ON THE GEOMETRIC APPROACH TO THE DISCRETE SERIES

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Dedicated to the memory of Harish-Chandra, in admiration.

1. Introduction

Let G_0 be a connected semisimple Lie group with finite center. Discrete series representations of G_0 are the irreducible unitary representations of G_0 with square integrable matrix coefficients. They appear discretely in the decomposition of the regular representation of G_0 on $L^2(G_0)$; i.e., their Plancherel measure is positive. In his two seminal papers [9] and [11], Harish-Chandra determined the necessary and sufficient condition for G_0 to admit discrete series representations. Let K_0 be a maximal compact subgroup of G_0. Then G_0 has discrete series if and only if the ranks of G_0 and K_0 are equal. In this situation, a maximal torus T_0 in K_0 is a (compact) Cartan subgroup in G_0. A discrete series representation has to have a regular real infinitesimal character. Moreover, the (distribution) character of a discrete series representation must be a tempered invariant eigendistribution on G_0.

In the first paper [9], Harish-Chandra establishes that every tempered invariant eigendistribution with regular real infinitesimal character is completely determined by its restriction to the elliptic set; i.e., the set of conjugacy classes represented by regular elements of T_0. He also describes all of these restrictions explicitly. While the first statement is a straightforward application of Harish-Chandra's "matching conditions", the exhaustion part in the second statement is technically difficult.

In the second paper [11], Harish-Chandra relates the above invariant eigendistributions to the characters of discrete series representations, by giving explicit formulas for the restriction of the latter to the elliptic set in G_0. In particular, the characters of discrete series span the space of tempered invariant eigendistributions with regular real infinitesimal character. This part of the proof is intertwined with the proof of the "discrete part" of the Plancherel formula for G_0 and an analogue of the orthogonality relations for discrete series characters (compare [18]). The flowchart of the argument is conceptually very similar to Hermann Weyl's proof of the character formula for irreducible finite-dimensional representations of connected compact Lie groups (though the adaptation to the discrete series requires new ideas and introduces tremendous technical complications).

In [15], a different approach to the discrete series is introduced. It is based on the localization theory of Harish-Chandra modules developed by Alexander Beilinson and Joseph Bernstein [2], [3]. Let $\mathfrak{g}$ be the complexified Lie algebra of G_0 and X the flag variety of $\mathfrak{g}$. Let K be the complexification of K_0. Under certain positivity conditions, they establish the equivalence of categories of Harish-Chandra sheaves (i.e., K-equivariant coherent $\mathcal{D}$-modules) on X with the categories of Harish-Chandra modules of the pair $(\mathfrak{g}, K)$. Using the connection of $\mathfrak{n}$-homology (for $\mathfrak{n}$ corresponding

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to the nilpotent subgroup in Iwasawa decomposition of G_0) with leading exponents in the Harish-Chandra expansion of matrix coefficients [23, II.2, Thm. 1], [6, 8.24], one establishes the relationship between the support of the localizations of an irreducible Harish-Chandra module and asymptotics of its matrix coefficients. This result gives a geometric proof of the existence criterion for discrete series. Moreover, it establishes that the discrete series are global sections of (irreducible) standard Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$ attached to closed K-orbits Q in the flag variety X and irreducible K-equivariant connections τ on Q compatible with a real regular infinitesimal character. A simple counting argument shows that Harish-Chandra's list of irreducible characters of discrete series agrees with the geometric construction via $\mathcal{D}$-modules, but the explicit connection is not immediately clear.

In this paper we make this connection explicit. Our argument is based on a simple geometric formula for the $\mathfrak{n}$-homology of modules over the enveloping algebra $\mathcal{U}(\mathfrak{g})$ for a regular infinitesimal character proved in Theorem 2.6. Let x be a point in the flag variety, $\mathfrak{b}_x$ the corresponding Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. Let N_x be the unipotent subgroup corresponding to $\mathfrak{n}_x$; its orbits in X define a Bruhat stratification of X. Roughly speaking, the weight subspaces of $\mathfrak{n}_x$-homology of a Harish-Chandra module are determined by the (derived) direct images to a point of restrictions of the corresponding Harish-Chandra sheaf to each Bruhat cell. In Section 4, we use this formula to reprove a result of Wilfried Schmid which describes the $\mathfrak{n}$-homology of the discrete series representations $\Gamma(X,\mathcal{I}(Q,\tau))$ with respect to nilpotent radical $\mathfrak{n}$ of a Borel subalgebra $\mathfrak{b}$ which is stable under the action of the Cartan involution attached to K_0. Using a special case of the Osborne conjecture [17], this gives the formula for the character of these representations on the elliptic set in G_0. This proof is formally similar to the deduction of the Weyl character formula from Kostant's formula for $\mathfrak{n}$-homology of irreducible finite-dimensional representations of connected compact Lie groups. Establishing the formula for the character of $\Gamma(X,\mathcal{I}(Q,\tau))$ makes the map from geometric parameters of discrete series into Harish-Chandra parameters completely explicit. As a consequence it also implies that the space of all tempered invariant eigendistributions on G_0 with a given regular infinitesimal character is spanned by the characters of discrete series; what immediately implies the most difficult technical result in [9] (the existence of tempered invariant eigendistributions Θ_λ in [9, Thm. 3]).

The machinery established above allows us to prove some other results on the discrete series by geometric methods. As an illustration, we explain in Section 6 how a natural K-equivariant filtration of standard Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$ leads to a very simple proof of Blattner's conjecture [16].

Our arguments can be extended to the Harish-Chandra class of reductive Lie groups as explained in an appendix to [14] and [12]. To simplify the notation and reduce a number of technical issues, we leave this as an exercise to an interested reader.

During two weeks in October 2023, one of us (D. M.) visited at Harish-Chandra Research Institute in Prayagraj, UP, India, during Centennial Celebration of the birth of Harish-Chandra. In the first week, during the workshop "Representation theory of real Lie groups and automorphic forms", he gave a series of lectures on the modern view of basic results of Harish-Chandra on representation theory of reductive Lie groups. During the lectures and in inspiring discussions after them, the audience asked a lot of questions about the relation between the original results

of Harish-Chandra and their "modern" interpretation. The present paper is an attempt to answer some of these questions related to discrete series in a coherent fashion. We thank the organizers for their hospitality and inspiring atmosphere during the workshop.

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2. A geometric formula for $\mathfrak{n}$-homology

2.1. **Geometric preliminaries.** We start with recalling some basic geometric setup (one can consult [22, Ch. II] for more details). Let $\mathfrak{g}$ be a complex semisimple Lie algebra and X its flag variety. For any $x \in X$, we denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra. Denote by $\mathcal{B}$ the tautological vector subbundle of the trivial bundle $X \times \mathfrak{g}$ over X with fiber $\mathfrak{b}_x$ over $x \in X$. For any $x \in X$, we denote by $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. Let $\mathcal{N}$ be the vector subbundle of $\mathcal{B}$ with fibers $\mathfrak{n}_x$, $x \in X$. Then the quotient vector bundle $\mathcal{H} = \mathcal{B}/\mathcal{N}$ is trivial. We denote by $\mathfrak{h}$ the space of its global sections. We call $\mathfrak{h}$ the *(abstract) Cartan algebra* of $\mathfrak{g}$. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. Then there is a canonical linear isomorphism of $\mathfrak{c}$ with $\mathfrak{h}$. The dual isomorphism of $\mathfrak{h}^*$ with $\mathfrak{c}^*$ we call the *specialization* at $x \in X$.

Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and $\mathcal{Z}(\mathfrak{g})$ its center. We have the canonical Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ [5, Ch. VII, §6, no. 4], defined in the following way. For any $x \in X$, $\mathcal{Z}(\mathfrak{g})$ is contained in the sum of the subalgebra $\mathcal{U}(\mathfrak{b}_x)$ and the right ideal $\mathfrak{n}_x\mathcal{U}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. Hence, we have the natural projection of $\mathcal{Z}(\mathfrak{g})$ into $\mathcal{U}(\mathfrak{b}_x)/(\mathfrak{n}_x U(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{b}_x)) = \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x\mathcal{U}(\mathfrak{b}_x) = \mathcal{U}(\mathfrak{c})$. Its composition with the natural isomorphism of $\mathcal{U}(\mathfrak{c})$ with $\mathcal{U}(\mathfrak{h})$ is independent of x and, by definition, equal to γ.

The specialization $\mathfrak{h}^* \longrightarrow \mathfrak{c}^*$ identifies the roots of the pair $(\mathfrak{g}, \mathfrak{c})$ with a reduced root system Σ in $\mathfrak{h}^*$ which we call the *root system* of $\mathfrak{g}$. We choose a positive set of roots Σ^+ in Σ such that, for any $x \in X$, the root subspaces corresponding to the specialization of these roots span $\mathfrak{n}_x$. Let Π be the set of simple roots determined by Σ^+. Denote by W the Weyl group of Σ and S the set of reflections with respect to simple roots in Π. We denote by $\ell : W \longrightarrow \mathbb{Z}_+$ the corresponding length function on W. Let ρ be the half sum of roots in Σ^+.

Since $\mathfrak{h}$ is abelian, the enveloping algebra $\mathcal{U}(\mathfrak{h})$ is equal to the symmetric algebra $S(\mathfrak{h})$ of $\mathfrak{h}$, which is isomorphic to the algebra of all polynomials on $\mathfrak{h}^*$. Therefore, for any $\lambda \in \mathfrak{h}^*$, the composition φ_λ of the homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ with the evaluation at $\lambda + \rho$ is a character of $\mathcal{Z}(\mathfrak{g})$. By a classic result of Harish-Chandra [5, Ch. VIII, §2, no. 5, Cor. 1 of Thm. 2], $\ker \varphi_\lambda = \ker \varphi_\mu$ if and only if there exists $w \in W$ such that $\lambda = w\mu$. This map establishes a bijection of W-orbits in $\mathfrak{h}^*$ and maximal ideals in $\mathcal{Z}(\mathfrak{g})$. Let $\theta = W\lambda \subset \mathfrak{h}^*$ be a W-orbit, and denote by $J_\theta = \ker \varphi_\lambda$ the corresponding maximal ideal in $\mathcal{Z}(\mathfrak{g})$. We denote by $\mathcal{U}_\theta$ the quotient $\mathcal{U}(\mathfrak{g})/J_\theta\mathcal{U}(\mathfrak{g})$.

2.2. **Localization of $\mathcal{U}_\theta$-modules.** For any $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein [2] constructed a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on the flag variety X with a natural algebra homomorphism $\mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D}_\lambda)$. Moreover, this homomorphism factors through $\mathcal{U}_\theta$ and the induced map $\mathcal{U}_\theta \longrightarrow \Gamma(X, \mathcal{D}_\lambda)$ is an isomorphism [22, Ch. II, Thm. 6.1.(i)].

Denote by $\mathcal{M}(\mathcal{U}_\theta)$ the category of $\mathcal{U}_\theta$-modules, and by $\mathcal{M}(\mathcal{D}_\lambda)$ the category of (quasicoherent) $\mathcal{D}_\lambda$-modules on X. One define the functors

$$\begin{tikzcd} \mathcal{M}(\mathcal{D}_\lambda) \arrow[r, bend left, "\Gamma(X,-)"] & \mathcal{M}(\mathcal{U}_\theta) \arrow[l, bend left, "\Delta_\lambda"] \end{tikzcd}$$

where $\Gamma(X,-)$ is the functor of global sections and

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for a module V in $\mathcal{M}(\mathcal{U}_\theta)$. The functor Δ_λ is called the *localization at* λ. Clearly, the functor Δ_λ is a left adjoint of $\Gamma(X,-)$.

Let $\Sigma\check{}$ be the dual root system of Σ. Denote by $\alpha\check{}$ the dual root of α. We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha\check{}(\lambda) \neq 0$ for any $\alpha \in \Sigma$.

Assume that the orbit θ consists of regular elements of $\mathfrak{h}^*$. Then $\mathcal{U}_\theta$ has finite cohomological dimension [22, Ch. III, Thm. 1.4]. Therefore there exists the left derived functor $L\Delta_\lambda$ between bounded derived categories $D^b(\mathcal{D}_\lambda)$ of $\mathcal{M}(\mathcal{D}_\lambda)$ and $D^b(\mathcal{U}_\theta)$ of $\mathcal{M}(\mathcal{U}_\theta)$. This functor is a left adjoint of the functor $R\Gamma$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{U}_\theta)$. Beilinson and Bernstein proved the following result [3].

