorphism group of $\mathfrak{g}$ considered as an algebraic group. Then the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$. Let K be an algebraic group and $\mathfrak{k}$ the Lie algebra of K. Let $\varphi : K \longrightarrow \mathrm{Aut}(\mathfrak{g})$ be a morphism of algebraic groups. Then the differential $L(\varphi)$ is a Lie algebra morphism of $\mathfrak{k}$ into $\mathrm{Der}(\mathfrak{g})$. We assume that the following condition holds:

(HC) There exists a Lie algebra monomorphism $i : \mathfrak{k} \longrightarrow \mathfrak{g}$ such that the diagram



commutes.

Therefore, we can identify $\mathfrak{k}$ with the Lie subalgebra $i(\mathfrak{k})$ of $\mathfrak{g}$.

We call such pair $(\mathfrak{g}, K)$ a *Harish-Chandra pair*.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let $\operatorname{Aut}(\mathfrak{g}, \mathfrak{h})$ be the subgroup of $\operatorname{Aut}(\mathfrak{g})$ consisting of all automorphisms which leave $\mathfrak{h}$ invariant. The restriction of elements of $\operatorname{Aut}(\mathfrak{g}, \mathfrak{h})$ to $\mathfrak{h}$ defines a algebraic group morphism of $\operatorname{Aut}(\mathfrak{g}, \mathfrak{h})$ into $\operatorname{Aut}(\mathfrak{h})$. Let T an algebraic subgroup of K. Assume that

(a) $\mathfrak{t} \subset \mathfrak{h}$;
(b) $\varphi(T) \subset \operatorname{Aut}(\mathfrak{g}, \mathfrak{h})$.

Then φ is a morphism of T into $\operatorname{Aut}(\mathfrak{g}, \mathfrak{h})$. By the above remark, by restriction, this defines a morphism $\psi : T \longrightarrow \operatorname{Aut}(\mathfrak{h})$, and we have the following diagram

$$\begin{array}{ccc} T & \xrightarrow{\ \varphi\ } & \operatorname{Aut}(\mathfrak{g}, \mathfrak{h}) \\ & \searrow^{\psi} & \downarrow{\scriptstyle res} \\ & & \operatorname{Aut}(\mathfrak{h}) \end{array} \quad .$$

Denote by $\operatorname{Der}(\mathfrak{g}, \mathfrak{h})$ the Lie subalgebra of all derivations of $\mathfrak{g}$ which leave $\mathfrak{h}$ invariant. By differentiation we get the commutative diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\ L(\varphi)\ } & \operatorname{Der}(\mathfrak{g}, \mathfrak{h}) \\ & \searrow^{L(\psi)} & \downarrow{\scriptstyle res} \\ & & \operatorname{Der}(\mathfrak{h}) \end{array} \quad .$$

The linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ is invariant for adjoint representation $\operatorname{ad}_{\mathfrak{g}}$ restricted to $\mathfrak{h}$. Moreover, the representation of $\mathfrak{h}$ on this invariant subspace is equal to $\operatorname{ad}_{\mathfrak{h}}$. Therefore, we see that $L(\psi)$ agrees with $\operatorname{ad}_{\mathfrak{h}} : \mathfrak{t} \longrightarrow \operatorname{Der}(\mathfrak{h})$. So, it follows that $(\mathfrak{h}, T)$ is a Harish-Chandra pair which we call a *Harish-Chandra subpair* of $(\mathfrak{g}, K)$.

Assume that $(\mathfrak{g}, K)$ is a Harish-Chandra pair. Then, for any algebraic subgroup T of K, $(\mathfrak{g}, T)$ is a Harish-Chandra subpair of $(\mathfrak{g}, K)$.

Moreover, if $(\mathfrak{h}, T)$ is a Harish-Chandra subpair of $(\mathfrak{g}, K)$, Harish-Chandra subpair $(\mathfrak{g}, T)$ has a $(\mathfrak{h}, T)$ as its Harish-Chandra subpair.

2.2. $(\mathfrak{g}, K)$**-modules.** A $(\mathfrak{g}, K)$*-module* (π, V) is a complex linear space V with representation π of $\mathfrak{g}$ and algebraic representation of K (which, by abuse of notation, we denote by the same letter) such that the following holds

(HCM1) the differential of the action of K is equal to the action of $\mathfrak{k}$ as a Lie subalgebra of $\mathfrak{g}$;

(HCM2) we have

$$\pi(\varphi(k)\xi)v = \pi(k)\pi(\xi)\pi(k^{-1})v$$

for any $\xi \in \mathfrak{g}$, $k \in K$ and $v \in V$.

We call the *category of* $(\mathfrak{g}, K)$*-modules* the category $\mathcal{M}(\mathfrak{g}, K)$ whose objects are $(\mathfrak{g}, K)$-modules and morphisms linear maps which are morphisms of representations of $\mathfrak{g}$ and K.

Clearly, $\mathcal{M}(\mathfrak{g}, K)$ is an abelian category. If $\mathfrak{g} = \mathfrak{k}$, a $(\mathfrak{k}, K)$-module is actually an algebraic representation of K, since the action of $\mathfrak{k}$ is obtained by differentiation of the action of K. Therefore, $\mathcal{M}(\mathfrak{k}, K)$ can be identified with $\mathcal{M}(K)$. If $K = \{1\}$, $\mathcal{M}(\mathfrak{g}, K)$ can be identified with the category $\mathcal{M}(\mathfrak{g})$ of $\mathfrak{g}$-modules.

3. Change of Lie algebras functors

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. Let $(\mathfrak{h}, K)$ be a Harish-Chandra subpair, i.e., $\mathfrak{h}$ is Lie subalgebra of $\mathfrak{g}$ which contains $\mathfrak{k}$ and the action of K normalizes $\mathfrak{h}$. We have a natural forgetful functor For : $\mathcal{M}(\mathfrak{g}, K) \longrightarrow \mathcal{M}(\mathfrak{h}, K)$. This exact functor has a right adjoint functor and a left adjoint functor. First, we want to construct these functors.

3.1. **The inducing functor.** Let (ν, U) be a $(\mathfrak{h}, K)$-module. If we consider $\mathcal{U}(\mathfrak{g})$ as a right $\mathfrak{h}$-module for right multiplication, we can form the tensor product $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$. Consider the bilinear map $\alpha : \mathcal{U}(\mathfrak{g}) \times U \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ given by $\alpha(\xi, u) = \xi \otimes u$ for $\xi \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. It induces a linear map $\beta : \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$. The tensor product $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U$ with the tensor product of the representation φ on $\mathcal{U}(\mathfrak{g})$ and the representation ν on U is an algebraic representation of K. Since the kernel of β is clearly K-invariant, $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ is an algebraic representation of K. We define the representation of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ by left multiplication on the first factor. Differentiation of the algebraic action of K on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ gives a representation of $\mathfrak{k}$ by

$$\xi \cdot (\eta \otimes u) = \operatorname{ad}(\xi)\eta \otimes u + \eta \otimes \xi u = [\xi, \eta] \otimes u + \eta\xi \otimes u = \xi\eta \otimes u$$

for any $\xi \in \mathfrak{k}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. Hence $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ is a $(\mathfrak{g}, K)$-module. We denote this module by $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(U)$ and call it the *induced module* of (ν, U). We denote by σ the $\mathfrak{g}$ and K actions on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$.

Let (ν', U') be another $(\mathfrak{h}, K)$-module and $\Psi : U \longrightarrow U'$ a morphism of $(\mathfrak{h}, K)$-modules. The biadditive map $(\eta, u) \longmapsto \eta \otimes \Psi(u)$ defines a linear map γ of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U$ into $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U'$ given by

$$\gamma(\eta \otimes u) = \eta \otimes \Psi(u)$$

for any $\eta \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. Since Ψ is a morphism of representations of K, γ is a morphism of representations of K. Moreover, we have

$$\sigma'(\xi)\gamma(\eta \otimes u) = \sigma'(\xi)(\eta \otimes \Psi(u)) = \xi\eta \otimes \Psi(u) = \gamma(\xi\eta \otimes u) = \gamma(\sigma(\xi)(\eta \otimes u))$$

for any $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. Therefore, γ is a morphism of induced modules.

It follows that $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is a functor from $\mathcal{M}(\mathfrak{h}, K)$ into $\mathcal{M}(\mathfrak{g}, K)$. By Poincaré-Birkhoff-Witt theorem, $\mathcal{U}(\mathfrak{g})$ is a free $\mathcal{U}(\mathfrak{k})$-module for the right multiplication. Therefore we have the following result.

3.1.1. **Theorem.** *The functor* $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is exact.*

We have the following version of Frobenius reciprocity.

3.1.2. **Theorem.** *The functor* $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is a left adjoint of the forgetful functor from* $\mathcal{M}(\mathfrak{g}, K)$ *into* $\mathcal{M}(\mathfrak{h}, K)$.

