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Localization and standard modules for real semisimple Lie groups I: The duality theorem

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§ 1. Introduction

In this paper we relate two constructions of representations of semisimple Lie groups – constructions that appear quite different at first glance.

Homogeneous vector bundles are one source of representations: if a real semisimple Lie group G_0 acts on a vector bundle $E \to M$ over a quotient space $M = G_0/H_0$, then G_0 acts also on the space of sections $C^\infty(M, E)$, and on any subspace $V \subset C^\infty(M, E)$ defined by a G_0-invariant system of differential equations. Ordinary induction, so-called cohomological induction and the construction of representations by "quantization" all fit into the framework of homogeneous vector bundles.

For any complex semisimple Lie algebra $\mathfrak{g}$, there is an equivalence of categories, due to Beilinson-Bernstein [1], between $\mathfrak{g}$-modules on the one hand, and sheaves of $\mathscr{D}$-modules over the flag variety X of $\mathfrak{g}$ on the other. In the context of real semisimple Lie groups this equivalence of categories associates infinitesimal representations to orbits in the flag variety of the complexified Lie algebra – orbits not of the group G_0 itself, but of the complexification of maximal compact subgroup $K_0 \subset G_0$.

We shall show that these Beilinson-Bernstein modules are naturally dual to modules attached to certain homogeneous vector bundles. In the special case of a compact group, both the Beilinson-Bernstein construction and the construction via homogeneous vector bundles reduce to the Borel-Weil-Bott theorem; our duality theorem is then a particular instance of Serre duality.

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To be more precise, we fix a connected semisimple Lie group G_0, with finite center, and a maximal compact subgroup $K_0 \subset G_0$. We write $\mathfrak{g}$ and $\mathfrak{k}$ for the complexified Lie algebras of G_0 and K_0. By definition, a Harish-Chandra module V is a module over the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, equipped with an action of K_0 such that

a) V is finitely generated over $\mathscr{U}(\mathfrak{g})$;
b) K_0 acts locally finitely and continuously (i.e., V is the union of finite dimensional K_0-invariant subspaces in which K_0 acts continuously);
c) the actions of K_0 and $\mathfrak{g}$ are compatible (i.e., $\mathfrak{k} \subset \mathfrak{g}$ operates by the differential of the K_0-action).

Harish-Chandra modules arise as infinitesimal representations corresponding to global representations of G_0.

For the purposes of this introduction, we suppose G_0 is linear, to ensure that its Cartan subgroups are abelian. The datum of a Cartan subgroup $H_0 \subset G_0$ and a character χ of H_0 determines a homogeneous line bundle $E \to G_0/H_0$, whose fibre at the identity coset is the representation space of χ. In the terminology of geometric quantization, the choice of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b} \supset \mathfrak{h}$ ($=$complexified Lie algebra of H_0) puts a G_0-invariant polarization on the symplectic manifold G_0/H_0. The sections of E which are annihilated by the polarization constitute a sheaf $\mathcal{O}_{\mathfrak{b}}(E)$. At one extreme, in the case of a real polarization, $\mathcal{O}_{\mathfrak{b}}(E)$ consists of all sections that drop to G_0/B_0, the quotient of G_0 by the largest subgroup with complexified Lie algebra $\mathfrak{b}$. At the opposite extreme, if $\mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{h}$, G_0/H_0 carries an invariant complex structure, $E \to G_0/H_0$ is a homogeneous holomorphic line bundle, and $\mathcal{O}_{\mathfrak{b}}(E)$ the sheaf of holomorphic sections. In all cases G_0 acts on the sheaf $\mathcal{O}_{\mathfrak{b}}(E)$, and thus also on its cohomology groups.