Theorem 2.1. *Let θ be a regular W-orbit in $\mathfrak{h}^*$ and $\lambda \in \theta$. Then the functors*

$$\begin{tikzcd} D^b(\mathcal{D}_\lambda) \arrow[r, bend left, "R\Gamma"] & D^b(\mathcal{U}_\theta) \arrow[l, bend left, "L\Delta_\lambda"] \end{tikzcd}$$

are mutually quasiinverse equivalences of categories.

We say that $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\alpha\check{}(\lambda) \notin \{1, 2, \ldots\}$ for all $\alpha \in \Sigma^+$. For antidominant λ, the above theorem is a direct consequence of the following result [2].

Theorem 2.2. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functors*

$$\begin{tikzcd} \mathcal{M}(\mathcal{D}_\lambda) \arrow[r, bend left, "\Gamma(X,-)"] & \mathcal{M}(\mathcal{U}_\theta) \arrow[l, bend left, "\Delta_\lambda"] \end{tikzcd}$$

are mutually quasi-inverse equivalences of categories.

One can view this result as a vast generalization of the Borel-Weil theorem.

2.3. **Intertwining functors.** Let $\lambda \in \mathfrak{h}^*$ be regular. For any $w \in W$, the functor $L\Delta_{w\lambda} \circ R\Gamma : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda})$ is an equivalence of categories by Theorem 2.1.

In [3], Beilinson and Bernstein describe this functor geometrically under some additional conditions. We sketch their construction here (compare [22, Ch. III, Sec. 3]). Let Z_w be the variety of pairs of Borel subalgebras in relative position w in $X \times X$. Then Z_w, $w \in W$, are the orbits for the diagonal action of the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ on $X \times X$. Denote by $p_1 : Z_w \longrightarrow X$ and $p_2 : Z_w \longrightarrow X$ the restrictions of the projections to the first, resp. second factor in $X \times X$. Then p_1 and p_2 are locally trivial fibrations with fibers which are affine spaces of dimension $\ell(w)$ [22, Ch. III, 3.2]. In particular, $\dim Z_w = \ell(w) + \dim X$ for any $w \in W$.

The twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X determines a compatible twisted sheaf $\mathcal{D}_\lambda^{p_i}$ on Z_w for $i = 1, 2$. Let $p_2^+ : \mathcal{M}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{D}_\lambda^{p_2})$ be the $\mathcal{D}$-module inverse image functor. This functor is exact and therefore determines trivially the functor between corresponding derived categories of $\mathcal{D}$-modules. Analogously, we have the direct image functor $p_{1,+} : D^b(\mathcal{D}_{w\lambda}^{p_1}) \longrightarrow D^b(\mathcal{D}_{w\lambda})$. Since the twisted sheaves of differential operators $\mathcal{D}_\lambda^{p_2}$ and $\mathcal{D}_{w\lambda}^{p_1}$ differ by a twist by the invertible $\mathcal{O}_{Z_w}$-module $\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho))$, we can define the functor

$$\mathcal{V}^{\cdot} \longrightarrow p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^{\cdot}))$$

from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_{w\lambda})$.

There exists a right exact functor $I_w : \mathcal{M}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{D}_{w\lambda})$ such that its left derived functor $LI_w : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda})$ is the functor above

$$LI_w(\mathcal{V}^{\cdot}) = p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^{\cdot}))$$

for any $\mathcal{V}^{\cdot}$ in $D^b(\mathcal{D}_\lambda)$. The functor LI_w is the *intertwining functor* corresponding to $w \in W$.

For any subset Θ of Σ^+, we say that $\lambda \in \mathfrak{h}^*$ is Θ-*antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer for any $\alpha \in \Theta$. Put

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\alpha \in -\Sigma^+\}$$

for $w \in W$. The following result is what we alluded to above.

Theorem 2.3. *Let $w \in W$ and let $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant and regular. Then the functors $LI_w \circ L\Delta_\lambda$ and $L\Delta_{w\lambda}$ are isomorphic.*

In particular, we have the following.

Corollary 2.4. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be regular antidominant. Then the functors $L\Delta_{w\lambda}$ and $LI_w \circ \Delta_\lambda$ are isomorphic.*

Hence, the localization $\Delta_\lambda(V)$ of a $\mathcal{U}_\theta$-module V, for a regular antidominant $\lambda \in \theta$, determines all (derived) localizations $L^p\Delta_{w\lambda}(V),\ p \in \mathbb{Z}_+$.

2.4. **$\mathfrak{n}$-homology.** Let θ be a W-orbit consisting of regular elements. Let V be an object in $\mathcal{M}(\mathcal{U}_\theta)$.

Let $x \in X$. As before, we pick a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. The Lie algebra homology groups $H_{\cdot}(\mathfrak{n}_x, V)$ are $\mathfrak{c}$-modules. Via the dual of the specialization map, we can view them as $\mathfrak{h}$-modules.

By a result of Casselman and Osborne [7], [24], (compare [22, Ch. III, 2.4]), we know that the Lie algebra homology groups $H_{\cdot}(\mathfrak{n}_x, V)$ are semisimple $\mathfrak{h}$-modules. If we denote by $H_p(\mathfrak{n}_x, V)_{(\mu)}$ the μ-eigenspace of $H_p(\mathfrak{n}_x, V)$ for any $\mu \in \mathfrak{h}^*$, we have

$$H_p(\mathfrak{n}_x, V) = \bigoplus_{w \in W} H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)}$$

for any $p \in \mathbb{Z}_+$.

Therefore, to calculate Lie algebra homology $H_{\cdot}(\mathfrak{n}_x, V)$ we have to calculate $H_{\cdot}(\mathfrak{n}_x, V)_{(w\lambda+\rho)}$ for all $w \in W$. In this section, we prove a formula for this calculation (Theorem 2.6 below).

For an abelian category $\mathcal{A}$ we denote by $D : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ the functor which sends an object A of $\mathcal{A}$ into the complex $D(A)$ which is A in degree 0 and 0 in other degrees.

Let $\mathcal{O}_X$ be the sheaf of regular functions on X and $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at $x \in X$. Let $\mathcal{V}$ be an $\mathcal{O}_X$-module. For $x \in X$, we denote by $\mathcal{V}_x$ the stalk of $\mathcal{V}$ at x.

Also, we denote by $\mathfrak{m}_x$ the maximal ideal in $\mathcal{O}_{X,x}$ consisting of germs vanishing at x. The *geometric fiber* $T_x(\mathcal{V})$ of $\mathcal{V}$ is $\mathcal{O}_{X,x}/\mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{V}_x$.

First, we observe that the above components of $\mathfrak{n}_x$-homology are related to derived geometric fibers of derived localizations. We have the following formula

$$H_p(\mathfrak{n}_x, V)_{(\mu+\rho)} = H^{-p}(LT_x(L\Delta_\mu(D(V))))$$

for any $\mu \in \theta$ and $p \in \mathbb{Z}_+$ [22, Ch. III, 2.6]. Combining this with Corollary 2.4, we see that for regular antidominant λ,

$$H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)} = H^{-p}(LT_x(LI_w(D(\Delta_\lambda(V)))))$$

for any $\mathcal{U}_\theta$-module V.

Let $i_x : \{x\} \longrightarrow X$ be the canonical inclusion. Then the $\mathcal{D}$-module inverse image functor Li_x^+ is equal to LT_x.

It follows that

$$(LT_x \circ LI_w)(\mathcal{V}^{\cdot}) = (Li_x^+ \circ LI_w)(\mathcal{V}^{\cdot}) = (Li_x^+ \circ p_{1+})(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^{\cdot}))$$

for any bounded complex $\mathcal{V}^{\cdot}$ in $D^b(\mathcal{D}_\lambda)$.

Moreover, $Y = p_1^{-1}(\{x\})$ is a closed $\ell(w)$-dimensional affine subspace in Z_w. The codimension of Y in Z_w is $\dim X$.

Denote by $j : Y \longrightarrow Z_w$ the natural inclusion. Let $q_i : Y \longrightarrow X$ the restriction of $p_i : Z_w \longrightarrow X$ to Y for $i = 1, 2$. Then we have the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z_w \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ \{x\} & \xrightarrow[i_x]{} & X \end{array} .$$

Hence, $Y = \{x\} \times_X Z_w$. Then, by base change ([4, Ch. 6, 8.4], [20, Ch. IV, Thm. 10.2]), we have

$$Li_x^+ \circ p_{1+} = q_{1+} \circ Lj^+.$$

This in turn implies that

$$\begin{aligned}(LT_x \circ LI_w)(\mathcal{V}^{\cdot}) &= (q_{1+} \circ Lj^+)(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^{\cdot})) = q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_Y} Lj^+(p_2^+(\mathcal{V}^{\cdot}))) \\ &= q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_Y} L(p_2 \circ j)^+(\mathcal{V}^{\cdot}))) = q_{1+}(j^*(\mathcal{T}_w) \otimes_{\mathcal{O}_Y} Lq_2^+(\mathcal{V}^{\cdot})))\end{aligned}$$

since p_2^+ is an exact functor.

Lemma 2.5. *We have*

$$j^*(\mathcal{T}_w) = \mathcal{O}_Y.$$

Proof. Let N_x be the unipotent subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{n}_x$. Clearly, Y is an N_x-orbit in Z_w under the restriction of the action of $\mathrm{Int}(\mathfrak{g})$ on $X \times X$. Also, $j^*(\mathcal{T}_w)$ is an N_x-equivariant invertible $\mathcal{O}_Y$-module. Since N_x is unipotent, it follows that it is isomorphic to $\mathcal{O}_Y$. □

Hence we finally get

$$(LT_x \circ LI_w)(\mathcal{V}^{\cdot}) = q_{1+}(Lq_2^+(\mathcal{V}^{\cdot}))).$$

Let B_x be the Borel subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to x. Then B_x acts on X and its orbits are the Bruhat cells $C(w)$, $w \in W$. Let $i_w : C(w) \longrightarrow X$ be the natural inclusion. Then the map $k : y \longmapsto (x, y)$ is an isomorphism of $C(w)$ onto Y. Moreover, we have $q_2 \circ k = p_2 \circ j \circ k = i_w$. It follows that

$$k^+(Lq_2^+(\mathcal{V}^\cdot))) = L(q_2 \circ k)^+(\mathcal{V}^\cdot) = Li_w^+(\mathcal{V}^\cdot)$$

and

$$Lq_2^+(\mathcal{V}^\cdot) = k_+(Li_w^+(\mathcal{V}^\cdot)).$$

Hence

$$(LT_x \circ LI_w)(\mathcal{V}^\cdot) = q_{1+}(k_+(Li_w^+(\mathcal{V}^\cdot))) = (q_1 \circ k)^+(Li_w^+(\mathcal{V}^\cdot)).$$

The map $q_1 \circ k$ is a map of $C(w)$ into the point x. If we denote this map by $\pi_w : C(w) \longrightarrow \{pt\}$, we finally get the following formula

$$(LT_x \circ LI_w)(\mathcal{V}^\cdot) = \pi_{w,+}(Li_w^+(\mathcal{V}^\cdot)).$$

Putting this together with the first part of the calculation, we get the following result.

Theorem 2.6. *Let $\lambda \in \theta$ be antidominant and regular. Then for any $x \in X$, $w \in W$, and $\mathcal{U}_\theta$-module V, we have*

$$H_p(\mathfrak{n}_x, V)_{(w\lambda+\rho)} = H^{-p}(\pi_{w,+}(Li_w^+(D(\Delta_\lambda(V)))))$$

for any $p \in \mathbb{Z}_+$.

3. Closed K-orbits in the equal rank case

Let G_0 be a connected semisimple Lie group with finite center. Let K_0 be a maximal compact subgroup of G_0. In this section we assume we are in the equal rank case; i.e., that $\operatorname{rank} G_0 = \operatorname{rank} K_0$. Let T_0 be a maximal torus in K_0. Then T_0 is a compact Cartan subgroup of G_0. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ be the complexified Lie algebras of G_0, K_0 and T_0 respectively. Denote by K the complexification of K_0. Then the connected algebraic group K acts naturally on the flag variety X of $\mathfrak{g}$. In this section we describe the structure of closed K-orbits in X, and their stratification given by a Bruhat stratification of X.