Proof. Let (π, V) be a $(\mathfrak{g}, K)$-module and (ν, U) a $(\mathfrak{h}, K)$-module. Let $\Phi : u \longrightarrow V$ be a morphism of $(\mathfrak{h}, K)$-modules. Then we can consider the biadditive map $\alpha : \mathcal{U}(\mathfrak{g}) \times U \longrightarrow V$ given by $\alpha(\xi, u) = \pi(\xi)\Phi(u)$ for $\xi \in \mathfrak{g}$ and $u \in U$. Clearly we have

$$\alpha(\xi\eta, u) = \pi(\xi\eta)\Phi(u) = \pi(\xi)\pi(\eta)\Phi(u) = \pi(\xi)\Phi(\nu(\eta)u) = \alpha(\xi, \nu(\eta)u)$$

for any $\xi \in \mathcal{U}(\mathfrak{g})$, $\eta \in \mathcal{U}(\mathfrak{h})$ and $u \in U$. Therefore, α induces a linear map $\bar{\Phi} : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} U \longrightarrow V$ given by $\bar{\Phi}(\xi \otimes u) = \pi(\xi)\Phi(u)$ for all $\xi \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. Moreover, we have

$$\bar{\Phi}(\sigma(\zeta)(\xi \otimes u)) = \bar{\Phi}(\zeta\xi \otimes u) = \pi(\zeta\xi)\Phi(u) = \pi(\zeta)\pi(\xi)\Phi(u) = \pi(\zeta)\bar{\Phi}(\xi \otimes u)$$

for all $\xi, \zeta \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. So, $\bar{\Phi}$ is a morphism of $\mathfrak{g}$-modules. In addition, we have

$$\bar{\Phi}(\sigma(k)(\xi\otimes u)) = \varphi(k)(\xi)\otimes\nu(k)u = \pi(\varphi(k)\xi)\Phi(\nu(k)u) = \pi(k)\pi(\xi)\Phi(u) = \pi(k)\bar{\Phi}(\xi\otimes u)$$

for all $\xi \in \mathcal{U}(\mathfrak{g})$, $k \in K$ and $u \in U$. Therefore, $\bar{\Phi}$ is a morphism of representations of K. It follows that $\bar{\Phi}$ is a morphism of $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(U)$ into V.

Conversely, if Ψ is a morphism of the $(\mathfrak{g}, K)$-module $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(U)$ into V, we can define a linear map $\tilde{\Psi} : U \longrightarrow V$ by $\tilde{\Psi}(u) = \Psi(1 \otimes u)$ for any $u \in U$. We have

$$\tilde{\Psi}(\nu(\xi)u) = \Psi(1 \otimes \nu(\xi)u) = \Psi(\xi \otimes u) = \pi(\xi)\Psi(1 \otimes u) = \pi(\xi)\tilde{\Psi}(u)$$

for any $\xi \in \mathcal{U}(\mathfrak{h})$ and $u \in U$. In addition,

$$\tilde{\Psi}(\nu(k)u) = \Psi(1 \otimes \nu(k)u) = \Psi(\sigma(k)(1 \otimes u)) = \pi(k)\Psi(1 \otimes u) = \pi(k)\tilde{\Psi}(u)$$

for any $k \in K$ and $u \in U$. Therefore, $\tilde{\Psi}$ is a morphism of $(\mathfrak{h}, K)$-modules.

Clearly, we have

$$(\bar{\Phi})\tilde{}(u) = \bar{\Phi}(1 \otimes u) = \Phi(u)$$

for all $u \in U$. Moreover, we have

$$(\tilde{\Psi})^{-}(\xi \otimes u) = \pi(x)\tilde{\Psi}(u) = \pi(\xi)\Psi(1 \otimes u) = \Psi(\sigma(\xi)(1 \otimes u)) = \Psi(\xi \otimes u)$$

for $\xi \in \mathcal{U}(\mathfrak{g})$ and $u \in U$. Therefore, the linear maps $\Phi \longmapsto \bar{\Phi}$ and $\Psi \longmapsto \tilde{\Psi}$ are mutually inverse. □

This has the following formal consequences.

3.1.3. **Corollary.** *The functor* $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *maps projective objects in* $\mathcal{M}(\mathfrak{h}, K)$ *into projective objects in* $\mathcal{M}(\mathfrak{g}, K)$.

Since $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is exact by 3.1.1 we also have:

3.1.4. **Corollary.** *The forgetful functor from* $\mathcal{M}(\mathfrak{g}, K) \longrightarrow \mathcal{M}(\mathfrak{h}, K)$ *maps injective objects in* $\mathcal{M}(\mathfrak{g}, K)$ *into injective objects in* $\mathcal{M}(\mathfrak{h}, K)$.

Assume now that K is reductive. Then the category of algebraic representations of K is semisimple, and every object in it is projective. By 3.1.3, $(\mathfrak{g}, K)$-modules $\operatorname{ind}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(U)$ are projective for arbitrary algebraic representation U of K.

This leads to the following result.

3.1.5. **Lemma.** *If* K *is a reductive algebraic group, the category* $\mathcal{M}(\mathfrak{g}, K)$ *has enough projectives.*

Proof. As we remarked, if K is reductive, $\operatorname{ind}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(U)$ is a projective object in $\mathcal{M}(\mathfrak{g}, K)$ for any algebraic representation U of K.

By 3.1.2, the functor $\operatorname{ind}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}$ is a left adjoint of the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)$ into $\mathcal{M}(K)$. Therefore, for a $(\mathfrak{g}, K)$-module V, we have the adjunction morphism $\operatorname{ind}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(V) \longrightarrow V$ given by $\xi \otimes v \longmapsto \nu(\xi)v$ for any $\xi \in \mathcal{U}(\mathfrak{h})$ and $v \in V$. clearly, this morphism is an epimorphism. By above remark, it follows that any object in $\mathcal{M}(\mathfrak{g}, K)$ is a quotient of a projective object. □

3.2. **The producing functor.** Let (ν, U) be a $\mathfrak{h}$-module. We consider $\mathcal{U}(\mathfrak{g})$ as a $\mathfrak{h}$-module for left multiplication. Equip the vector space $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ with action of $\mathfrak{g}$ given by

$$(\xi \cdot A)(\eta) = A(\eta\xi)$$

for any $A \in \operatorname{Hom}_{\mathfrak{k}}(\mathcal{U}(\mathfrak{g}), U)$, $\xi \in \mathfrak{g}$ and $\eta \in \mathcal{U}(\mathfrak{g})$. By Poincaré-Birkhoff-Witt theorem, the $\mathfrak{h}$-module $\mathcal{U}(\mathfrak{g})$ is free. Therefore, the functor $U \longmapsto \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ is an exact functor from the category of $\mathfrak{h}$-modules into the category of $\mathfrak{g}$-modules.

Assume that U is a $(\mathfrak{h}, K)$-module. Let $A \in \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$. Let $k \in K$. We define a linear map from $\mathcal{U}(\mathfrak{g})$ into U by

$$(k \cdot A)(\eta) = \nu(k) A(\varphi(k^{-1})\eta)$$

for $\eta \in \mathcal{U}(\mathfrak{g})$. We have

$$\begin{aligned}(k \cdot A)(\zeta\eta) &= \nu(k) A(\varphi(k^{-1})(\zeta\eta)) = \nu(k) A((\varphi(k^{-1})\zeta)(\varphi(k^{-1})\eta)) \\ &= \nu(k)\nu(\varphi(k^{-1})\zeta) A(\varphi(k^{-1})\eta) = \nu(\zeta)\nu(k) A(\varphi(k^{-1})\eta) = \nu(\zeta)(k \cdot A)(\eta)\end{aligned}$$

for any $\zeta \in \mathfrak{h}$ and $\eta \in \mathcal{U}(\mathfrak{g})$. This implies that $k \cdot A$ is in $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$. Hence, this defines a representation of K on $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$.

Now we need the following result.

3.2.1. **Lemma.** *Let V be a vector space with representation σ of $\mathfrak{g}$ and representation τ of K. Assume that the action map $\Psi : \mathfrak{g} \times V \longrightarrow V$ given by $\Psi(\xi, v) = \sigma(\xi)v$ for $\xi \in \mathfrak{g}$ and $v \in V$, is K-equivariant, i.e.,*

$$\Psi(\varphi(k)\xi, \tau(k)v) = \tau(k)\Psi(\xi, v)$$

for any $\xi \in \mathfrak{g}$, $k \in K$ and $v \in V$. Then the largest algebraic representation $V_{[K]}$ is a $\mathfrak{g}$-submodule of (σ, V).

Proof. The action map $\Psi : \mathfrak{g} \times V \longrightarrow V$ is bilinear and induces a linear map $\Phi : \mathfrak{g} \otimes_{\mathbb{C}} V \longrightarrow V$. If we consider $\mathfrak{g} \otimes_{\mathbb{C}} V$ as a tensor product of representations of K, the map Φ is a morphism of of representations. Since the representation φ of K on $\mathfrak{g}$ is algebraic, for any algebraic vector $v \in V$ and $\xi \in \mathfrak{g}$, $\xi \otimes v$ is an algebraic vector in $\mathfrak{g} \otimes_{\mathbb{C}} V$ and $\Psi(\xi \otimes v)$ is also algebraic. It follows that $\Psi(\mathfrak{g} \otimes_{\mathbb{C}} V_{[K]}) \subset V_{[K]}$. This means that $V_{[K]}$ is invariant for σ. ☐

Consider the action map $\Psi : \mathfrak{g} \times \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U) \longrightarrow \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$. For any $k \in K$, we have

$$\begin{aligned}\Psi(\varphi(k)\xi, k \cdot A)(\eta) &= (\varphi(k)\xi) \cdot (k \cdot A))(\eta) = (k \cdot A)(\eta\varphi(k)\xi) \\ &= \nu(k) A(\varphi(k^{-1})(\eta\varphi(k)\xi)) = \nu(k) A(\varphi(k^{-1})(\eta)\xi) \\ &= \nu(k)(\xi \cdot A)(\varphi(k^{-1})\eta) = (k \cdot (\xi \cdot A))(\eta)\end{aligned}$$

for any $A \in \operatorname{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}), U)$, $\xi \in \mathfrak{g}$ and $\eta \in \mathcal{U}(\mathfrak{g})$. Therefore Ψ is K-equivariant.