3.1. **Description of closed orbits.** Denote by σ the Cartan involution of $\mathfrak{g}$ determined by $\mathfrak{k}$. Then σ fixes $\mathfrak{k}$. Therefore, it also fixes $\mathfrak{t}$. Let R be the root system in $\mathfrak{t}^*$ of the pair $(\mathfrak{g}, \mathfrak{t})$. Let $\alpha \in R$, and ξ in the root subspace $\mathfrak{g}_\alpha$. Then for any $\eta \in \mathfrak{t}$, we have

$$\alpha(\eta)\sigma(\xi) = \sigma(\alpha(\eta)\xi) = \sigma([\eta, \xi]) = [\eta, \sigma(\xi)].$$

Hence $\sigma(\xi)$ is also in $\mathfrak{g}_\alpha$. In particular, it has to be proportional to ξ. Hence, σ is either 1 or -1 on $\mathfrak{g}_\alpha$. A root $\alpha \in R$ is *compact (imaginary)* in the first case, and *noncompact (imaginary)* in the second case. Denote by $R_c \subset R$ the set of all compact roots. Since

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_c} \mathfrak{g}_\alpha,$$

the set R_c is naturally identified with the root system of $(\mathfrak{k}, \mathfrak{t})$. The complement $R - R_c = R_n$ is the set of all noncompact roots.

It is well known that the group K acts on X with finitely many orbits (see, for example, [22, Ch. IV, Prop. 2.2]). Since we want to describe the K-orbits in the flag variety X of $\mathfrak{g}$, without any loss of generality we can assume that K is a subgroup of the group $\operatorname{Int}(\mathfrak{g})$.

Let Ω be the subvariety of all σ-stable Borel subalgebras in X. Clearly, Ω is a union of K-orbits. More precisely, by [15, Lem. 6.16], Ω is the union of all closed K-orbits in X. Hence, the closed K-orbits are the connected components of Ω.

Fix a closed K-orbit Q in X and a point $x \in Q$. Let S_x be the stabilizer of x in K. Its Lie algebra $\mathfrak{s}_x$ is equal to $\mathfrak{k} \cap \mathfrak{b}_x$. Hence, it is a solvable Lie algebra. On the other hand, since Q is a closed K-orbit, Q is a projective variety. Hence S_x must be a parabolic subgroup of K. It follows that S_x is a Borel subgroup of K and $\mathfrak{s}_x = \mathfrak{k} \cap \mathfrak{b}_x$ is a Borel subalgebra of $\mathfrak{k}$. Therefore the orbit map $K \ni k \longmapsto k \cdot x \in Q$ factors through the flag variety X_K of $\mathfrak{k}$. The induced map $X_K \longrightarrow Q$ is an isomorphism which is the inverse of the map $Q \ni y \longmapsto \mathfrak{k} \cap \mathfrak{b}_y \in X_K$.

We have proved the following result.

Theorem 3.1. (i) *The variety Ω is the disjoint union of all closed K-orbits in the flag variety X.*

(ii) *Let Q be a closed K-orbit in X. Then the map $\mathfrak{b}_x \longmapsto \mathfrak{b}_x \cap \mathfrak{k}$ defines a K-equivariant isomorphism of Q with the flag variety X_K of $\mathfrak{k}$.*

Now we remark that any closed K-orbit Q contains a point x such that $\mathfrak{b}_x$ contains the Cartan subalgebra $\mathfrak{t}$. Let y be a point in Q. Then $\mathfrak{b}_y \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{k}$ contained in $\mathfrak{b}_y$. Then, since all Cartan subalgebras of $\mathfrak{k}$ are K-conjugate, $\mathfrak{c}$ is K-conjugate of $\mathfrak{t}$. It follows that a K-conjugate $\mathfrak{b}_x$ of $\mathfrak{b}_y$ contains $\mathfrak{t}$ and $x \in Q$.

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{t}$. Then any Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ contains the Cartan subalgebra $\mathfrak{t}$ and it is σ-stable. Therefore, it is in Ω. Moreover, by Theorem 3.1, the number of closed K-orbits is equal to the cardinality of the set $\{x \in X \mid \mathfrak{b}_x \cap \mathfrak{k} = \mathfrak{b}\}$. This number is equal to the number of sets of positive roots R^+ in R which contain a fixed set of positive roots R_c^+ in R_c. It follows that it is equal to the number of Weyl chambers of R contained in a fixed Weyl chamber of R_c. Let W_K be the Weyl group of $\mathfrak{k}$. By specialization at x, it can be identified with a subgroup of W. This immediately implies that the number of closed K-orbits is equal to $\operatorname{Card}(W/W_K) = \operatorname{Card}(W)/\operatorname{Card}(W_K)$.

Proposition 3.2. *The number of closed K-orbits in X is equal to* $\operatorname{Card}(W/W_K)$.

3.2. **A stratification of closed K-orbits.** Let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{t}$. Then it is σ-stable and therefore determines a point in a closed K-orbit in X. Moreover, $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{t}$. We consider the specialization $\mathfrak{h}^* \longrightarrow \mathfrak{t}^*$ determined by $\mathfrak{b}$. This defines an isomorphism of W with the Weyl group of the root system R in $\mathfrak{t}^*$. Moreover, it identifies W_K with a subgroup of W generated by reflections with respect to compact roots. The set of positive roots Σ^+ determines a set of compact positive roots. This set determines a set of simple roots in the root system of compact roots. Corresponding reflections generate W_K and define the length function ℓ_K on W_K.

Let B be the Borel subgroup of $\operatorname{Int}(\mathfrak{g})$ corresponding to $\mathfrak{b}$. Denote by $C(w)$ the B-orbit in X corresponding the element $w \in W$. Let B_K be the Borel subgroup of K corresponding to $\mathfrak{b} \cap \mathfrak{k}$. Then B_K is a subgroup of B.

We denote by $C_K(w)$, $w \in W_K$, the Bruhat cells in the flag variety X_K of $\mathfrak{k}$ corresponding to the action of B_K. As we proved in Theorem 3.1.(ii), for any closed K-orbit Q, the map $x \longmapsto \mathfrak{b}_x \cap \mathfrak{k}$ is a K-equivariant isomorphism of Q onto X_K. Clearly, for any $w \in W_K$, it induces by restriction an isomorphism of a B_K-orbit $D_Q(w)$ in Q onto $C_K(w)$. Therefore, Q is the union of $\operatorname{Card}(W_K)$ B_K-orbits.

By Theorem 3.1.(i) and Proposition 3.2, we see that Ω is the union of Card(W) B_K-orbits. Clearly, the Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$ represent all Bruhat cells $C(w)$, $w \in W$, in X with respect to the action of B. Therefore, the intersection $\Omega \cap C(w)$ is nonempty for any $w \in W$. Moreover, for any $w \in W$, the intersection $\Omega \cap C(w)$ is B_K-invariant and therefore a union of B_K-orbits. This in turn implies that $\Omega \cap C(w)$ is a B_K-orbit for any $w \in W$.

Let Q be a closed K-orbit. By the above discussion and Theorem 3.1, we see that there exists a unique point $x \in Q$ which is fixed by B_K. Therefore, there exists a unique $u \in W$ such that $Q \cap C(u) = \{x\}$. We summarize this discussion in the following lemma.

Lemma 3.3. *Let Q be a closed K-orbit in X.*

(i) *There is a unique $u \in W$ such that $Q \cap C(u)$ is a point.*
(ii) *The orbit Q intersects Bruhat cell $C(v)$, $v \in W$, if and only if $v = wu$ for some $w \in W_K$.*
(iii) *The variety $Q \cap C(wu)$ is a B_K-orbit in Q for any $w \in W_K$. More precisely, we have $D_Q(w) = Q \cap C(wu)$ for all $w \in W_K$.*
(iv) *We have $\dim D_Q(w) = \ell_K(w)$.*

4. Calculation of $\mathfrak{n}$-homology for the discrete series

Using Theorem 2.6, we can prove a result of Wilfried Schmid on $\mathfrak{n}$-homology of discrete series representations [25]. To illustrate the simplicity of our argument, we first treat the special case of irreducible finite-dimensional representations of a connected compact semisimple Lie groups due to Bertram Kostant [19]. The method of the proof in both cases is essentially identical.

4.1. **Kostant's theorem.** Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$. Let F be an irreducible finite-dimensional representation with lowest weight λ. Then, by the Borel-Weil theorem, we have $F = \Gamma(X, \mathcal{O}(\lambda))$ [22, Ch. II, Thm. 5.1]. Moreover, $\mathcal{O}(\lambda)$ is a $\mathcal{D}_{\lambda-\rho}$-module. As we remarked before, since $\lambda - \rho$ is regular, we have

$$H_p(\mathfrak{n}, F) = \bigoplus_{w \in W} H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)}.$$

Moreover, we have $\Delta_{\lambda-\rho}(F) = \mathcal{O}(\lambda)$. Therefore, it follows that $Li^+_w(D(\mathcal{O}(\lambda))) = D(\mathcal{O}_{C(w)})$. This implies that

$$H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} = H^{-p}(\pi_{w,+}(D(\mathcal{O}_{C(w)}))).$$

Lemma 4.1. *We have*

$$\pi_{w,+}(D(\mathcal{O}_{C(w)})) = D(\mathbb{C})[\ell(w)].$$

Proof. This follows from [20, Ch. I, Thm. 11.2 and Lem. 11.4]. $\square$

This implies that $H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} = 0$ if $p \neq \ell(w)$ and $H_{\ell(w)}(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} = \mathbb{C}$.

Putting this all together, we get

$$H_p(\mathfrak{n}, F) = \bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}$$

where $W(p)$ is the subset of W consisting of all elements w such that $\ell(w) = p$. We have proven the following result.

Theorem 4.2 (Kostant)**.** *Let F be an irreducible finite-dimensional representation with lowest weight λ. Then*

$$H_p(\mathfrak{n}, F) = \bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho}$$

for $p \in \mathbb{Z}_+$.

This is Kostant's result mentioned above.

4.2. **Schmid's result.** Now we discuss a generalization of Kostant's result corresponding to the $\mathfrak{n}$-homology of discrete series representations. This result has been proved by Schmid [25, Thm. 4.1].

We first recall the geometric version of the classification of discrete series representations of a connected semisimple Lie group G_0 with finite center [15]. First, the discrete series representations exist if and only if the rank of G_0 is equal to the rank K_0. In this situation, we follow the notation from the introduction to the preceding section.

Let V be the Harish-Chandra module of a discrete series representation of G_0. Assume that V is in $\mathcal{M}(\mathcal{U}_\theta)$. Then θ is a regular W-orbit of a unique real strongly antidominant $\lambda \in \mathfrak{h}^*$; i.e., $\alpha\check{}(\lambda) < 0$ for any α in Σ^+ (see [15, Thm. 12.4]). Moreover, there exists a unique closed K-orbit Q and a unique irreducible K-equivariant connection on Q compatible with $\lambda+\rho$ such that $V = \Gamma(X, \mathcal{I}(Q, \tau))$ [15, Thm. 12.5].

Let $\mathfrak{b}$ be a σ-stable Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Denote by $i_Q : Q \longrightarrow X$ the natural immersion of Q, and, for any $v \in W$, by $i_v : C(v) \longrightarrow X$ the natural immersion of the Bruhat cell (with respect to $\mathfrak{b}$) $C(v)$ into X. Since Q is the support of $\mathcal{I}(Q, \tau)$, the restriction of $\mathcal{I}(Q, \tau)$ to the complement of Q is 0. Therefore $Li_v^+(\mathcal{I}(Q, \tau)) = 0$ for any v such that $C(v) \cap Q = \emptyset$. Hence, in this case, $H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q, \tau)))_{(v\lambda+\rho)} = 0$ for $p \in \mathbb{Z}_+$, by Theorem 2.6.