By 3.2.1, the largest algebraic subrepresentation $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$ of the representation $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ is $\mathfrak{g}$-invariant.

By differentiating the representation of K on $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$ we get the representation of $\mathfrak{k}$ given by

$$(\zeta \cdot A)(\eta) = \zeta \cdot A(\eta) - A(\operatorname{ad}(\zeta)\eta) = \zeta \cdot A(\eta) - A(\zeta\eta - \eta\zeta) = A(\eta\zeta) = (\zeta \cdot A)(\eta)$$

for any $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$, $\zeta \in \mathfrak{k}$ and $\eta \in \mathcal{U}(\mathfrak{g})$. Hence, the differential of the action of K agrees with the action of $\mathfrak{k}$, i.e., $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$ is a $(\mathfrak{g}, K)$-module. We put

$$\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(U) = \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$$

and denote the action of $\mathfrak{g}$ and K by ρ. We call this module the *produced module* of (ν, U).

Let (ν', U') be another $(\mathfrak{h}, K)$-module and $\omega : U \longrightarrow U'$ a morphism of $(\mathfrak{h}, K)$-modules. Then $A \longmapsto \omega \circ A$ is a linear map from $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ into $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), V')$.

For any $\xi \in \mathfrak{g}$, we have

$$(\xi \cdot (\omega \circ A))(\eta) = (\omega \circ A)(\eta\xi) = \omega(A(\eta\xi)) = \omega((\xi \cdot A)(\eta)) = (\omega \circ (\xi \cdot A))(\eta)$$

for any $A \in \mathrm{Hom}_{\mathfrak{b}}(\mathcal{U}(\mathfrak{g}), U)$ and $\eta \in \mathcal{U}(\mathfrak{g})$, i.e., $A \longmapsto \omega \circ A$ is a morphism of $\mathfrak{g}$-modules.

For any $k \in K$ we have

$$\begin{aligned}(k \cdot (\omega \circ A))(\eta) &= \nu'(k)(\omega \circ A)(\varphi(k^{-1})\eta) = \nu'(k)\omega(A(\varphi(k^{-1})\eta)) \\ &= \omega(\nu(k)A(\varphi(k^{-1})\eta)) = \omega(((k \cdot A)(\eta))) = (\omega \circ (k \cdot A))(\eta)\end{aligned}$$

for all $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$, $\eta \in \mathcal{U}(\mathfrak{g})$. Hence, $A \longmapsto \omega \circ A$ is a morphism of representations of K. Therefore, it induces a morphism of $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$ into $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), V)_{[K]}$. Hence, $A \longmapsto \omega \circ A$ is a morphism of $(\mathfrak{g}, K)$-modules.

This implies that $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is an additive functor from the category $\mathcal{M}(\mathfrak{h}, K)$ into the category $\mathcal{M}(\mathfrak{g}, K)$.

We have the following version of Frobenius reciprocity.

3.2.2. **Theorem.** *The functor* $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is a right adjoint of the forgetful functor from* $\mathcal{M}(\mathfrak{g}, K)$ *into* $\mathcal{M}(\mathfrak{h}, K)$.

Proof. Let (ν, U) be a $(\mathfrak{h}, K)$-module. We define a map $e : \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]} \longrightarrow U$ by $e(A) = A(1)$ for any $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$. We claim that e is a morphism of $(\mathfrak{h}, K)$-modules. First, for $\xi \in \mathfrak{h}$, we have

$$\nu(\xi)e(A) = \nu(\xi)A(1) = A(\xi) = (\rho(\xi)A)(1) = e(\rho(\xi)A)$$

for any $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$. In addition, for $k \in K$, we have

$$\nu(k)e(A) = \nu(k)A(1) = \nu(k)A(\varphi(k^{-1})1) = (\rho(k)A)(1) = e(\rho(k)A)$$

for any $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$.

Let (π, V) be a $(\mathfrak{g}, K)$-module. We define a map

$$\alpha : \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}(U)) \longrightarrow \mathrm{Hom}_{(\mathfrak{h},K)}(V, U)$$

by $\alpha(\Psi) = e \circ \Psi$.

We claim that α is injective. Assume that Ψ is in the kernel of α. Then for $v \in V$, we have

$$0 = \alpha(\Psi)(\pi(\eta)v) = e(\Psi(\pi(\eta)v)) = e(\rho(\eta)\Psi(v)) = (\rho(\eta)\Psi(v))(1) = \Psi(v)(\eta)$$

for $\eta \in \mathcal{U}(\mathfrak{g})$. Therefore $\Psi(v) = 0$ for any $v \in V$, i.e., $\Psi = 0$.

Let $\Phi \in \mathrm{Hom}_{(\mathfrak{h},K)}(V, U)$. For any $v \in V$, we define the linear map $\Phi_v : \mathcal{U}(\mathfrak{g}) \longrightarrow V$ by $\Phi_v(\eta) = \Phi(\pi(\eta)v)$. Then, for $\zeta \in \mathfrak{h}$, we have

$$\Phi_v(\zeta\eta) = \Phi(\pi(\zeta\eta)v) = \Phi(\pi(\zeta)\pi(\eta)v)\nu(\zeta)\Phi(\pi(\eta)v) = \nu(\zeta)\Phi_v(\eta)$$

for any $\eta \in \mathcal{U}(\mathfrak{g})$. Therefore, Φ_v is in $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$. Moreover, for $k \in K$, we have

$$\begin{aligned}(k \cdot \Phi_v)(\eta) = \nu(k)\Phi_v(\varphi(k^{-1})\eta) = \nu(k)\Phi(\pi(\varphi(k^{-1})\eta)v) \\ = \Phi(\pi(k)\pi(\varphi(k^{-1})\eta)v) = \Phi(\pi(\eta)\pi(k)v) = \Phi_{\pi(k)v}(\eta)\end{aligned}$$

for any $\eta \in \mathcal{U}(\mathfrak{g})$. Therefore, we have $k \cdot \Phi_v = \Phi_{\pi(k)v}$ for any $k \in K$. It follows that $\Phi_v \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$. The map $\Omega : v \longmapsto \Phi_v$ is clearly a linear map from V into $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]}$. In addition, for $\xi \in \mathfrak{g}$, we have

$$\begin{aligned}(\rho(\xi)\Omega(v))(\eta) = (\rho(\xi)\Phi_v)(\eta) = \Phi_v(\eta\xi) \\ = \Phi(\pi(\eta\xi)v) = \Phi(\pi(\eta)\pi(\xi)v) = \Phi_{\pi(\xi)v}(\eta) = \Omega(\pi(\xi)v)(\eta),\end{aligned}$$

for any $\eta \in \mathcal{U}(\mathfrak{g})$. Hence, we have $\rho(\xi)\Omega(v) = \Omega(\pi(\xi)v)$ for any $v \in V$. It follows that $\Omega \in \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]})$.

Let $\beta : \mathrm{Hom}_{(\mathfrak{h},K)}(V, U) \longrightarrow \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]})$ be the map which sends Φ into Ω. Then we have

$$(\alpha \circ \beta)(\Phi)(v) = (\alpha(\Omega))(v) = e(\Omega(v)) = e(\Phi_v) = \Phi_v(1) = \Phi(v)$$

for any $v \in V$. Hence, $\alpha \circ \beta$ is the identity on $\mathrm{Hom}_{(\mathfrak{h},K)}(V, U)$, i.e., α is surjective. This implies that α is a linear isomorphism and β its inverse. □

The above result has the following formal consequences.

3.2.3. **Corollary.** *The functor* $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is left exact.*

3.2.4. **Corollary.** *The functor* $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *maps injective objects in* $\mathcal{M}(\mathfrak{h}, K)$ *into injective objects in* $\mathcal{M}(\mathfrak{g}, K)$.

Assume now that K is reductive. Then the category of algebraic representations of K is semisimple and every object (ν, U) is injective. Therefore, Harish-Chandra modules $\mathrm{pro}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(U)$ are injective in $\mathcal{M}(\mathfrak{g}, K)$ for arbitrary algebraic representation U of K.

This leads to the following result.

3.2.5. **Corollary.** *If* K *is a reductive algebraic group, the category* $\mathcal{M}(\mathfrak{g}, K)$ *has enough injectives.*

Proof. Let (π, V) be a $(\mathfrak{g}, K)$-module. By adjointness, the identity morphism on algebraic representation V of K, determines the adjointness morphism $V \longrightarrow \mathrm{pro}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(V)$. From the discussion from the proof 3.2.2, that morphism sends a vector $v \in V$ to the map $\eta \longmapsto \pi(\eta)v$, $\eta \in \mathcal{U}(\mathfrak{g})$. Hence, the above adjointness morphism is a monomorphism. By the above remark, the $(\mathfrak{g}, K)$-module $\mathrm{pro}_{(\mathfrak{k},K)}^{(\mathfrak{g},K)}(V)$ is injective. □

Later, in 4.1.5, we are going to prove the above result without the assumption that K is reductive.

3.2.6. **Theorem.** *If* K *is a reductive algebraic group, the functor* $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is exact.*

To prove this theorem we need some preparation. The group K acts on $\mathfrak{g}$ via φ. Since K is reductive algebraic group and $\mathfrak{h}$ is K-invariant there exists a K-invariant vector subspace $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$.

Let $S(\mathfrak{s})$ be the symmetric algebra of $\mathfrak{s}$. The action φ of K on $\mathfrak{s}$ extends to the action on $S(\mathfrak{s})$ by automorphisms.

Also, the action φ of K on $\mathfrak{g}$ (resp. $\mathfrak{h}$) extends to the action on $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{h})$) by automorphisms.

Therefore, the tensor product $\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})$ is an algebraic representation of K. Moreover, it is a free left $\mathcal{U}(\mathfrak{h})$-module by left multiplication in the first factor.

Clearly, $\mathcal{U}(\mathfrak{g})$ is also an algebraic representation of K, and a left $\mathcal{U}(\mathfrak{h})$-module by left multiplication. By Poincaré-Birkhoff-Witt theorem, it is a free $\mathcal{U}(\mathfrak{h})$-module.

Let $\lambda : S(\mathfrak{s}) \longrightarrow \mathcal{U}(\mathfrak{g})$ be the linear map given by

$$\lambda(\xi_1 \xi_2 \cdots \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \xi_{\sigma(2)} \cdots \xi_{\sigma(n)}$$

for $\xi_1, \xi_2, \dots, \xi_n$. Moreover, we define $\Lambda : \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}) \longrightarrow \mathcal{U}(\mathfrak{g})$ by $\Lambda(\eta \otimes \zeta) = \eta \cdot \lambda(\zeta)$ for $\eta \in \mathcal{U}(\mathfrak{h})$ and $\zeta \in S(\mathfrak{s})$.

3.2.7. **Lemma.** *The linear map $\Lambda : \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}) \longrightarrow \mathcal{U}(\mathfrak{g})$ is an isomorphism of $\mathcal{U}(\mathfrak{h})$-modules and algebraic representations of K.*

Proof. Let $(\mathcal{U}_p(\mathfrak{g}); p \in \mathbb{Z}_+)$ be the standard filtration of the enveloping algebra of $\mathfrak{g}$. We consider $\mathcal{U}(\mathfrak{g})$ as a left $\mathcal{U}(\mathfrak{h})$-module for the left multiplication. Define an increasing filtration $(F_p\mathcal{U}(\mathfrak{g}); p \in \mathbb{Z}_+)$ such that $F_p\mathcal{U}(\mathfrak{g})$ is the $\mathcal{U}(\mathfrak{h})$-submodule generated by $\mathcal{U}_p(\mathfrak{g})$.

Let $(S_p(\mathfrak{s}); p \in \mathbb{Z}_+)$ be the standard filtration of the symmetric algebra $S(\mathfrak{s})$. We define a filtration $(F_p(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})); p \in \mathbb{Z}_+)$ by

$$F_p(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})) = \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S_p(\mathfrak{s})$$

for $p \in \mathbb{Z}_+$. Clearly, we have

$$\Lambda(F_p(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})) \subset F_p\mathcal{U}(\mathfrak{g})$$

for all $p \in \mathbb{Z}_+$. By Poincaré-Birkhoff-Witt theorem, we see that $\operatorname{Gr} \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s})$ and $\operatorname{Gr} \Lambda$ is the identity map. This easily implies that $\Lambda : \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}) \longrightarrow \mathcal{U}(\mathfrak{g})$ is an isomorphism. □

Now we can prove 3.2.6. By 3.2.7, we see that

$$\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U) = \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}), U)$$

as representations of K. By restriction to $\mathbb{C} \otimes_{\mathbb{C}} S(\mathfrak{s})$, we have a linear isomorphism of $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}), U)$ with $\operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U)$. This linear isomorphism is an intertwining map of the representation of K on $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} S(\mathfrak{s}), U)$ with the natural representation on $\operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U)$. Therefore we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U') & \longrightarrow & \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U) & \longrightarrow & \operatorname{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U') & \longrightarrow & \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U) & \longrightarrow & \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U'') & \longrightarrow & 0
\end{array}$$

Here the horizontal lines are exact sequences and vertical morphisms are isomorphisms of representations of K. Applying the algebrazation functor, by 1.2.5 and 1.3.2, we get the diagram of algebraic representations of K

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U')_{[K]} & \longrightarrow & \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)_{[K]} & \longrightarrow & \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U'')_{[K]} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U')_{[K]} & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U)_{[K]} & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{s}), U'')_{[K]} & \longrightarrow & 0 \end{array}$$

where horizontal lines are exact and vertical are isomorphisms. This implies that the top line as a short exact sequence, i.e., the functor $\mathrm{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ is exact. This completes the proof of 3.2.6.

4. Zuckerman functor

Now we want to study change of groups in Harish-Chandra pairs. Let T be a closed algebraic subgroup of K. Then we have a natural forgetful functor $\mathcal{M}(\mathfrak{g}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, T)$. The Zuckerman functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is by definition the right adjoint functor to this forgetful functor.

First we describe a construction of this functor.

4.1. **Zuckerman functor.** Let $R(K)$ be the ring of regular functions on K. Then for any vector space V, we can view $R(K) \otimes V$ as the vector space of all regular maps from K into V and denote it by $R(K, V)$. We define an algebraic representation ρ of K on $R(K, V)$ as the tensor product of the right regular representation of K on $R(K)$ and trivial action on V.

Let now V be an algebraic representation of K. Then we have the natural *matrix coefficient* map $c : V \longrightarrow R(K, V)$ which maps a vector $v \in V$ into the function $k \longmapsto \pi(k)v$. Then, if we define ρ as above, we have

$$c(\pi(k)v)(h) = \pi(h)\pi(k)v = \pi(hk)v = c(v)(hk) = (\rho(k)c(v))(h)$$

for any $k, h \in K$, i.e.,

$$c(\pi(k)v) = \rho(k)c(v), \quad v \in V,$$

and c is a monomorphism of representations of K.

If V is a $(\mathfrak{g}, K)$-module, we also have

$$c(\pi(\xi)v)(k) = \pi(k)\pi(\xi)v = \pi(\varphi(k)\xi)\pi(k)v = \pi(\varphi(k)\xi)c(v)(k)$$

for $\xi \in \mathfrak{g}$, $k \in K$ and $v \in V$. Therefore, if we define a representation ν of $\mathfrak{g}$ on $R(K, V)$ by

$$(\nu(\xi)F)(k) = \pi(\varphi(k)\xi)F(k), \quad k \in K,$$

for $\xi \in \mathfrak{g}$ and $v \in V$, we see that $c : V \longrightarrow R(K, V)$ intertwines the actions of $\mathfrak{g}$ and K.

In addition, if we define the representation τ of K on $R(K, V)$ as the tensor product of the left regular representation λ of K on $R(K)$ with the natural action π on V, we have

$$(\tau(k)c(v))(h) = \pi(k)c(v)(k^{-1}h) = \pi(k)\pi(k^{-1}h)v = \pi(h)v = c(v)(h), \quad h \in K,$$

for all $k \in K$ and $v \in V$. Therefore, the image of c is in the space of all τ-invariant functions in $R(K, V)$.

Let Ψ be a τ-invariant function in $R(K, V)$. Then

$$\Psi(k) = (\tau(h)\Psi)(k) = \pi(k)\Psi(k^{-1}h)$$

for any $h, k \in K$. Therefore, $\Psi(kh) = \pi(k)\Psi(h)$ for any $h, k \in K$. In particular, we have $\Psi(k) = \pi(k)\Psi(1)$ for all $k \in K$, and Ψ is a matrix coefficient of $\Psi(1) \in V$. Therefore, the matrix coefficient map is an isomorphism of $(\mathfrak{g}, K)$-module V onto the $(\mathfrak{g}, K)$-module of the τ-invariants of $R(K, V)$.

We shall use this observation to construct the Zuckerman functor.

Let now (π, V) be a $(\mathfrak{g}, T)$-module only. Then, as above, we can define the structure of $\mathcal{U}(\mathfrak{g})$-module on $R(K, V)$ by

$$(\nu(\xi)F)(k) = \pi(\varphi(k)\xi)F(k), \quad k \in K,$$

for $\xi \in \mathfrak{g}$ and $F \in R(K, V)$, and the structure of an algebraic representation of K by

$$(\rho(k)F)(h) = F(hk), \quad h \in K,$$

for $k \in K$. Let τ be the tensor product of the left regular representation λ of $\mathfrak{k}$ and T on $R(K)$ with the natural action π on V. This defines a structure of Harish-Chandra module for $(\mathfrak{k}, T)$ on $R(K, V)$. We claim that these actions of $\mathfrak{k}$ and T commute with the representations ν and ρ. For ρ this is evident. For ν, we first have

$$\begin{aligned}(\tau(t)\nu(\xi)F)(k) &= \pi(t)(\nu(\xi)F)(t^{-1}k) = \pi(t)\pi(\varphi(t^{-1}k)\xi)F(t^{-1}k)\\ &= \pi(\varphi(k)\xi)\pi(t)F(t^{-1}k) = \pi(\varphi(k)\xi)(\lambda(t)F)(k) = (\nu(\xi)\tau(t)F)(k), \quad k \in K,\end{aligned}$$

for $t \in T$, $\xi \in \mathfrak{g}$ and $F \in R(K, V)$. Therefore, the ν-action of $\mathfrak{g}$ commutes with the τ-action of T.