Let $v \in W$ be such that $Q \cap C(v) \neq \emptyset$. As we explained in Lemma 3.3, there exists a unique $u \in W$ such that $Q \cap C(u)$ is a point. Then $v = wu$, for some $w \in W_K$, and $D_Q(w)$ is a smooth subvariety of Q. Denote by $a : D_Q(w) \longrightarrow C(wu)$ and $b : D_Q(w) \longrightarrow Q$ the natural immersions. Then we have the commutative diagram

$$\begin{tikzcd} D_Q(w) \arrow[r, "a"] \arrow[d, "b"'] & C(wu) \arrow[d, "i_{wu}"] \\ Q \arrow[r, "i_Q"'] & X \end{tikzcd}$$

i.e., $D_Q(w)$ is the fiber product $Q \otimes_X C(wu)$. By base change [20, Ch. IV, Thm. 10.2], we have

$$Ri^!_{wu} \circ i_{Q,+} = a_+ \circ Rb^!.$$

Moreover, we have $Li^+_{wu} = Ri^!_{wu}[\dim X - \dim C(wu)]$ and $Lb^+ = Rb^![\dim Q - \dim D_Q(w)]$. Therefore, we have

$$\begin{aligned} Li^+_{wu}(D(\mathcal{I}(Q, \tau))) &= Li^+_{wu}(D(i_{Q,+}(\tau))) = Li^+_{wu}(i_{Q,+}(D(\tau))) \\ &= Ri^!_{wu}(i_{Q,+}(D(\tau)))[\dim X - \ell(wu)] \\ &= a_+(Rb^!(D(\tau)))[\dim X - \ell(wu)] \\ &= a_+(Lb^+(D(\tau)))[\dim X - \ell(wu)][-\dim Q + \dim D_Q(w)] \\ &= a_+(Lb^+(D(\tau)))[\dim X - \dim X_K - \ell(wu) + \ell_K(w)]. \end{aligned}$$

Since τ is a connection on Q, we have

$$Li^{+}_{wu}(D(\mathcal{I}(Q,\tau))) = a_{+}(D(b^{+}(\tau)))[\dim X - \dim X_K - \ell(wu) + \ell_K(w)].$$

Denote by c the map of $D_Q(w)$ into a point $\{pt\}$. Using Lemma 4.1 this finally leads to

$$\begin{aligned}\pi_{w,+}(Li^{+}_{wu}(D(\mathcal{I}(Q,\tau)))) &= \pi_{w,+}(a_{+}(D(b^{+}(\tau)))[\dim X - \dim X_K - \ell(wu) + \ell_K(w)] \\ &= (\pi_w \circ a)_{+}(D(\mathcal{O}_{D_Q(w)}))[\dim X - \dim X_K - \ell(wu) + \ell_K(w)] \\ &= c_{+}(D(\mathcal{O}_{D_Q(w)}))[\dim X - \dim X_K - \ell(wu) + \ell_K(w)] \\ &= D(\mathbb{C})[\dim X - \dim X_K - \ell(wu) + \ell_K(w)][\ell_K(w)] \\ &= D(\mathbb{C})[\dim X - \dim X_K - \ell(wu) + 2\ell_K(w)].\end{aligned}$$

Finally, $\dim X$ is equal to the number of all positive roots; i.e., to half of the number of all roots. Analogously, $\dim X_K$ is equal to the half of the number of all compact roots. Hence, the difference $\dim X - \dim X_K$ is equal to the half of the number of all noncompact roots, i.e., $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})$.

Applying again Theorem 2.6, we have

$$\begin{aligned}H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(wu\lambda+\rho)} &= H^{-p}(\pi_{w,+}(Li^{+}_{wu}(D(\mathcal{I}(Q,\tau))))) \\ &= \begin{cases} 0 & \text{if } p \neq \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w); \\ \mathbb{C} & \text{if } p = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w) \end{cases}\end{aligned}$$

for any $w \in W_K$.

This proves the following result.[1]

Theorem 4.3 (Schmid)**.** *Let V be a discrete series representation such that $V = \Gamma(X, \mathcal{I}(Q,\tau))$.*

Then, if $v \notin W_K u$,

$$H_p(\mathfrak{n}, V)_{(v\lambda+\rho)} = 0,$$

for all $p \in \mathbb{Z}_+$. Moreover, for $w \in W_K$, we have

$$H_p(\mathfrak{n}, V)_{(wu\lambda+\rho)} = \begin{cases} 0 & \textit{if } p \neq \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w); \\ \mathbb{C} & \textit{if } p = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w). \end{cases}$$

4.3. **Kostant's result and the BGG resolution.** In this section we interpret the calculation of $\mathfrak{n}$-homology of finite-dimensional representations using the BGG resolution. This gives us a more precise version of the result, we see that each cohomology class corresponds to a Bruhat cell in X.

Let λ be an antidominant weight. Then, the $\mathcal{D}_{\lambda-\rho}$-module $\mathcal{O}(\lambda)$ can be represented in $D^b(\mathcal{D}_{\lambda-\rho})$ by its Cousin resolution corresponding to the stratification of X by Bruhat cells $C(w)$ as explained in Appendix A.

Let $C(w)$ be a Bruhat cell in X and $i_w : C(w) \longrightarrow X$ the natural inclusion. The cell $C(w)$ admits unique irreducible N-equivariant connection $\mathcal{O}_{C(w)}$. Its direct image $\mathcal{I}(w, \lambda-\rho) = i_{w,+}(\mathcal{O}_{C(w)})$ is a *standard* $\mathcal{D}_{\lambda-\rho}$-module attached to $C(w)$ [22, Ch. V, Sec. 1].

The Cousin resolution (see Theorem A.3 in Appendix) is a complex $\mathcal{C}^{\cdot}$ such that

$$\mathcal{C}^p = \bigoplus_{w \in W(\dim X - p)} \mathcal{I}(w, \lambda-\rho)$$

[1] Clearly, if G_0 is compact, this specializes to Kostant's result.

for any $p \in \mathbb{Z}_+$, with explicitly given differentials.

Clearly, there is a natural monomorphism from $\mathcal{O}(\lambda)$ into the standard $\mathcal{D}_{\lambda-\rho}$-module $\mathcal{I}(w_0, \lambda-\rho)$ attached to the open Bruhat cell $C(w_0)$ (where w_0 is the longest element of the Weyl group W). Therefore, there is a natural morphism ϵ of $D(\mathcal{O}(\lambda))$ into $\mathcal{C}^{\cdot}$ in $D^b(\mathcal{D}_{\lambda-\rho})$. Theorem A.3 says that ϵ is an isomorphism.

Let θ be the Weyl group orbit of $\lambda-\rho$. Since the functor Γ is exact for antidominant λ, $C^{\cdot} = \Gamma(X, \mathcal{C}^{\cdot})$ is isomorphic to $D(F)$ in $D^b(\mathcal{U}_\theta)$ where F is the finite-dimensional $\mathcal{U}(\mathfrak{g})$-module with lowest weight λ; i.e., we get a resolution of F by modules C^p, $p \in \mathbb{Z}_+$. By [22, Ch. V, 1.14], we know that

$$C^p = \bigoplus_{w \in W(\dim X - p)} I(w(\lambda-\rho))$$

for all $p \in \mathbb{Z}_+$. Here $I(\mu)$ are the *duals* of the Verma modules $M(\mu)$ in the category of highest weight modules. This is the *BGG resolution* of F. [2]

By Theorem 2.6, we have

$$H_p(\mathfrak{n}, I(w(\lambda-\rho)))_{(v(\lambda-\rho)+\rho)} = H^{-p}(\pi_{v,+}(Li_v^+(D(\mathcal{I}(w,\lambda-\rho))))).$$

By base change,

$$Li_v^+(D(\mathcal{I}(w,\lambda-\rho))) = 0$$

if $v \neq w$. Hence, we have

$$H_p(\mathfrak{n}, I(w(\lambda-\rho)))_{(v(\lambda-\rho)+\rho)} = 0$$

for all $p \in \mathbb{Z}_+$ if $v \neq w$.

On the other hand, if $v = w$ we have $Li_w^+ = Ri_w^![\dim X - \ell(w)]$, and

$$Li_w^+(D(\mathcal{I}(w,\lambda-\rho))) = Ri_w^!(D(\mathcal{I}(w,\lambda-\rho)))[\dim X - \ell(w)] = D(\mathcal{O}_{C(w)})[\dim X - \ell(w)]$$

by Kashiwara's equivalence of categories. By Lemma 4.1, this implies that

$$\pi_{w,+}(Li_w^+(D(\mathcal{I}(w,\lambda-\rho)))) = D(\mathbb{C})[\dim X].$$

Hence, we conclude that

$$H_p(\mathfrak{n}, I(w(\lambda-\rho)))_{(w(\lambda-\rho)+\rho)} = \begin{cases} \mathbb{C} & \text{if } p = \dim X; \\ 0 & \text{if } p \neq \dim X. \end{cases}$$

Therefore, we proved the following result.

Lemma 4.4. *Let λ be an antidominant weight. Then*

$$H_p(\mathfrak{n}, I(w(\lambda-\rho))) = \begin{cases} \mathbb{C}_{w(\lambda-\rho)+\rho} & \text{if } p = \dim X; \\ 0 & \text{if } p \neq \dim X. \end{cases}$$

This implies the following result.

Corollary 4.5. *Let λ be an antidominant weight. Then*

$$H_p(\mathfrak{n}, C^q) = \begin{cases} \bigoplus_{w \in W(\dim X - q)} \mathbb{C}_{w(\lambda-\rho)+\rho} & \text{if } p = \dim X; \\ 0 & \text{if } p \neq \dim X. \end{cases}$$

[2] More precisely, this is the dual of the original BGG resolution.

By the above discussion, the $\mathfrak{n}$-homology of F is given by the hypercohomology of the $\mathfrak{n}$-homology functor for the complex $C^\cdot$.

More precisely, the $\mathfrak{n}$-homology is the left derived functor of the functor $V \longmapsto V/\mathfrak{n}V$ from $\mathcal{M}(\mathcal{U}_\theta)$ into the category $\mathcal{M}_{ss}(\mathcal{U}(\mathfrak{h}))$ of semisimple $\mathcal{U}(\mathfrak{h})$-modules. We can view this functor as an exact functor from $D^b(\mathcal{U}_\theta)$ into $D^b(\mathcal{M}_{ss}(\mathcal{U}(\mathfrak{h})))$.

Let $w \in W$. Then we can consider the composition of this functor with the functor of taking the weight subspace of the weight $w(\lambda - \rho) + \rho$. This is an exact functor S from $D^b(\mathcal{U}_\theta)$ into the bounded derived category of vector spaces $D^b(\mathbb{C})$. Clearly, we have

$$H_p(\mathfrak{n}, V)_{(w(\lambda-\rho)+\rho)} = H_{-p}(S(D(V))).$$

Therefore, we see that

$$\begin{aligned} H_{-p}(S(D(C^q))) &= H_p(\mathfrak{n}, C^q)_{(w(\lambda-\rho)+\rho)} \\ &= \begin{cases} 0 & \text{if } p \neq \dim X \text{ or } q \neq \dim X - \ell(w); \\ \mathbb{C} & \text{if } p = \dim X \text{ and } q = \dim X - \ell(w). \end{cases} \end{aligned}$$

It follows that

$$S(D(C^q)) = 0$$

if $q \neq \dim X - \ell(w)$; and

$$S(D(C^{\dim X - \ell(w)})) = D(\mathbb{C})[\dim X].$$

We interrupt the proof to prove a simple result in homological algebra. An interested reader can easily supply an alternate argument via spectral sequences.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. We denote by $D^b(\mathcal{A})$ and $D^b(\mathcal{B})$ their bounded derived categories. Let $F : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$ be an exact functor between these triangulated categories.

Lemma 4.6. *Let $C^\cdot$ be a complex in $D^b(\mathcal{A})$. Assume that there exists an integer p_0 such that $F(D(C^p)) = 0$ for all $p \neq p_0$. Then*

$$F(C^\cdot) = F(D(C^{p_0}))[-p_0].$$

Proof. We use freely the results on stupid truncations from [21, Ch. III, 4.4 and 4.5].

For any complex $X^\cdot$ in $D^b(\mathcal{A})$ and an integer s we denote by $\sigma_{\geq s}(X^\cdot)$ and $\sigma_{\leq s}(X^\cdot)$ its stupid truncations. Then we have the distinguished triangle of stupid truncations

$$\begin{tikzcd} & \sigma_{\leq s-1}(X^\cdot) \arrow[dl, "{[1]}"'] & \\ \sigma_{\geq s}(X^\cdot) \arrow[rr] & & X^\cdot \arrow[ul] \end{tikzcd} .$$

Assume first that $F(D(C^p)) = 0$ for all $p \in \mathbb{Z}$. We claim that $F(C^\cdot) = 0$ in this case.

The proof is by induction the length of $C^\cdot$. If $C^\cdot = 0$, the claim is obvious. If $\ell(C^\cdot) = 1$, we have $C^\cdot = D(C^q)[-q]$ for some $q \in \mathbb{Z}$. Therefore, we have

$$F(C^\cdot) = F(D(C^q)[-q]) = F(D(C^q))[-q] = 0.$$

This establishes the claim in this case. Assume now that $\ell(C^\cdot) > 1$. Then there exists $s \in \mathbb{Z}$ such that the lengths of $\sigma_{\leq s-1}(C^\cdot)$ and $\sigma_{\geq s}(C^\cdot)$ are strictly less than $\ell(C^\cdot)$. By the induction assumption, we see that $F(\sigma_{\leq s-1}(C^\cdot)) = F(\sigma_{\geq s}(C^\cdot)) = 0$. From the distinguished triangle of stupid truncations we conclude that $F(C^\cdot) = 0$.