On the other hand, the differential of the action τ of $\mathfrak{k}$ on $R(K, V)$ is given by

$$(\tau(\eta)F)(k) = (\lambda(\eta)F)(k) + \pi(\eta)F(k)$$

for $\eta \in \mathfrak{g}$, $k \in K$ and $F \in R(K, V)$. For $h \in K$, we have

$$(\nu(\xi)F)(h^{-1}k) = \pi(\varphi(h^{-1}k)\xi)F(h^{-1}k) = \pi(\varphi(h^{-1})\varphi(k)\xi)F(h^{-1}k)$$

for $h, k \in K$. Therefore, by differentiation with respect to h at 1, we get

$$\begin{aligned}(\lambda(\eta)\nu(\xi)F)(k) &= -\pi(\operatorname{ad}(\eta)(\varphi(k)\xi))F(k) + \pi(\varphi(k)\xi)(\lambda(\eta)F)(k)\\ &= -\pi(\eta)\pi(\varphi(k)\xi)F(k) + \pi(\varphi(k)\xi)\pi(\eta)F(k) + \pi(\varphi(k)\xi)(\lambda(\eta)F)(k)\\ &= -\pi(\eta)(\nu(\xi)F)(k) + \pi(\varphi(k)\xi)((\lambda(\eta)F)(k) + \pi(\eta)F(k))\\ &= -\pi(\eta)(\nu(\xi)F)(k) + \pi(\varphi(k)\xi)(\tau(\eta)F)(k) = -\pi(\eta)(\nu(\xi)F)(k) + (\nu(\xi)\tau(\eta)F)(k)\end{aligned}$$

for $\xi \in \mathfrak{g}$, $\eta \in \mathfrak{k}$ and $k \in K$. Therefore, we have

$$(\tau(\eta)\nu(\xi)F)(k) = (\lambda(\eta)\nu(\xi)F)(k) + \pi(\eta)(\nu(\xi)F)(k) = (\nu(\xi)\tau(\eta)F)(k)$$

for $F \in R(K, V)$, $\xi \in \mathfrak{g}$, $\eta \in \mathfrak{k}$ and $k \in K$. Therefore, the ν-action of $\mathfrak{k}$ commutes with τ-action of $\mathfrak{k}$.

It follows that the subspace of $(\mathfrak{k}, T)$-invariants

$$\Gamma_{K,T}(V) = R(K, V)^{(\mathfrak{k},T)}$$

in $R(K, V)$ with respect to τ is an invariant subspace for the actions of $\mathfrak{g}$ and K.

4.1.1. **Lemma.** *Let V be a $(\mathfrak{g}, T)$-module. Then $\Gamma_{K,T}(V)$ is a $(\mathfrak{g}, K)$-module.*

Proof. First, for $k \in K$, $\xi \in \mathfrak{g}$ and $F \in R(K,W)$ we have

$$(\nu(\varphi(k)\xi)F)(h) = \pi(\varphi(h)\varphi(k)\xi)F(h) = \pi(\varphi(hk)\xi)F(h) \\ = \pi(\varphi(hk)\xi)(\rho(k^{-1})F)(hk) = (\nu(\xi)\rho(k^{-1})F)(hk) \\ = (\rho(k)\nu(\xi)\rho(k^{-1})F)(h), \quad h \in K;$$

i.e.,

$$\nu(\phi(k)\xi) = \rho(k)\nu(\xi)\rho(k^{-1}).$$

Also, for $F \in \Gamma_{K,T}(V)$, we have

$$F(kh) = F((\mathrm{Int}(k)(h))k)$$

for any $h, k \in K$. Since F is invariant for τ-action of K, by differentiation with respect to h at 1 we get

$$(\rho(\xi)F)(k) = -(\lambda(\mathrm{Ad}(k)\xi)F)(k) = -(\tau(\varphi(k)\xi)F)(k) + \pi(\varphi(k)\xi)F(k) = (\nu(\xi)F)(k)$$

for $\xi \in \mathfrak{k}$ and $k \in K$. Hence, we have $\nu(\xi) = \rho(\xi)$ on $\Gamma_{K,T}(V)$. Therefore, the actions ν and ρ define a structure of Harish-Chandra module on $\Gamma_{K,T}(V)$. □

Let (π, V) and (π', V') be two Harish-Chandra modules for $(\mathfrak{g}, T)$ and $\alpha \in \mathrm{Hom}_{(\mathfrak{g},T)}(V, V')$. Then α induces a linear map $1 \otimes \alpha : R(K,V) \longrightarrow R(K,V')$. Clearly, $1 \otimes \alpha$ intertwines the actions ν, ρ and τ on these modules. Hence, it induces a morphism $\Gamma_{K,T}(\alpha) : \Gamma_{K,T}(V) \longrightarrow \Gamma_{K,T}(V')$. It follows that $\Gamma_{K,T}$ is an additive functor from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{g}, K)$.

Let (π, V) be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, K)$. Then, as we saw above, the matrix coefficient map c_V of V is a $(\mathfrak{g}, K)$-morphism of V into $\Gamma_{K,T}(V)$. It is easy to check that the maps c_V actually define a natural transformation of the identity functor on $\mathcal{M}(\mathfrak{g}, K)$ into the composition of $\Gamma_{K,T}$ with the forgetful functor.

On the other hand, let (π, V) be a Harish-Chandra module for $(\mathfrak{g}, T)$ and $e_V : \Gamma_{K,T}(V) \longrightarrow V$ the linear map given by $e_V(F) = F(1)$. Then e_V is a $(\mathfrak{g}, T)$-morphism from $\Gamma_{K,T}(V)$ into V. To see this, we observe that for any $F \in \Gamma_{K,T}(V)$,

$$e_V(\nu(\xi)F) = (\nu(\xi)F)(1) = \pi(\xi)F(1) = \pi(\xi)e_V(F)$$

for $\xi \in \mathfrak{g}$ and

$$e_V(\nu(t)F) = (\rho(t)F)(1) = F(t) = \pi(t)F(1) = \pi(t)e_V(F)$$

for $t \in T$. Clearly, the maps e_v define a natural transformation of the composition of the forgetful functor with the functor $\Gamma_{K,T}$ into the identity functor on $\mathcal{M}(\mathfrak{g}, T)$.

Using these natural transformations, we get the following result.

4.1.2. **Theorem.** *The functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is right adjoint to the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)$ into $\mathcal{M}(\mathfrak{g}, T)$.*

Proof. Let (π, V) be a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, K)$ and (π', V') a Harish-Chandra module in $\mathcal{M}(\mathfrak{g}, T)$. For $\alpha \in \mathrm{Hom}_{(\mathfrak{g},T)}(V, V')$, the composition $\bar{\alpha} = \Gamma_{K,T}(\alpha) \circ c_V : V \longrightarrow \Gamma_{K,T}(V')$ is in $\mathrm{Hom}_{(\mathfrak{g},K)}(V, \Gamma_{K,T}(V'))$. Thus we have a linear map $\alpha \longmapsto \bar{\alpha}$ of $\mathrm{Hom}_{(\mathfrak{g},T)}(V, V')$ into $\mathrm{Hom}_{(\mathfrak{g},K)}(V, \Gamma_{K,T}(V'))$.

Also, if $\beta \in \mathrm{Hom}_{(\mathfrak{g},K)}(V, \Gamma_{K,T}(V'))$, $\tilde{\beta} = e_{V'} \circ \beta \in \mathrm{Hom}_{(\mathfrak{g},T)}(V, V')$. Thus we have a linear map $\beta \longmapsto \tilde{\beta}$ of $\mathrm{Hom}_{(\mathfrak{g},K)}(V, \Gamma_{K,T}(V'))$ into $\mathrm{Hom}_{(\mathfrak{g},T)}(V, V')$.

These maps are inverse to each other. Namely, for any $v \in V$,

$$(\tilde{\beta})^{-}(v)(k) = ((1 \otimes \tilde{\beta}) \circ c_V)(v)(k) = \tilde{\beta}(c_V(v)(k))$$
$$= e_{V'}(\beta(\pi(k)v)) = \beta(\pi(k)v)(1) = (\rho(k)\beta(v))(1) = \beta(v)(k),$$

for all $k \in K$. Therefore, $(\tilde{\beta})^{-} = \beta$.

On the other hand, for $\alpha \in \operatorname{Hom}_{(\mathfrak{g},T)}(V, V')$, we have

$$(\bar{\alpha})\tilde{}(v) = (\bar{\alpha})(v)(1) = ((1 \otimes \alpha) \circ c_V)(v)(1) = \alpha(c_V(v)(1)) = \alpha(v)$$

for $v \in V$; i.e., $(\bar{\alpha})\tilde{} = \alpha$.

Hence, the maps $\beta \longmapsto \tilde{\beta}$ and $\alpha \longmapsto \bar{\alpha}$ are mutually inverse bijections and this proves the proposition. □

The functor $\Gamma_{K,T}$ is called the *Zuckerman functor*.

The above result has the following immediate consequence.

4.1.3. **Corollary.** *The functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is left exact.*

Moreover, since the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)$ into $\mathcal{M}(\mathfrak{g}, T)$ is exact, the following result holds.

4.1.4. **Corollary.** *The functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ maps injective objects in $\mathcal{M}(\mathfrak{g}, T)$ into injective objects in $\mathcal{M}(\mathfrak{g}, K)$.*

Let $\Gamma_K = \Gamma_{K,\{1\}}$. Then we have the following consequence.

4.1.5. **Theorem.** *The category $\mathcal{M}(\mathfrak{g}, K)$ has enough injectives.*

Proof. Let (π, V) be a $(\mathfrak{g}, K)$-module. Then there exists an injective object I in $\mathcal{M}(\mathfrak{g})$ and a $\mathfrak{g}$-monomorphism $i : V \longrightarrow I$. By 4.1.4, $\Gamma_K(I)$ is an injective $(\mathfrak{g}, K)$-module.

Since Γ_K is left exact, $\Gamma_K(i) : \Gamma_K(V) \longrightarrow \Gamma_K(I)$ is also a monomorphism. By the discussion before 4.1.2, the adjunction morphism $V \longrightarrow \Gamma_K(V)$ is also a monomorphism. Therefore, the composition of these two morphisms defines a monomorphism $V \longrightarrow \Gamma_K(I)$ of V into an injective object in $\mathcal{M}(\mathfrak{g}, K)$. □

4.2. **A remark on notation.** Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair and $(\mathfrak{h}, K)$ be a Harish-Chandra subpair. Let T be an algebraic subgroup of K. Then $(\mathfrak{g}, T)$ is a Harish-Chandra pair and $(\mathfrak{h}, T)$ is its Harish-Chandra subpair. Therefore, we can consider the functors $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ and $\Gamma_{K,T} : \mathcal{M}(\mathfrak{h}, T) \longrightarrow \mathcal{M}(\mathfrak{h}, K)$ and our notation seems ambiguous. On the other hand, by inspection of the construction of Zuckerman functor $\Gamma_{K,T}$ it is evident that the following diagram of functors commutes:

$$\begin{array}{ccc}
\mathcal{M}(\mathfrak{g}, T) & \xrightarrow{\Gamma_{K,T}} & \mathcal{M}(\mathfrak{g}, K) \\
{\scriptstyle \mathrm{For}}\downarrow & & \downarrow{\scriptstyle \mathrm{For}} \\
\mathcal{M}(\mathfrak{h}, T) & \xrightarrow[\Gamma_{K,T}]{} & \mathcal{M}(\mathfrak{h}, K)
\end{array}$$

This eliminates the above ambiguity.

5. Derived categories of $(\mathfrak{g}, K)$-modules

5.1. **Derived categories of $(\mathfrak{g}, K)$-modules.** Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. We denote by $D^*(\mathfrak{g}, K)$ the derived category of the abelian category $(\mathfrak{g}, K)$ consisting of bounded (resp. bounded from below, bounded from above) complexes for $* = b, +, -, \emptyset$.

Let $(\mathfrak{h}, K)$ be a Harish-Chandra subpair of $(\mathfrak{g}, K)$. First we observe that that forgetful functors from $D^*(\mathfrak{g}, K)$ into $D^*(\mathfrak{h}, K)$ have left and right adjoints under certain conditions.

First, by 3.1.1, the functor $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is exact and lifts to exact functors $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^*(\mathfrak{h}, K) \longrightarrow D^*(\mathfrak{g}, K)$.

Moreover, by 3.1.2 and [1, Ch. 5, 1.7.1], we have the following consequence.

5.1.1. **Theorem.** *The functor* $\operatorname{ind}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^*(\mathfrak{h}, K) \longrightarrow D^*(\mathfrak{g}, K)$ *is a left adjoint of the forgetful functor from* $D^*(\mathfrak{g}, K)$ *into* $D^*(\mathfrak{h}, K)$.

The category $\mathcal{M}(\mathfrak{h}, K)$ has enough of injective objects by 4.1.5. Moreover, the functor $\operatorname{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is left exact by 3.2.3. Hence, by [1, Ch. 5, 3.1.3], the derived functor $\operatorname{Rpro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^+(\mathfrak{h}, K) \longrightarrow D^+(\mathfrak{g}, K)$ exists. In addition, by [1, Ch. 5, 1.7.1], we have the following result.

5.1.2. **Theorem.** *The functor* $\operatorname{Rpro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^+(\mathfrak{h}, K) \longrightarrow D^+(\mathfrak{g}, K)$ *is a right adjoint of the forgetful functor from* $D^+(\mathfrak{g}, K)$ *into* $D^+(\mathfrak{h}, K)$.

If K is reductive, the functor $\operatorname{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, K) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is exact by 3.2.6. Hence, it lifts to exact functors $\operatorname{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^*(\mathfrak{h}, K) \longrightarrow D^*(\mathfrak{g}, K)$.

Moreover, by [1, Ch. 5, 1.7.1], we have the following result.

5.1.3. **Theorem.** *Assume that the group K is reductive. Then the functor* $\operatorname{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} : D^*(\mathfrak{h}, K) \longrightarrow D^*(\mathfrak{g}, K)$ *is a right adjoint of the forgetful functor from* $D^*(\mathfrak{g}, K)$ *into* $D^*(\mathfrak{h}, K)$.

In other words, $\operatorname{Rpro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)} = \operatorname{pro}_{(\mathfrak{h},K)}^{(\mathfrak{g},K)}$ in this situation.

Assume now that T is a closed subgroup of the algebraic group K. Since the category $\mathcal{M}(\mathfrak{g}, T)$ has enough of injective objects by 4.1.5, the Zuckerman functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ has a right derived functor $\mathrm{R}\,\Gamma_{K.T} : D^+(\mathfrak{g}, T) \longrightarrow D^+(\mathfrak{g}, K)$.

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair and $(\mathfrak{h}, K)$ be a Harish-Chandra subpair. Let T be an algebraic subgroup of K. Then $(\mathfrak{g}, T)$ is a Harish-Chandra pair and $(\mathfrak{h}, T)$ is its Harish-Chandra subpair. Since the forgetful functor from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{h}, T)$ maps injective $(\mathfrak{g}, T)$-modules into injective $(\mathfrak{h}, T)$-modules by 3.1.4, it follows that the diagram of functors

$$\begin{array}{ccc} \mathcal{D}^+(\mathfrak{g}, T) & \xrightarrow{R\Gamma_{K,T}} & \mathcal{D}^+(\mathfrak{g}, K) \\ {\scriptstyle \mathrm{For}}\downarrow & & \downarrow{\scriptstyle \mathrm{For}} \\ \mathcal{D}^+(\mathfrak{h}, T) & \xrightarrow[R\Gamma_{K,T}]{} & \mathcal{D}^+(\mathfrak{h}, K) \end{array}$$

commutes, and explains the ambiguity in notation.

Moreover, by [1, Ch. 5, 1.7.1], we have the following result.

5.1.4. **Theorem.** *The functor* $\mathrm{R}\,\Gamma_{K,T} : D^+(\mathfrak{g}, T) \longrightarrow D^+(\mathfrak{g}, K)$ *is a right adjoint of the forgetful functor from* $D^+(\mathfrak{g}, K)$ *into* $D^+(\mathfrak{g}, T)$.

5.2. **Cohomological induction.** Let $(\mathfrak{h}, T)$ be a Harish-Chandra subpair of $(\mathfrak{g}, K)$. Consider the forgetful functor from $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{h}, T)$. We can view it as a composition of two forgetful functors, the forgetful functor from $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{g}, T)$ followed by the forgetful functor from $D^+(\mathfrak{g}, T)$ into $D^+(\mathfrak{h}, T)$. By 5.1.2 and 5.1.4, it follows that the forgetful functor $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{h}, T)$ has a right adjoint $\mathrm{R}\,\Gamma_{K,T} \circ \mathrm{Rpro}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}$.

We define the *cohomological induction functor* $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} : D^+(\mathfrak{h}, T) \longrightarrow D^+(\mathfrak{g}, K)$ as the composition $\mathrm{R}\,\Gamma_{K,T} \circ \mathrm{Rpro}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}$.

5.2.1. **Theorem.** *The cohomological induction functor* $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} : D^+(\mathfrak{h}, T) \longrightarrow D^+(\mathfrak{g}, K)$ *is a right adjoint of the forgetful functor from* $D^+(\mathfrak{g}, K)$ *into* $D^+(\mathfrak{h}, T)$.

5.2.2. **Proposition.** *If the group* T *is reductive, we have*

$$\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} = \mathrm{R}\,\Gamma_{K,T} \circ \mathrm{pro}^{(\mathfrak{g},T)}_{(\mathfrak{h},T)}.$$

5.3. **Cohomological induction in stages.** Let $(\mathfrak{h}, T)$ be a Harish-Chandra subpair of $(\mathfrak{g}, K)$. Let $(\mathfrak{l}, S)$ be a Harish-Chandra subpair of $(\mathfrak{h}, T)$. Then $(\mathfrak{l}, S)$ is a Harish-Chandra subpair of $(\mathfrak{g}, K)$.

Therefore, if we consider the forgetful functor from $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{l}, S)$, we can view it as a composition of the forgetful functor from $D^+(\mathfrak{g}, K)$ into $D^+(\mathfrak{h}, T)$ followed by the forgetful functor from $D^+(\mathfrak{h}, T)$ into $D^+(\mathfrak{l}, S)$. Considering the adjoint functors, we immediately get the following result.