Now we can discuss the general case. If $C^{p_0} \neq 0$, $\ell(C^\cdot)$ is greater or equal to 1. If $\ell(C^\cdot) = 1$, we have $C^\cdot = D(C^{p_0})[-p_0]$. Therefore, we have $F(C^\cdot) = F(D(C^{p_0}))[-p_0]$. Then there exists $s \in \mathbb{Z}$ such that the lengths of $\sigma_{\leq s-1}(C^\cdot)$ and $\sigma_{\geq s}(C^\cdot)$ are strictly less than $\ell(C^\cdot)$. Moreover, either $p_0 < s$ or $p_0 \geq s$.

In the first case, by the induction assumption, $F(\sigma_{\leq s-1}(C^\cdot)) = F(D(C^{p_0}))[-p_0]$. Moreover, by the above claim $F(\sigma_{\geq s}(C^\cdot)) = 0$. From the distinguished triangle of stupid truncations we conclude that $F(C^\cdot) = F(\sigma_{\leq s-1}(C^\cdot)) = F(D(C^{p_0}))[-p_0]$.

In the second case, by the induction assumption, $F(\sigma_{\geq s}(C^\cdot)) = F(D(C^{p_0}))[-p_0]$. Moreover, by the above claim $F(\sigma_{\leq s-1}(C^\cdot)) = 0$. From the distinguished triangle of stupid truncations we conclude that $F(C^\cdot) = F(\sigma_{\geq s}(C^\cdot)) = F(D(C^{p_0}))[-p_0]$. $\square$

Now we go back to the discussion of $\mathfrak{n}$-homology. By Lemma 4.6 it follows that

$$S(C^\cdot) = D(\mathbb{C})[\dim X][-\dim X + \ell(w)] = D(\mathbb{C})[\ell(w)].$$

Since $D(F)$ is isomorphic to its BGG-resolution $C^\cdot$ in $D^b(\mathcal{U}_\theta)$, we see that $S(D(F)) = D(\mathbb{C})[\ell(w)]$. This implies that

$$\begin{aligned} H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} &= H_{-p}(S(D(F))) = H_{-p}(S(C^\cdot)) \\ &= H_{-p}(D(\mathbb{C})[\ell(w)]) = \begin{cases} 0 & \text{if } p \neq \ell(w); \\ \mathbb{C} & \text{if } p = \ell(w). \end{cases} \end{aligned}$$

This finally leads to

$$H_p(\mathfrak{n}, F) = \bigoplus_{w \in W} H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} = \bigoplus_{w \in W(p)} \mathbb{C}_{w(\lambda-\rho)+\rho},$$

i.e., Kostant's theorem (Theorem 4.2).

Therefore, this calculation shows that each dual Verma module in the BGG-resolution of F gives exactly one cohomology class in $\mathfrak{n}$-homology of F.

4.4. **Schmid's result and the Trauber resolution.** In this section we interpret the calculation of $\mathfrak{n}$-homology of discrete series representations using the Trauber resolution [26]. This argument is very similar to that in Section 4.3.

Let λ be antidominant and regular. Then, the $\mathcal{D}_\lambda$-module $\mathcal{I}(Q, \tau)$ can be represented in $D^b(\mathcal{D}_\lambda)$ by its Cousin resolution corresponding to the stratification of Q by B_K-orbits $D_Q(w)$ as explained in the Appendix A.4.

Let $D_Q(w)$, $w \in W_K$, be a B_K-orbit in X. Denote by N_K the unipotent radical of B_K. Then $D_Q(w)$ admits unique irreducible N_K-equivariant connection $\mathcal{O}_{D_Q(w)}$. Its direct image $\mathcal{J}(w, \lambda)$ is a standard $\mathcal{D}_\lambda$-module attached to $D_Q(w)$.

The Cousin resolution of $\mathcal{I}(Q, \tau)$ from Theorem A.4 is a complex $\mathcal{D}^\cdot$ such that

$$\mathcal{D}^p = \bigoplus_{w \in W_K(\dim Q - p)} \mathcal{J}(w, \lambda)$$

for any $p \in \mathbb{Z}_+$, with explicitly given differentials.

Clearly, there is a natural monomorphism $\mathcal{I}(Q, \tau)$ into the standard $\mathcal{D}_\lambda$-module $\mathcal{J}(w_0, \lambda)$ attached to the open B_K-orbit $D_Q(w_0)$ (here w_0 is the longest element of the Weyl group W_K now). Therefore, there is a natural morphism ϵ of $D(\mathcal{I}(Q,\tau))$ into $\mathcal{D}^{\cdot}$ in $D^b(\mathcal{D}_\lambda)$. The main result of A.4 says that ϵ is an isomorphism.

Let θ be the Weyl group orbit of λ. Since the functor Γ is exact for antidominant λ, $D^{\cdot} = \Gamma(X,\mathcal{D}^{\cdot})$ is isomorphic to $D(\Gamma(X,\mathcal{I}(Q,\tau)))$ in $D^b(\mathcal{U}_\theta)$, i.e., we get a resolution of $\Gamma(X,\mathcal{I}(Q,\tau))$ by modules D^p, $p \in \mathbb{Z}_+$. This is the *Trauber resolution* of the discrete series $\Gamma(X,\mathcal{I}(Q,\tau))$ [26].

We put

$$J(w,\lambda) = \Gamma(X,\mathcal{J}(w,\lambda))$$

for any $w \in W_K$.

By Theorem 2.6, we have

$$H_p(\mathfrak{n}, J(w,\lambda))_{(v\lambda+\rho)} = H^{-p}(\pi_{v,+}(Li_v^+(D(\mathcal{J}(w,\lambda)))))$$

for any $v \in W$. By base change, we get

$$Li_v^+(D(\mathcal{J}(w,\lambda))) = 0$$

if $v \neq wu$.

Hence, we have

$$H_p(\mathfrak{n}, J(w,\lambda))_{(v\lambda+\rho)} = 0$$

for all $p \in \mathbb{Z}_+$ if $v \neq wu$.

On the other hand, if $v = wu$ we have

$$Li_{wu}^+(D(\mathcal{J}(w,\lambda))) = Ri_{wu}^!(D(\mathcal{J}(w,\lambda)))[\dim X - \ell(wu)].$$

Denote by j_w the immersion of $D_Q(w)$ into $C(wu)$. Then we have

$$D(\mathcal{J}(w,\lambda)) = i_{Q,+}(j_{w,+}(D(\mathcal{O}_{D_Q(w)}))),$$

since $D_Q(w)$ is affine and Q is affinely imbedded. Therefore, using the commutative diagram from Section 4.2, we have

$$\begin{aligned} Ri_{wu}^!(D(\mathcal{J}(w,\lambda))) &= Ri_{wu}^!(i_{Q,+}(j_{w,+}(D(\mathcal{O}_{D_Q(w)})))) \\ &= a_+(Rb^!(j_{w,+}(D(\mathcal{O}_{D_Q(w)})))) = a_+(D(\mathcal{O}_{D_Q(w)})) \end{aligned}$$

by base change and Kashiwara's equivalence of categories. If we denote $c = \pi_w \circ a$ by Lemma 4.1, this implies that

$$\begin{aligned} \pi_{w,+}(Li_{wu}^+(D(\mathcal{J}(w,\lambda)))) &= \pi_{w,+}(a_+(D(\mathcal{O}_{D_Q(w)})))[\dim X - \ell(wu)] \\ &= c_+(D(\mathcal{O}_{D_Q(w)}))[\dim X - \ell(wu)] \\ &= D(\mathbb{C})[\dim X - \ell(wu)][\ell_K(w)] \\ &= D(\mathbb{C})[\dim X - \ell(wu) + \ell_K(w)]. \end{aligned}$$

Hence, we conclude that

$$H_p(\mathfrak{n}, J(w,\lambda))_{(wu\lambda+\rho)} = \begin{cases} \mathbb{C} & \text{if } p = \dim X - \ell(wu) + \ell_K(w); \\ 0 & \text{if } p \neq \dim X - \ell(wu) + \ell_K(w). \end{cases}$$

Therefore, we proved the following result.

Lemma 4.7. *Let λ be an antidominant weight and $w \in W_K$. Then*

$$H_p(\mathfrak{n}, J(w,\lambda)) = \begin{cases} \mathbb{C}_{wu\lambda+\rho} & \text{if } p = \dim X - \ell(wu) + \ell_K(w); \\ 0 & \text{if } p \neq \dim X - \ell(wu) + \ell_K(w). \end{cases}$$

By the above discussion, the $\mathfrak{n}$-homology of $\Gamma(X, \mathcal{I}(Q,\tau))$ is given by the hypercohomology of the $\mathfrak{n}$-homology functor for the complex $D^{\cdot}$.

Let $v \in W$. As in Section 4.3, we consider the exact functor S from $D^b(\mathcal{U}_\theta)$ into bounded derived category of vector spaces $D^b(\mathbb{C})$.

Therefore, we see that

$$\begin{aligned} &H_{-p}(S(D(D^q))) = H_p(\mathfrak{n}, D^q)_{(v\lambda+\rho)} \\ &= \begin{cases} 0 & \text{if } v \neq wu,\ w \in W_K, \text{ or } p \neq \dim X - \ell(wu) + \ell_K(w) \text{ or } q \neq \dim Q - \ell_K(w); \\ \mathbb{C} & \text{if } v = wu,\ w \in W_K,\ p = \dim X - \ell(wu) + \ell_K(w) \text{ and } q = \dim Q - \ell_K(w). \end{cases} \end{aligned}$$

If $v \notin W_K u$, we see that $S(D(D^q)) = 0$. By Lemma 4.6 it follows that $S(D^{\cdot}) = 0$. Since $D(\Gamma(X, \mathcal{I}(Q,\tau)))$ is isomorphic to its Trauber resolution $D^{\cdot}$ in $D^b(\mathcal{U}_\theta)$, we see that $S(D(\Gamma(X, \mathcal{I}(Q,\tau)))) = 0$. Therefore, we see that $H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(v\lambda+\rho)} = 0$ if $v \notin W_K u$ for all $p \in \mathbb{Z}_+$.

Assume that $v = wu$, $w \in W_K$. It follows that

$$S(D(D^q)) = 0$$

if $q \neq \dim Q - \ell_K(w)$; and

$$S(D(D^{\dim Q - \ell_K(w)})) = D(\mathbb{C})[\dim X - \ell(wu) + \ell_K(w)].$$

By Lemma 4.6 it follows that

$$\begin{aligned} S(D^{\cdot}) &= D(\mathbb{C})[\dim X - \ell(wu) + \ell_K(w)][-\dim Q + \ell_K(w)] \\ &= D(\mathbb{C})[\dim X - \dim X_K - \ell(wu) + 2\ell_K(w)] \\ &= D(\mathbb{C})[\tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w)]. \end{aligned}$$

Since $D(\Gamma(X, \mathcal{I}(Q,\tau)))$ is isomorphic to its Trauber resolution $D^{\cdot}$ in $D^b(\mathcal{U}_\theta)$, we see that $S(D(\Gamma(X, \mathcal{I}(Q,\tau)))) = D(\mathbb{C})[\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wv) + 2\ell_K(w)]$. This implies that

$$\begin{aligned} H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(wu\lambda+\rho)} &= H_{-p}(S(D(\Gamma(X, \mathcal{I}(Q,\tau))))) = H_{-p}(S(D^{\cdot})) \\ &= H_{-p}(D(\mathbb{C})[\tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w)]) \\ &= \begin{cases} 0 & \text{if } p \neq \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w); \\ \mathbb{C} & \text{if } p = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(wu) + 2\ell_K(w). \end{cases} \end{aligned}$$

This is exactly Schmid's theorem (Theorem 4.3).

Therefore, this calculation shows that each module $J(w,\lambda)$ in the Trauber resolution contributes exactly one cohomology class in $\mathfrak{n}$-homology of $\Gamma(X, \mathcal{I}(Q,\tau))$.

5. Character formulas

In this section we calculate the characters of discrete series representations $\Gamma(X, \mathcal{I}(Q,\tau))$ on the elliptic set. We start with the trivial (and well-known) example of a compact Lie group G_0. In this case, our result is just the well-known Weyl character formula.