5.3.1. **Theorem.** *The cohomological induction functor* $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{l},S)} : D^+(\mathfrak{l}, S) \longrightarrow D^+(\mathfrak{g}, K)$ *is isomorphic to* $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},K)}_{(\mathfrak{h},T)} \circ \mathrm{R}\,\mathrm{I}^{(\mathfrak{h},T)}_{(\mathfrak{l},S)}$.

5.4. **Induction of algebraic representations of reductive groups.** In this section we explain how "classic" induction of representations fits into the cohomological induction picture.

Let K be an algebraic group and H its closed subgroup. Then we can consider the category $\mathcal{M}(G)$ of algebraic representations of G and the category $\mathcal{M}(H)$ of algebraic representations of H. Moreover, we can identify these categories with the categories $\mathcal{M}(\mathfrak{g}, G)$ and $\mathcal{M}(\mathfrak{h}, H)$, respectively. Analogously, we can define the derived categories $D^+(G)$ and $D^+(H)$, respectively. Moreover, we have the cohomological induction functor $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{h},H)} : D^+(H) \longrightarrow D^+(G)$. This functor is a right adjoint of the forgetful functor from $D^+(G)$ into $D^+(H)$ by 5.2.1.

Assume now that G and H are reductive groups. Then, by 3.2.6, the functor $\mathrm{pro}^{(\mathfrak{g},H)}_{(\mathfrak{h},H)} : \mathcal{M}(H) \longrightarrow \mathcal{M}(\mathfrak{g}, H)$ is exact. Moreover, by 5.2.2, we have $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{h},H)} = \mathrm{R}\,\Gamma_{G,H} \circ \mathrm{pro}^{(\mathfrak{g},H)}_{(\mathfrak{h},H)}$. On the other hand, the functor $\Gamma_{G,H}$ is a right adjoint of the restriction functor from $\mathcal{M}(G)$ into $\mathcal{M}(H)$. Therefore, for any irreducible algebraic representation V of G we have

$$\mathrm{Hom}_G(V, \Gamma_{G,H}(\mathrm{pro}^{(\mathfrak{g},H)}_{(\mathfrak{h},H)}(U))) = \mathrm{Hom}_{(\mathfrak{g},H)}(V, \mathrm{pro}^{(\mathfrak{g},H)}_{(\mathfrak{h},H)}(U)) = \mathrm{Hom}_H(V, U)$$

for any algebraic representation U of H. Since H is reductive, the functor $U \longmapsto \mathrm{Hom}_H(V, U)$ is exact. This in turn implies that the functor

$$U \longmapsto \mathrm{Hom}_G(V, \Gamma_{G,H}(\mathrm{pro}_{(\mathfrak{h},H)}^{(\mathfrak{g},H)}(U)))$$

is exact for any V. Since K is reductive, any algebraic representation is a direct sum of irreducible representations, and therefore the functor $I_{G,H} = \Gamma_{G,H} \circ \mathrm{pro}_{(\mathfrak{h},H)}^{(\mathfrak{g},H)} : \mathcal{M}(H) \longrightarrow \mathcal{M}(G)$ is exact. We call this functor the *induction functor*. By 3.2.2, $I_{G,H}$ is a right adjoint of the restriction functor from $\mathcal{M}(G)$ into $\mathcal{M}(H)$. In particular, we have a version of "classical" Frobenius reciprocity.

5.4.1. **Theorem.** *Let V be an irreducible algebraic representation of G. Then we have*

$$\mathrm{Hom}_G(V, \mathrm{I}_H^G(U)) = \mathrm{Hom}_H(V, U)$$

for any algebraic representation of U.

Since I_H^G is exact, it lifts to the functor between corresponding derived categories $\mathrm{I}_H^G : D^*(H) \longrightarrow D^*(G)$. In particular, we see that the following result holds.

5.4.2. **Theorem.** *The functor $\mathrm{I}_H^G : D^+(H) \longrightarrow D^+(G)$ is isomorphic to the functor $\mathrm{R}\,\mathrm{I}_{(\mathfrak{h},H)}^{(\mathfrak{g},G)} : D^+(H) \longrightarrow D^+(G)$.*

5.5. **A vanishing theorem.** Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. Let T be an algebraic subgroup of K and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Assume that

(i) $\mathfrak{h}$ is invariant for the action of T;
(ii) $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}$;
(iii) $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}$.

Then $(\mathfrak{h}, T)$ is a Harish-Chandra subpair of $(\mathfrak{g}, T)$.

Consider the biadditive map $\alpha : \mathcal{U}(\mathfrak{h}) \times \mathcal{U}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g})$ given by $\alpha(\xi, \eta) = \xi \cdot \eta$, for $\xi \in \mathcal{U}(\mathfrak{h})$ and $\eta \in \mathcal{U}(\mathfrak{k})$. Since

$$\alpha(\xi \cdot \zeta, \eta) = \xi \cdot \zeta \cdot \eta = \alpha(\xi, \zeta \cdot \eta)$$

for $\xi \in \mathcal{U}(\mathfrak{h})$, $\eta \in \mathcal{U}(\mathfrak{k})$ and $\zeta \in \mathcal{U}(\mathfrak{t})$; the biadditive map α defines a linear map $\beta : \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g})$ where we consider $\mathcal{U}(\mathfrak{h})$ as a right $\mathcal{U}(\mathfrak{t})$-module for right multiplication and $\mathcal{U}(\mathfrak{k})$ as a left $\mathcal{U}(\mathfrak{t})$-module for left multiplication.

Using the conditions (ii) and (iii) and Poincaré-Birkhoff-Witt theorem we immediately get the following result.

5.5.1. **Lemma.** *The linear map $\beta : \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g})$ is an isomorphism.*

If we consider $\mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{t})} \mathcal{U}(\mathfrak{k})$ as a left $\mathcal{U}(\mathfrak{h})$-module for left multiplication in the first factor, and $\mathcal{U}(\mathfrak{g})$ as a left $\mathcal{U}(\mathfrak{h})$-module for the left multiplication, β is obviously an isomorphism of $\mathcal{U}(\mathfrak{h})$-modules.

Let $i : \mathcal{U}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g})$ be the natural inclusion. Let U be a $\mathfrak{g}$-module. Then the composition of $A \in \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ with i defines a linear map γ from $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U)$ into $\mathrm{Hom}_{\mathfrak{t}}(\mathcal{U}(\mathfrak{k}), U)$. By 5.5.1, this map is a linear isomorphism.

If U is a $(\mathfrak{h}, T)$-module, then the isomorphism $\gamma : \mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U) \longrightarrow \mathrm{Hom}_{\mathfrak{t}}(\mathcal{U}(\mathfrak{k}), U)$ intertwines the action of T on both modules described in the definition of produced functor. Therefore, γ induces and isomorphism δ of algebraic representations $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), U_{[T]}$ and $\mathrm{Hom}_{\mathfrak{t}}(\mathcal{U}(\mathfrak{k}), U)_{[T]}$ of T. Clearly, it also intertwines the $\mathfrak{k}$-actions on these spaces.

Therefore we have the following result.

5.5.2. **Lemma.** *The following diagram of functors commutes:*

$$\begin{array}{ccc} \mathcal{M}(\mathfrak{h}, T) & \xrightarrow{\operatorname{pro}_{(\mathfrak{h},T)}^{(\mathfrak{g},T)}} & \mathcal{M}(\mathfrak{g}, T) \\ \text{For}\downarrow & & \downarrow\text{For} \\ \mathcal{M}(\mathfrak{t}, T) & \xrightarrow[\operatorname{pro}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}]{} & \mathcal{M}(\mathfrak{k}, T) \end{array}.$$

Since forgetful functors map injectives into injectives by 3.1.4, this implies the following result.

5.5.3. **Lemma.** *The following diagram of functors commutes:*

$$\begin{array}{ccc} D^+(\mathfrak{h}, T) & \xrightarrow{\operatorname{Rpro}_{(\mathfrak{h},T)}^{(\mathfrak{g},T)}} & D^+(\mathfrak{g}, T) \\ \text{For}\downarrow & & \downarrow\text{For} \\ D^+(\mathfrak{t}, T) & \xrightarrow[\operatorname{Rpro}_{(\mathfrak{t},T)}^{(\mathfrak{k},T)}]{} & D^+(\mathfrak{k}, T) \end{array}.$$

Applying the functor $R\Gamma_{K,T}$ to this diagram, and using the remark that it commutes with forgetful functors we conclude that the following diagram of functors commutes

$$\begin{array}{ccc} D^+(\mathfrak{h}, T) & \xrightarrow{\mathrm{R\,I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}} & D^+(\mathfrak{g}, K) \\ \text{For}\downarrow & & \downarrow\text{For} \\ D^+(T) & \xrightarrow[\mathrm{R\,I}_{(\mathfrak{t},T)}^{(\mathfrak{k},K)}]{} & D^+(K) \end{array}.$$

Assume now that in addition

(iv) algebraic groups K and T are reductive;

holds. Then by 5.4.2 we see that

$$\begin{array}{ccc} D^+(\mathfrak{h}, T) & \xrightarrow{\mathrm{R\,I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}} & D^+(\mathfrak{g}, K) \\ \text{For}\downarrow & & \downarrow\text{For} \\ D^+(T) & \xrightarrow[\mathrm{I}_T^K]{} & D^+(K) \end{array}$$

commutes. Since the induction functor $I_T^K : \mathcal{M}(T) \longrightarrow \mathcal{M}(K)$ is exact, we conclude that higher derived functors of $\mathrm{I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)}$ vanish and the following result holds.