5.1. **Case of compact groups.** Let G_0 be a connected compact semisimple Lie group with complexified Lie algebra $\mathfrak{g}$. We fix a maximal tours T_0 in G_0. Let $\mathfrak{t} \subset \mathfrak{g}$ be its complexified Lie algebra. Fix a set of positive roots R^+ in the root system R of $(\mathfrak{g}, \mathfrak{t})$. Then, $\mathfrak{t}$ and the root subspaces of $\mathfrak{g}_\alpha$ corresponding to roots $\alpha \in R^+$ span a Borel subalgebra $\mathfrak{b}$. We put $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Let F be the irreducible finite-dimensional representation of G_0 with lowest weight $\lambda \in \mathfrak{h}^*$. We want to determine the character $\mathrm{ch}(F)$ of F. Since all conjugacy classes in G_0 intersect T_0, it is enough to give a formula for $\mathrm{ch}(F)$ on T_0.

Clearly, F and the standard complex $C^{\cdot}(\mathfrak{n}, F)$, with $C^{-p}(\mathfrak{n}, F) = \bigwedge^p \mathfrak{n} \otimes F$, $p \in \mathbb{Z}$, have natural structures of finite-dimensional representations of T. Therefore, their characters ch_T are well-defined. By Euler's principle, we have

$$\sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(C^{-p}(\mathfrak{n}, F)) = \sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(H_p(\mathfrak{n}, F))$$

where $n = \dim \mathfrak{n}$.

Moreover, we have

$$\mathrm{ch}_T(C^{-p}(\mathfrak{n}, F)) = \mathrm{ch}_T\left(\bigwedge^p \mathfrak{n} \otimes F\right) = \mathrm{ch}_T\left(\bigwedge^p \mathfrak{n}\right) \cdot \mathrm{ch}(F)$$

on T_0. Hence, it follows that

$$\begin{aligned}\sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(H_p(\mathfrak{n}, F)) &= \sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(C^{-p}(\mathfrak{n}, F)) \\ &= \mathrm{ch}(F) \cdot \sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T\left(\bigwedge^p \mathfrak{n}\right).\end{aligned}$$

This finally implies the following formula

$$\mathrm{ch}(F) = \frac{\sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(H_p(\mathfrak{n}, F))}{\sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T(\bigwedge^p \mathfrak{n})}$$

on T_0.

Let $e^\mu : T \longrightarrow \mathbb{C}^*$ be the morphism with the differential corresponding under specialization to $\mu \in \mathfrak{h}^*$. Then the weights in $\bigwedge^p \mathfrak{n}$ are the sums of all sets of p different roots from R^+; i.e, we have

$$\mathrm{ch}_T\left(\bigwedge^p \mathfrak{n}\right) = \sum_{P \subset R^+,\ \mathrm{Card}(P) = p} \left(\prod_{\alpha \in P} e^\alpha\right).$$

Therefore, we have

$$\begin{aligned}\sum_{p=0}^{n} (-1)^p \,\mathrm{ch}_T\left(\bigwedge^p \mathfrak{n}\right) &= \sum_{p=0}^{n} (-1)^p \left(\sum_{P \subset R^+,\ \mathrm{Card}(P) = p} \left(\prod_{\alpha \in P} e^\alpha\right)\right) \\ &= \sum_{p=0}^{n} \left(\sum_{P \in R^+,\ \mathrm{Card}(P) = p} \left(\prod_{\alpha \in P} (-e^\alpha)\right)\right) \\ &= \sum_{P \subset R^+} \left(\prod_{\alpha \in P} (-e^\alpha)\right) = \prod_{\alpha \in R^+} (1 - e^\alpha).\end{aligned}$$

Hence, we have

$$\operatorname{ch}(F)=\frac{\sum_{p=0}^{n}(-1)^p\operatorname{ch}_T(H_p(\mathfrak{n},F_\lambda))}{\prod_{\alpha\in R^+}(1-e^\alpha)}$$

on T_0. Using Kostant's theorem 4.2, we get

$$\operatorname{ch}_T(H_p(\mathfrak{n},F))=\sum_{w\in W(p)}e^{w(\lambda-\rho)+\rho}$$

and finally

$$\sum_{p=0}^{n}(-1)^p\operatorname{ch}_T(H_p(\mathfrak{n},F))=\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda-\rho)+\rho}.$$

Putting everything together, we conclude that

$$\operatorname{ch}(F)=\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda-\rho)+\rho}}{\prod_{\alpha\in R^+}(1-e^\alpha)}.$$

This finally implies the following result.

Theorem 5.1 (Weyl)**.** *Let F be the irreducible finite-dimensional representation of $\mathfrak{g}$ with lowest weight λ. Then*

$$\operatorname{ch}(F)=\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda-\rho)+\rho}}{\prod_{\alpha\in R^+}(1-e^\alpha)}$$

on the maximal torus T_0 in G_0.

5.2. **Discrete series.** The formal Euler characteristic argument in the case of compact G_0 clearly does not work in the case of the discrete series of a noncompact G_0 since they are infinite-dimensional. It is replaced by an argument based on the Osborne formula [17].[3]

Let G_0 be a connected semisimple Lie group and K_0 its maximal compact subgroup. Let $C_0^\infty(G_0)$ be the space of complex-valued smooth functions on G_0 with compact support, equipped with the usual topology. The continuous dual of that space is the space of distributions on G_0. Let V be a Harish-Chandra module of finite length, then the Harish-Chandra character $\operatorname{ch}(V)$ of V is a distribution on G_0 [8].[4]

The characters are invariant under inner automorphisms of G_0, therefore they are *invariant* distributions on G_0. The center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is naturally identified with the invariant differential operators on G_0. If V is an irreducible Harish-Chandra module in $\mathcal{M}(\mathcal{U}_\theta)$, its character $\operatorname{ch}(V)$ is annihilated by the maximal ideal J_θ in $\mathcal{Z}(\mathfrak{g})$; i.e., it is an *invariant eigendistribution* on G_0. We denote the space of all invariant eigendistributions annihilated by J_θ by $\mathcal{E}(G_0,\theta)$. Harish-Chandra has shown that any distribution T in $\mathcal{E}(G_0,\theta)$ is given by

$$C_0^\infty(G_0)\ni f\longmapsto T(f)=\int_{G_0}f(g)\Theta_T(g)\,d\mu(g)$$

[3]Actually, we need only a special case for compact Cartan subgroups [17, 7.27], which is already implicit in [25].

[4]Harish-Chandra defines the character for a representation of G_0 on a Hilbert space. By [6, Proposition 8.23], every Harish-Chandra module V of finite length is the module of K_0-finite vectors of a subrepresentation of a representation of G_0 induced by a finite dimensional representation of a minimal parabolic subgroup P_0 of G_0. Therefore, Harish-Chandra's construction applies to the closure of V. Moreover, the character is the sum of K_0-finite matrix coefficients which are completely determined by the Harish-Chandra module [6, Thm. 8.7].

where μ is a fixed Haar measure on G_0 and Θ_T is a locally integrable real analytic function on the set G_0' of all regular elements in G_0 [10, Thm. 2].

Therefore, for any irreducible Harish-Chandra module V the character $\operatorname{ch}(V)$ is given by

$$C_0^\infty(G_0) \ni f \longmapsto \operatorname{ch}(V)(f) = \int_{G_0} f(g)\Theta_V(g)\, d\mu(g)$$

for a locally integrable real analytic function Θ_V on the set G_0' of all regular elements in G_0.

From now on, we assume that ranks of G_0 and K_0 are equal. Let T_0 be a maximal torus in K_0. An element $g \in G_0$ is *elliptic* if its adjoint action $\operatorname{Ad}(g)$ on $\mathfrak{g}$ is semisimple and its eigenvalues are complex numbers of absolute value 1. Denote by E the set of all regular elliptic elements in G_0. Also denote by T_0' the set of regular elements in T_0. Clearly, E is an open set in G_0, invariant under conjugation by elements of G_0, and every conjugacy class in E intersects T_0.

The restriction to E of the function Θ_V for an irreducible Harish-Chandra module is a real analytic function on E constant on conjugacy classes. The Osborne formula describes this function explicitly. Let $\mathfrak{b}$ be a Borel subalgebra in X such that $\mathfrak{t} \subset \mathfrak{b}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Then, we have

$$\Theta_V|_{T_0'} = \frac{\sum_{p=0}^n (-1)^p \operatorname{ch}_T(H_p(\mathfrak{n}, V))}{\prod_{\alpha \in R^+}(1 - e^\alpha)}$$

on the regular elements in a compact Cartan subgroup T_0 of G_0. This is the special case of the Osborne formula we alluded to above. It is a generalization of the formula we used in calculations for compact group G_0 in the preceding section.

Now, we are going to apply this formula to the case of the discrete series representation $V = \Gamma(X, \mathcal{I}(Q,\tau))$ as discussed in Section 4. Assume that $\mathfrak{b}$ is a Borel subalgebra corresponding to a point in Q.

Let $\epsilon : W \longrightarrow \{\pm 1\}$ be the character of the Weyl group W given by $\epsilon(w) = \det(w)$, $w \in W$. Then, $\epsilon(s) = -1$ for any reflection s in W. Therefore, we have $\epsilon(w) = (-1)^{\ell(w)}$ for any $w \in W$. In addition, since $\epsilon(s) = -1$ for any reflection s with respect to a compact root, the restriction of ϵ to W_K is equal to the corresponding character of Weyl group W_K. First, by Theorem 4.3, the numerator in the Osborne formula is equal to

$$\begin{aligned}
&\sum_{p \in \mathbb{Z}_+} (-1)^p \operatorname{ch}_T(H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))) \\
&\qquad = \sum_{p \in \mathbb{Z}_+} (-1)^p \Big(\sum_{w \in W_K} \operatorname{ch}_T(H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(w\lambda+\rho)}) \Big) \\
&\qquad = \sum_{w \in W_K} \Big(\sum_{p \in \mathbb{Z}_+} (-1)^p \operatorname{ch}_T(H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(w\lambda+\rho)}) \Big) \\
&\qquad = \sum_{w \in W_K} (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k}) - \ell(w) + 2\ell_K(w)} e^{w\lambda+\rho} \\
&\qquad = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})} \sum_{w \in W_K} (-1)^{\ell(w)} e^{w\lambda+\rho} \\
&\qquad = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})} \sum_{w \in W_K} (-1)^{\ell_K(w)} e^{wu\lambda+\rho}
\end{aligned}$$

This finally implies the following result.

Theorem 5.2. *On the regular elements of the compact Cartan subgroup T_0 the character of the discrete series representation $V = \Gamma(X, \mathcal{I}(Q,\tau))$ is given by*

$$\Theta_V|_{T_0'} = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})} \frac{\sum_{w\in W_K} (-1)^{\ell_K(w)} e^{w\lambda+\rho}}{\prod_{\alpha\in R^+}(1-e^\alpha)} \quad .$$

5.3. **Relation with Harish-Chandra parametrization.** In this section, we relate the information from 5.2 with Harish-Chandra's results on discrete series characters.

In [11], Harish-Chandra introduced the Schwartz space $\mathcal{C}(G_0)$ consisting of complex-valued rapidly decreasing smooth functions on G_0. It contains $C_0^\infty(G_0)$ as a dense subspace. A distribution on G_0 is *tempered* if it extends to a continuous linear form on $\mathcal{C}(G_0)$. We denote by $\mathcal{E}_{temp}(G_0,\theta)$ the subspace of $\mathcal{E}(G_0,\theta)$ consisting of tempered invariant eigendistributions.

If G_0 and K_0 have the same rank, the set of regular elliptic elements E is open in G_0 and one can consider the restriction map from the space of invariant eigendistributions on G_0 to the space of invariant eigendistributions on E.

Harish-Chandra proved the following result:[5]

Theorem 5.3. *Let θ be a W-orbit in $\mathfrak{h}^*$ consisting of regular real elements. Then the restriction map to E is injective on $\mathcal{E}_{temp}(G_0,\theta)$.*

Since the characters of discrete series are tempered invariant eigendistributions [11, Lem. 76], they are completely determined by their restrictions to E. In particular, we have the following version of 5.2:

Theorem 5.4. *The character of the discrete series representation $V = \Gamma(X, \mathcal{I}(Q,\tau))$ is the unique tempered invariant eigendistribution $\mathrm{ch}(V)$ satisfying*

$$\Theta_V|_{T_0'} = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})} \frac{\sum_{w\in W_K} (-1)^{\ell_K(w)} e^{w\lambda+\rho}}{\prod_{\alpha\in R^+}(1-e^\alpha)} \quad .$$

This is equivalent to the Harish-Chandra formula for the characters of discrete series [11, Theorem 16]. This establishes a precise connection between the geometric parametrization and Harish-Chandra's parametrization of the discrete series.

Moreover, this proves the existence of tempered invariant distributions Θ_λ in [9, Thm. 3] which is the central result of that paper.

In addition, it implies that for any W-orbit θ in $\mathfrak{h}^*$ consisting of regular real elements, the space of all tempered invariant eigendistributions on G_0 is spanned by the characters of the discrete series which are in $\mathcal{M}(\mathcal{U}_\theta)$.

6. The Blattner Conjecture

As before, let G_0 be a connected semisimple Lie group with finite center and K_0 its maximal compact subgroup. Also, we assume that the ranks of G_0 and K_0 are equal, so G_0 admits discrete series representations. The formula for the multiplicity of a finite-dimensional irreducible representations of K_0 in a discrete series representation of G_0 was conjectured by Robert Blattner. His conjecture was proved by Hecht and Schmid in [16].[6] We shall give a very simple proof of that formula using the geometric realization of discrete series.

[5] An equivalent result was proven later by Atiyah and Schmid in [1] by different methods.

[6] Their proof assumes that the group G_0 is linear. Our argument removes that condition.

6.1. **Filtration by normal degree.** First we recall a natural filtration of a $\mathcal{D}$-module direct image for a closed immersion. Let Y be a smooth algebraic variety and Z a closed smooth subvariety of Y. Denote by $i : Z \longrightarrow Y$ the inclusion morphism of Z into Y. Let $\mathcal{T}_Y$ (resp. $\mathcal{T}_Z$) be the tangent sheaf to Y (resp. Z). Denote by $i^*(\mathcal{T}_Y)$ the $\mathcal{O}$-module inverse image of $\mathcal{T}_Y$. Define the *normal sheaf* to Z as $\mathcal{N}_{Z|Y} = i^*(\mathcal{T}_Y)/\mathcal{T}_Z$. Moreover, we denote by ω_Y (resp. ω_Z) the invertible $\mathcal{O}$-module of sections top degree differential forms on Y (resp. Z). Also, we put $\omega_{Z|Y} = i^*(\omega_Y) \otimes_{\mathcal{O}_Z} (\omega_Z)^{-1}$.

Let $\mathcal{D}$ be a twisted sheaf of differential operators on Y. Let $\mathcal{D}^i$ be the corresponding twisted sheaf of differential operators on Z [22]. Let $i_\bullet$ be the sheaf direct image functor and i_* the $\mathcal{O}$-module direct image functor. Then the $\mathcal{D}$-module direct image functor

$$i_+(\mathcal{V}) = i_\bullet(\mathcal{V} \otimes_{\mathcal{D}^i} \mathcal{D}_{Z\to Y})$$

is an exact functor from the category of right $\mathcal{D}^i$-modules into the category of right $\mathcal{D}$-modules. As explained in the Appendix to [14], one defines a natural filtration of $\mathcal{D}_{Z\to Y}$ by left $\mathcal{D}^i$- and $i^{-1}\mathcal{O}_Y$-modules. By tensoring, we get a natural $\mathcal{O}_Y$-module filtration of $i_+(\mathcal{V})$ which we call the filtration by *normal degree*. The corresponding graded module is

$$\operatorname{Gr} i_+(\mathcal{V}) = i_\bullet(\mathcal{V} \otimes_{\mathcal{O}_Z} S(\mathcal{N}_{Z|Y})) = i_*(\mathcal{V} \otimes_{\mathcal{O}_Z} S(\mathcal{N}_{Z|Y}))$$

(here $S(\mathcal{N}_{Z|Y})$ is the symmetric algebra of $\mathcal{N}_{Z|Y}$). Since the opposite sheaf of rings of $\mathcal{D}$ is also a sheaf of twisted differential operators, we can easily adapt this construction to left $\mathcal{D}$-modules as explained in [3]. Going to left modules contributes a twist by $\omega_{Z|Y}$ in above formula and we get that

$$\operatorname{Gr} i_+(\mathcal{V}) = i_*(\mathcal{V} \otimes_{\mathcal{O}_Z} \omega_{Z|Y} \otimes_{\mathcal{O}_Z} S(\mathcal{N}_{Z|Y})).$$

6.2. **Proof of the Blattner formula.** Let T_0 be the Cartan subgroup of K_0. Denote, as before, by K and T the complexifications of K_0 and T_0. Denote by $\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of K and T.

Let Q be a closed K-orbit in the flag variety X of $\mathfrak{g}$. Let R be the root system of $(\mathfrak{g}, \mathfrak{t})$ in $\mathfrak{t}^*$. We fix a set of positive roots R_c^+ of the system of compact roots R_c in R. Then, by the discussion in Section 3, for any closed K-orbit in the flag variety X there exists a unique set of positive roots R^+ in R such that $R_c^+ = R_c \cap R^+$ and the Borel subalgebra $\mathfrak{b}$ spanned by $\mathfrak{t}$ and root subspaces corresponding to roots in R^+ is in Q.

Let x be a point in Q corresponding to the Borel subalgebra $\mathfrak{b}$. The tangent space $T_x(X)$ at x to X is identified with $\bar{\mathfrak{n}}$. This isomorphism identifies the tangent space to the orbit Q at x with $\bigoplus_{\alpha\in R_c^+} \mathfrak{g}_{-\alpha}$. This implies the following result.

Lemma 6.1. (i) *The geometric fiber* $T_x(\omega_{Q|X})$ *of* $\omega_{Q|X}$ *at* x *is isomorphic to* $-2\rho + 2\rho_c = -2\rho_n$.
(ii) *The geometric fiber of* $\mathcal{N}_{Q|X}$ *is isomorphic to* $\bigoplus_{\alpha\in R_n^+} \mathfrak{g}_{-\alpha}$.

Let τ be an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. If we consider the normal degree filtration of the standard Harish-Chandra sheaf $\mathcal{I}(Q, \tau)$, we get the short exact sequence

$$0 \to \mathrm{F}_{p-1}\,\mathcal{I}(Q,\tau) \to \mathrm{F}_p\,\mathcal{I}(Q,\tau) \to i_\bullet(\omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X})) \to 0.$$

The last sheaf is an $\mathcal{O}_X$-module and by the Leray spectral sequence

$$\begin{aligned} H^p(X, i_\bullet(\omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}))) &= H^p(X, i_*(\omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}))) \\ &= H^p(Q, \omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X})) \end{aligned}$$

since i is an affine morphism.

Clearly, we have $\tau = \mathcal{O}_Q(\lambda+\rho)$. Moreover, by Lemma 6.1.(i), we have

$$\omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}) = \mathcal{O}_Q(\lambda+\rho_c-\rho_n) \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}).$$

This K-equivariant coherent $\mathcal{O}_Q$-module has finite-dimensional cohomology; i.e., $H^p(Q, \omega_{Q|X} \otimes_{\mathcal{O}_Q} \tau \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}))$ are finite-dimensional algebraic representations of K for any $p \in \mathbb{Z}_+$. From the long exact sequence of cohomology associated to the above short exact sequence, we conclude that $H^q(X, \mathrm{F}_p \mathcal{I}(Q,\tau))$ are finite-dimensional algebraic representations of K. Since K is reductive, all representations in this long exact sequence are semisimple finite-dimensional algebraic representation of K.

Let ν be a R_c^+-antidominant weight and F_ν the irreducible finite-dimensional representation with lowest weight ν. By applying the functor $\mathrm{Hom}_K(F_\nu, -)$ we get again a long exact sequence of finite-dimensional vector spaces. Using the Euler principle, we conclude that

$$\begin{aligned} &\sum_{q\in\mathbb{Z}} (-1)^q \dim \mathrm{Hom}_K(F_\nu, H^q(X, \mathrm{F}_p \mathcal{I}(Q,\tau))) \\ &\qquad = \sum_{q\in\mathbb{Z}} (-1)^q \dim \mathrm{Hom}_K(F_\nu, H^q(X, \mathrm{F}_{p-1} \mathcal{I}(Q,\tau))) \\ &\qquad + \sum_{q\in\mathbb{Z}} (-1)^q \dim \mathrm{Hom}_K(F_\nu, H^q(Q, \mathcal{O}_Q(\lambda+\rho_c-\rho_n) \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}))). \end{aligned}$$

The geometric fiber $T_x(S^p(\mathcal{N}_{Q|X}))$ is a representation of the stabilizer of x in K. Since it is a Borel subgroup of K, by Lie's theorem, there exists a finite filtration of $T_x(S^p(\mathcal{N}_{Q|X}))$ by K-equivariant $\mathcal{O}_Q$-modules. By Lemma 6.1.(ii), the graded $\mathcal{O}_Q$-module is the direct sum of $\mathcal{O}_Q(\lambda+\rho_c-\rho_n-\kappa)$ where κ is a sum of p positive noncompact roots.

By induction on the length of this filtration, using again the Euler principle, we conclude that

$$\begin{aligned} &\sum_{q\in\mathbb{Z}} (-1)^q \dim \mathrm{Hom}_K(F_\nu, H^q(Q, \mathcal{O}_Q(\lambda+\rho_c-\rho_n) \otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X}))) \\ &\qquad = \sum_\kappa \sum_{q\in\mathbb{Z}} (-1)^q \dim \mathrm{Hom}_K(F_\nu, H^q(Q, \mathcal{O}_Q(\lambda+\rho_c-\rho_n-\kappa))) \end{aligned}$$

where the sum goes over all sums κ of p noncompact positive roots.

By the Borel-Weil-Bott theorem, the cohomology of $\mathcal{O}_Q(\lambda+\rho_c-\rho_n-\kappa)$ vanishes if $\lambda-\rho_n-\kappa$ is not regular with respect to R_c. Otherwise, if $\lambda-\rho_n-\kappa$ is regular and equal to $w(\nu-\rho_c)$, $w \in W_K$, it follows that $H^{\ell(w)}(Q, O_Q(\lambda+\rho_c-\rho_n-\kappa)) = F_\nu$ and the other cohomologies vanish.

Let P be the function on $\mathfrak{h}^*$ defined in the following way: For any μ we set $P(\mu)$ to be the number of ways one can represent μ as a non-negative integer linear combination of the positive noncompact roots. Also, set P_p to be the function on $\mathfrak{h}^*$ defined in the following way: For any μ, $P_p(\mu)$ is the number of ways one can represent μ as a sum of p positive noncompact roots. Clearly, we have $P = \sum_{p=0}^\infty P_p$.

Therefore, we have

$$\sum_{q\in\mathbb{Z}}(-1)^q \dim \operatorname{Hom}_K(F_\nu, H^q(Q, \mathcal{O}_Q(\lambda+\rho_c-\rho_n)\otimes_{\mathcal{O}_Q} S^p(\mathcal{N}_{Q|X})))$$
$$= \sum_{w\in W_K}(-1)^{\ell(w)}P_p(\lambda-\rho_n-w(\nu-\rho_c)).$$

By induction we see that

$$\sum_{q\in\mathbb{Z}}(-1)^q \dim \operatorname{Hom}_K(F_\nu, H^q(X, \mathrm{F}_p\,\mathcal{I}(Q,\tau))) = \sum_{s=0}^{p}\sum_{w\in W_K}(-1)^{\ell(w)}P_s(\lambda-\rho_n-w(\nu-\rho_c)).$$

Since cohomology commutes with direct limits, by taking the limit as $p\to\infty$, we get

$$\sum_{q\in\mathbb{Z}}(-1)^q \dim \operatorname{Hom}_K(F_\nu, H^q(X, \mathcal{I}(Q,\tau))) = \sum_{w\in W_K}(-1)^{\ell(w)}P(\lambda-\rho_n-w(\nu-\rho_c)).$$

Finally, since λ is antidominant, higher cohomologies of the standard Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ vanish and we see that

$$\dim \operatorname{Hom}_K(F_\nu, \Gamma(X, \mathcal{I}(Q,\tau))) = \sum_{w\in W_K}(-1)^{\ell(w)}P(\lambda-\rho_n-w(\nu-\rho_c)).$$

Theorem 6.2 (Hecht-Schmid)**.** *Let ν be an antidominant weight for the root system R_c and F_ν the finite-dimensional irreducible algebraic representation of K. Then the multiplicity of the representation F_ν in the restriction of the discrete series representation $V=\Gamma(X,\mathcal{I}(Q,\tau))$ to K is equal to*

$$\sum_{w\in W_K}(-1)^{\ell(w)}P(\lambda-\rho_n-w(\nu-\rho_c)).$$

This is Blattner's formula.