5.5.4. **Theorem.** *Assume that conditions (i)-(iv) hold. Then the functor* $\mathrm{I}_{(\mathfrak{h},T)}^{(\mathfrak{g},K)} : \mathcal{M}(\mathfrak{h}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ *is exact.*

6. Formal Borel-Weil-Bott theorem

In this section we construct irreducible finite-dimensional representations of a connected semisimple algebraic group G via cohomological induction. This construction is a formal analogue of the Borel-Weil-Bott theorem.

Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and R the root system of the pair $(\mathfrak{g}, \mathfrak{t})$ in $\mathfrak{t}^*$. For any $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the corresponding root subspace in $\mathfrak{g}$. Denote by R^+ a set of positive roots in R. We put $\mathfrak{n} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$. Also let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ its nilpotent radical.

Let W be the Weyl group of the root system R. The set R^+ of positive roots in R determines a basis B of R. The basis B determines a set of simple reflections in W and defines a length function $\ell : W \longrightarrow \mathbb{Z}_+$ on W. We put

$$W(p) = \{w \in W \mid \ell(w) = p\}$$

for $p \in \mathbb{Z}_+$.

Let T be the Cartan subgroup of G corresponding to $\mathfrak{t}$. Then T is an algebraic torus.

6.1. **$\mathfrak{n}$-homology as an adjoint functor.** Clearly $\mathrm{Ad}(T)$ normalizes $\mathfrak{n}$. Hence, for any $(\mathfrak{b}, T)$-module V the module $V_\mathfrak{n} = V/(\mathfrak{n}\cdot V)$ is a $(\mathfrak{b}, T)$-module with trivial action of $\mathfrak{n}$. Therefore, we can view it as a $(\mathfrak{t}, T)$-module, or as an algebraic representation of T. Therefore, we can view $V \longmapsto V_\mathfrak{n}$ as an additive functor form $\mathcal{M}(\mathfrak{b}, T)$ into $\mathcal{M}(T)$.

On the other hand, any algebraic representation of T we can view as a $(\mathfrak{b}, T)$-module on which $\mathfrak{n}$ acts trivially. This defines an exact "forgetful" functor F from $\mathcal{M}(T)$ into $\mathcal{M}(\mathfrak{b}, T)$.

6.1.1. **Lemma.** *The functor $V \longmapsto V_\mathfrak{n}$ is a left adjoint functor of the functor F from $\mathcal{M}(T)$ into $\mathcal{M}(\mathfrak{b}, T)$.*

Proof. Let V be a $(\mathfrak{b}, T)$-module and U an algebraic representation of T. Then we have

$$\mathrm{Hom}_{(\mathfrak{b},T)}(V, F(U)) = \mathrm{Hom}_{(\mathfrak{b},T)}(F(V_\mathfrak{n}), F(U)) = \mathrm{Hom}_T(V_\mathfrak{n}, U).$$

□

If we forget the T-action and restrict the action of $\mathfrak{b}$ to $\mathfrak{n}$, the functor $V \longmapsto V_\mathfrak{n}$ is the zeroth Lie algebra homology of V with respect to $\mathfrak{n}$.

On the other hand, since T is reductive, for an algebraic representation U of T, the module $\mathrm{ind}_{(\mathfrak{t},T)}^{(\mathfrak{b},T)}(U) = \mathcal{U}(\mathfrak{b}) \otimes_{\mathcal{U}(\mathfrak{t})} U$ is a projective object in $\mathcal{M}(\mathfrak{g}, T)$. Moreover, the adjointness morphism $\mathrm{ind}_{(\mathfrak{t},T)}^{(\mathfrak{b},T)}(U) \longrightarrow U$ is an epimorphism, as we discussed in 3.1.5. This implies that the functor $V \longmapsto V_\mathfrak{n}$ has a derived functor from $D^-(\mathfrak{b}, T)$ into $D^-(T)$.

We can consider $\mathcal{U}(\mathfrak{b})$ as a left $\mathfrak{n}$-module for left multiplication and right $\mathfrak{t}$-module for right multiplication. By Poincaré-Birkhoff-Witt theorem, we see that $\mathrm{ind}_{(\mathfrak{t},T)}^{(\mathfrak{b},T)}(U)$ is a free $\mathfrak{n}$-module. Therefore, the composition of the left derived functors of the functor $V \longmapsto V_\mathfrak{n}$ from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(T)$ composed with the forgetful functor from $\mathcal{M}(T)$ into the category of vector spaces agree with Lie algebra homology with respect to $\mathfrak{n}$. Therefore, by abuse of notation, we denote by $H_\bullet(\mathfrak{n}, -) : D^-(\mathfrak{b}, T) \longrightarrow D^-(T)$ the left derived functor of $V \longmapsto V_\mathfrak{n}$.

From the standard complex, we see that the left cohomological dimension of the functor zeroth $\mathfrak{n}$-homology functor is finite. Therefore, the left cohomological dimension of the functor $V \longmapsto V_\mathfrak{n}$ is also finite. Therefore, the functor $H_\bullet(\mathfrak{n}, -)$ extends to an exact functor from $D(\mathfrak{b}, T)$ into $D(T)$ of finite amplitude [1, Ch. 5, 3.4.5]. In particular, we can view it as a functor from $D^+(\mathfrak{b}, T)$ into $D^+(T)$.

By 6.1.1 and [1, Ch. V, 1.7.1] it follows that the following result holds.

6.1.2. **Theorem.** *The functor $H_\bullet(\mathfrak{n}, -) : D^+(\mathfrak{g}, T) \longrightarrow D^+(T)$ is a left adjoint of the forgetful functor from $\mathcal{D}^+(T)$ into $\mathcal{D}^+(\mathfrak{b}, T)$.*

6.2. **A Frobenius reciprocity.** Combining the adjoint pairs from 5.2.1 and 6.1.2, we get the following diagram of functors

$$D^+(T) \underset{H_\bullet(\mathfrak{n},-)}{\overset{\text{For}}{\rightleftarrows}} D^+(\mathfrak{b}, T) \underset{\text{For}}{\overset{\mathrm{R}\, I^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}}{\rightleftarrows}} D^+(G)$$

where red arrows represent right adjoints of the blue arrows. This implies that the composition of red arrows is a right adjoint of the composition of blue arrows. Therefore, we have the following result.

6.2.1. **Theorem.** *The functor $\mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)} : D^b(T) \longrightarrow D^b(G)$ is a right adjoint of the functor $H_\bullet(\mathfrak{n}, -) : D^b(G) \longrightarrow D^b(T)$.*

6.3. **Borel-Weil-Bott theorem.** Let μ be a weight of T. We denote by $\mathbb{C}_\mu$ the one-dimensional representation of T corresponding to the weight μ.

Let F_λ be the irreducible finite-dimensional representation of G with lowest weight λ. Since the category $\mathcal{M}(T)$ is semisimple, by Kostant's theorem, we know that

$$H_\bullet(\mathfrak{n}, D(F_\lambda)) = \bigoplus_{p=0}^{\dim \mathfrak{n}} D\left(\bigoplus_{w\in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}\right)[p].$$

Let μ be a weight of T. Then, by 6.2.1, we have

$$\begin{aligned}\mathrm{Hom}_{D^+(G)}(D(F_\lambda), \mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(D(\mathbb{C}_\mu))[q]) &= \mathrm{Hom}_{D^+(T)}(H_\bullet(\mathfrak{n}, D(F_\lambda)), D(\mathbb{C}_\mu)[q]) \\ &= \mathrm{Hom}_{D^+(T)}\left(\bigoplus_{p=0}^{\dim \mathfrak{n}} D\left(\bigoplus_{w\in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}\right)[p], D(\mathbb{C}_\mu)[q]\right)\end{aligned}$$

Therefore, $\mathrm{Hom}_{D^+(G)}(D(F_\lambda), \mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(D(\mathbb{C}_\mu))[q]) = 0$ if μ is not equal to $w(\lambda-\rho)+\rho$ for some $w \in W$. If $\mu = w(\lambda-\rho)+\rho$ for $w \in W(p)$,

$$\mathrm{Hom}_{D^+(G)}(D(F_\lambda), \mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(D(\mathbb{C}_\mu))[q]) = \mathbb{C}$$

if $q = p$ and

$$\mathrm{Hom}_{D^+(G)}(D(F_\lambda), \mathrm{R}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(D(\mathbb{C}_\mu))[q]) = 0$$

otherwise.

This implies that $\mathrm{R}^p\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(\mathbb{C}_{w(\lambda-\rho)+\rho}) = F_\lambda$ for $p = \ell(w)$ and zero otherwise.

6.3.1. **Theorem** (Borel-Weil-Bott). *Let λ is an antidominant weight of T and $w \in W$. Then we have*

$$\mathrm{R}^{\ell(w)}\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(\mathbb{C}_{w(\lambda-\rho)+\rho}) = F_\lambda$$

and

$$\mathrm{R}^p\,\mathrm{I}^{(\mathfrak{g},G)}_{(\mathfrak{b},T)}(\mathbb{C}_{w(\lambda-\rho)+\rho}) = 0$$

for $p \neq \ell(w)$.

References

[1] Dragan Miličić, *Lectures on Derived Categories*, unpublished manuscript.