Appendix A. Cousin resolution

In this appendix, for the convenience of the reader, we prove the well known results about Cousin resolutions in $\mathcal{D}$-module setting (for references, see [13], [26]).

A.1. **Stratifications.** Let X be a smooth algebraic variety[7] over $\mathbb{C}$. A subvariety Y of X is *affinely imbedded* if for any affine open set U in X the set $U\cap Y$ is affine. Clearly, if Y is an affinely imbedded subvariety and U an open set in X, then $Y\cap U$ is affinely imbedded in U.

Closed subvarieties of X are affinely imbedded in X. Also, affine subvarieties are affinely imbedded in X. The canonical inclusions of affinely imbedded subvarieties are affine morphisms.

Let

$$X=F_0\supset F_1\supset F_2\supset\cdots\supset F_n\supset F_{n+1}=\emptyset$$

be a decreasing filtration of X by closed algebraic subvarieties satisfying the following condition:

(S) F_p-F_{p+1} is a nonempty smooth affinely imbedded subvariety of X of dimension $\dim X-p$ for $0\le p\le n$.

[7]Our varieties do not have to be connected.

Then we call $(F_p; 0 \leq p \leq n+1)$ a *stratification* of X of *length* n. We call the smooth affinely imbedded subvariety $S_p = Z_p - Z_{p+1}$ the $(\dim X - p)$-dimensional *stratum* of X.

Let $0 \leq q \leq n$ and $X_q = X - F_{q+1}$. Then X_q is smooth subvariety of X and $(F_p - F_{q+1}; 0 \leq p \leq q+1)$ is a stratification of X_q of length q.

A.2. **Local cohomology.** Let $\mathcal{D}_X$ the sheaf of differential operators in X. Let Y be a closed smooth subvariety of X. Let $U = X - Y$ be the complement of Y. Let $i : Y \longrightarrow X$ and $j : U \longrightarrow X$ be the canonical inclusions.

Denote by $\mathcal{M}(\mathcal{D}_X)$, $\mathcal{M}(\mathcal{D}_Y)$ and $\mathcal{M}(\mathcal{D}_U)$ the corresponding categories of quasicoherent $\mathcal{D}$-modules, and by $D^b(\mathcal{D}_X)$, $D^b(\mathcal{D}_Y)$ and $D^b(\mathcal{D}_U)$ the corresponding derived categories. Then the direct image functor $i_+ : \mathcal{M}(\mathcal{D}_Y) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ is exact and lifts to an exact functor $i_+ : D^b(\mathcal{D}_Y) \longrightarrow D^b(\mathcal{D}_X)$. Also, the direct image functor $j_+ : \mathcal{M}(\mathcal{D}_U) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ is the ordinary sheaf-theoretic direct image functor $j_\bullet$, it is left exact and its derived functor is $Rj_\bullet : D^b(\mathcal{D}_U) \longrightarrow D^b(\mathcal{D}_X)$.

Let $\Gamma_Y : \mathcal{M}(\mathcal{D}_X) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ be the functor which associates to each quasicoherent $\mathcal{D}_X$-module $\mathcal{F}$ the subsheaf of all local sections $\Gamma_Y(\mathcal{F})$ with support in Y. Let $R\Gamma_Y : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathcal{D}_X)$ the corresponding functor of local cohomology.

Let $\mathcal{F}^\cdot$ be an object in $D^b(\mathcal{D}_X)$. Then we have the canonical distinguished triangle

$$\begin{tikzcd} & Rj_\bullet(\mathcal{F}^\cdot|_U) \arrow[dl, "{[1]}"'] & \\ R\Gamma_Y(\mathcal{F}^\cdot) \arrow[rr] & & \mathcal{F}^\cdot \arrow[ul] \end{tikzcd}$$

by [4, VI.8.3]. In addition, we have

$$R\Gamma_Y(\mathcal{F}^\cdot) = i_+(Ri^!(\mathcal{F}^\cdot)).$$

If $m = \dim X - \dim Y$, we have

$$R^p i^!(\mathcal{O}_X) = \begin{cases} 0 & \text{for } p \neq m; \\ \mathcal{O}_Y & \text{for } p = m; \end{cases}$$

i.e.,

$$R\Gamma_Y(D(\mathcal{O}_X)) = D(i_+(\mathcal{O}_Y))[-m].$$

Specializing the above distinguished triangle for $\mathcal{F}^\cdot = D(\mathcal{O}_X)$ we get the distinguished triangle

$$\begin{tikzcd} & Rj_\bullet(D(\mathcal{O}_U)) \arrow[dl, "{[1]}"'] & \\ D(i_+(\mathcal{O}_Y))[-m] \arrow[rr] & & D(\mathcal{O}_X) \arrow[ul] \end{tikzcd}$$

Assume that Y is of codimension 1 in X; i.e., $m = 1$. Then, applying the cohomology functor to this distinguished triangle, we get the long exact sequence:

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X \longrightarrow j_\bullet(\mathcal{O}_U) \longrightarrow i_+(\mathcal{O}_Y) \longrightarrow 0 \longrightarrow \dots$$
$$\cdots \longrightarrow 0 \longrightarrow R^p j_\bullet(\mathcal{O}_U) \longrightarrow 0 \longrightarrow \dots .$$

Therefore, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow j_\bullet(\mathcal{O}_U) \longrightarrow i_+(\mathcal{O}_Y) \longrightarrow 0$$

and $R^p j_\bullet(\mathcal{O}_Y) = 0$ for $p \geq 1$.

Analogously, if $m > 1$, we get that

$$R^p j_\bullet(\mathcal{O}_U) = \begin{cases} \mathcal{O}_X & \text{for } p = 0; \\ i_+(\mathcal{O}_Y) & \text{for } p = m-1; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we proved the following result.

Lemma A.1. *Let $m = \dim X - \dim Y$. Then:*

(i) *If $m = 1$, we have the exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow j_\bullet(\mathcal{O}_U) \longrightarrow i_+(\mathcal{O}_Y) \longrightarrow 0$$

and $R^p j_\bullet(\mathcal{O}_U) = 0$ for $p > 0$.

(ii) *If $m > 1$, we have*

$$R^p j_\bullet(\mathcal{O}_U) = \begin{cases} \mathcal{O}_X & \textit{for } p = 0; \\ i_+(\mathcal{O}_Y) & \textit{for } p = m-1; \\ 0 & \textit{otherwise.} \end{cases}$$

A.3. **Cousin resolution.** Let X be a smooth algebraic variety with a stratification $(F_p; 0 \leq p \leq n+1)$. Denote by $i_p : S_p \longrightarrow X$, $0 \leq p \leq n$, the canonical inclusions of strata into X.

Theorem A.2. *There exists a canonical exact sequence*

$$0 \to \mathcal{O}_X \to i_{0,+}(\mathcal{O}_{S_0}) \to i_{1,+}(\mathcal{O}_{S_1}) \to \cdots \to i_{n,+}(\mathcal{O}_{S_n}) \to 0.$$

Therefore, the complex

$$\cdots \to 0 \to i_{0,+}(\mathcal{O}_{S_0}) \to i_{1,+}(\mathcal{O}_{S_1}) \to \cdots \to i_{n,+}(\mathcal{O}_{S_n}) \to 0 \to \dots$$

is a resolution of $\mathcal{O}_X$ in $\mathcal{M}(\mathcal{D}_X)$. This resolution is called the *Cousin resolution* of $\mathcal{O}_X$.

We prove the theorem by induction in n. If $n = 1$, we have $F_0 = X$ and $F_1 = Y$ is a smooth closed subvariety in X. Moreover, $S_0 = X - Y = U$ and $S_1 = Y$. Therefore, the Cousin resolution reduces to Lemma A.1.(i).

Assume that $n > 1$ and that the statement holds for $n - 1$. Let $U = X_{n-1}$ and denote by $j : U \longrightarrow X$ the natural inclusion. Then $(F_p - F_n; 0 \leq p \leq n)$ define a stratification of U. Denote by $j_p : S_p \longrightarrow U$ the natural inclusions. By the induction assumption, we have the exact sequence

$$0 \to \mathcal{O}_U \to j_{0,+}(\mathcal{O}_{S_0}) \to j_{1,+}(\mathcal{O}_{S_1}) \to \cdots \to j_{n-1,+}(\mathcal{O}_{S_{n-1}}) \to 0 \quad .$$

Therefore, we have a quasiisomorphism of $D(\mathcal{O}_U)$ into the complex

$$\cdots \to 0 \to j_{0,+}(\mathcal{O}_{S_0}) \to j_{1,+}(\mathcal{O}_{S_1}) \to \cdots \to j_{n-1,+}(\mathcal{O}_{S_{n-1}}) \to 0 \to \dots \quad .$$

By our assumption $j_p : S_p \longrightarrow U$ and $i_p : S_p \longrightarrow X$ are affine morphisms. Therefore, the functors $j_{p,+} : \mathcal{M}(\mathcal{D}_{S_p}) \longrightarrow \mathcal{M}(\mathcal{D}_U)$ and $i_{p,+} : \mathcal{M}(\mathcal{D}_{S_p}) \longrightarrow \mathcal{M}(\mathcal{D}_X)$ are exact. Since $Rj_\bullet \circ j_{p,+} = i_{p,+}$, it follows that the modules $j_{p,+}(\mathcal{V})$ are acyclic for $j_\bullet$ for any $\mathcal{D}_{S_p}$-module $\mathcal{V}$. Therefore, by acting on the above complex by $j_\bullet$, we get the complex

$$\cdots \to 0 \to i_{0,+}(\mathcal{O}_{S_0}) \to i_{1,+}(\mathcal{O}_{S_1}) \to \cdots \to i_{n-1,+}(\mathcal{O}_{S_{n-1}}) \to 0 \to \dots$$

which computes $Rj_\bullet(D(\mathcal{O}_U))$. Since $U = X - S_n$, by Lemma A.1.(ii), the kernel of the morphism $i_{0,+}(\mathcal{O}_{S_0}) \to i_{1,+}(\mathcal{O}_{S_1})$ has to be $\mathcal{O}_X$, and the cokernel of the morphism $i_{n-2,+}(\mathcal{O}_{S_{n-2}}) \to i_{n-1,+}(\mathcal{O}_{S_{n-1}})$ has to be $i_{n,+}(\mathcal{O}_{S_n})$. This clearly establishes the induction step.

A.4. **Variants.** In the main text we use the "twisted" versions of Theorem A.2.

First, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X and $\mathcal{D}_\mathcal{L}$ the sheaf of differential operators on $\mathcal{L}$. By tensoring by $\mathcal{L}$, we get immediately the following version of Theorem A.2.

Theorem A.3. *There exists a canonical exact sequence*

$$0 \to \mathcal{L} \to i_{0,+}(i_0^*(\mathcal{L})) \to i_{1,+}(i_1^*(\mathcal{L})) \to \cdots \to i_{n,+}(i_n^*(\mathcal{L})) \to 0$$

in $\mathcal{M}(\mathcal{D}_\mathcal{L})$.

Finally, let Y be a smooth closed subvariety of X. Denote by $i : Y \longrightarrow X$ the canonical inclusion and by $\mathcal{D}^i$ the twisted sheaf of differential operators on Y induced by the twisted sheaf of differential operators $\mathcal{D}$ on X [22, 1.1]. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module on Y. Assume that $\mathcal{D}$ is the twisted sheaf of differential operators on X such that $\mathcal{D}^i = \mathcal{D}_\mathcal{L}$.

Let $(F_p; 0 \le p \le n+1)$ be a stratification of Y. Denote by $i_p : S_i \longrightarrow X$ and $k_p : S_p \longrightarrow Y$ the natural inclusions of strata S_p, $0 \le p \le n$, into X (resp. Y). Then, applying the Kashiwara equivalence of categories to closed immersion i, from Theorem A.3 we get the following result.

Theorem A.4. *There exists a canonical exact sequence*

$$0 \to i_+(\mathcal{L}) \to i_{0,+}(k_0^*(\mathcal{L})) \to i_{1,+}(k_1^*(\mathcal{L})) \to \cdots \to i_{n,+}(k_n^*(\mathcal{L})) \to 0$$

in $\mathcal{M}(\mathcal{D})$.

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