For technical reasons, it is simpler to work not directly with the global representation of G_0 on the cohomology groups, but rather with certain analogously defined Harish-Chandra modules. The idea is due to Zuckerman (see [22]); translated back into geometric terms, it can be explained as follows. The cohomology of $\mathcal{O}_{\mathfrak{b}}(E)$ is computed by a complex of E-valued differential forms. Conjugating H_0 if necessary, we may suppose that $K_0 \cap H_0$ is maximal compact in H_0. Both $\mathfrak{g}$ and K_0 act on the complex of K_0-finite, E-valued forms on a formal neighborhood of $K_0/(K_0 \cap H_0)$ in G_0/H_0 – in other words, differential forms with coefficients which are formal power series in directions normal to $K_0/(K_0 \cap H_0)$, but smooth along $K_0/(K_0 \cap H_0)$ itself. The cohomology groups of this formal complex, equipped with the induced actions of $\mathfrak{g}$ and K_0, are Harish-Chandra modules, the standard Zuckerman modules corresponding to the homogeneous line bundle $E \to G_0/H_0$ and the polarization $\mathfrak{b}$. These are the modules that appear on one side of our duality theorem.

The assignment $(G_0/H_0, \mathfrak{b}) \to$ orbit through $\mathfrak{b}$ sets up a bijection between pairs $(G_0/H_0, \mathfrak{b})$ as above, modulo isomorphism, and G_0-orbits in the flag variety X of $\mathfrak{g}$. The bijection extends to homogeneous line bundles: the G_0-stabilizer of $\mathfrak{b} \in X$ contains H_0 as Levi component, so the character of H_0 that defines E can be continued uniquely to the stabilizer. In this manner the standard Zuckerman modules are parametrized by triples (S, E, q), consisting of a G_0-orbit $S \subset X$, a G_0-homogeneous line bundle $E \to S$, and an integer q (the degree of the cohomology).

The complexification K of K_0 acts on X via the adjoint homomorphism. A K-orbit $Q \subset X$ will be said to be dual to a G_0-orbit $S \subset X$ if K_0 acts transitively on the intersection $Q \cap S$; according to [17], this notion of duality pairs the two types of orbits in a bijective, order-reversing fashion. Each G_0-homogeneous line bundle E over a G_0-orbit S is carried along by the duality: there exists a unique K-homogeneous algebraic line bundle F over the dual orbit Q, such that E and F coincide as K_0-homogeneous line bundles over the intersection $Q \cap S$. In addition to F, the line bundle E (equivalently, the character of H_0 that defines E) determines a $\mathfrak{g}$-equivariant twisted sheaf of differential operators[1] $\mathscr{D}$ on the flag variety X, whose restriction to Q operates on sections of F. The additional datum of the sheaf $\mathscr{D}$ makes the correspondence $E \to (F, \mathscr{D})$ bijective.

For the moment, we consider a particular K-orbit $Q \subset X$, a K-homogeneous algebraic line bundle $F \to Q$, and a $\mathfrak{g}$-equivariant twisted sheaf of differential operators $\mathscr{D}$ on X which, when restricted to Q, acts on sections of F. Then $\mathcal{O}_Q(F)$ can be pushed forward to a sheaf of $\mathscr{D}$-modules $j_+\mathcal{O}_Q(F)$ on X, the $\mathscr{D}$-module direct image with respect to the inclusion $j: Q \to X$. Both $\mathfrak{g}$ and K_0 act on $j_+\mathcal{O}_Q(F)$, and hence also on the cohomology groups $H^q(X, j_+\mathcal{O}_Q(F))$. These cohomology groups are Harish-Chandra modules, the Beilinson-Bernstein modules corresponding to the data $(Q, F, \mathscr{D}, q)$.

The duality theorem is now easily stated: if S and Q are dual orbits, and if the duality relates E to $(F, \mathscr{D})$, the Beilinson-Bernstein module $H^q(X, j_+\mathcal{O}_Q(F))$ is canonically dual, in the category of Harish-Chandra modules, to the standard Zuckerman module in degree $s-q$, corresponding to the bundle $E^* \otimes \Omega_X \to S$; here $s = \dim_{\mathbb{R}}(Q \cap S) - \dim_{\mathbb{C}} Q$ depends on the orbit Q, Ω_X denotes the canonical bundle of X, and E^* the dual of E.

The example of $G_0 = SL(2, \mathbb{R})$, $K_0 = SO(2)$ may help to clarify the theorem and its setting. In this case $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Up to conjugacy, G_0 contains two Cartan subgroups, namely K_0 and the diagonal subgroup D_0. The choice of an invariant polarization for G_0/K_0 amounts to an identification with the G_0-orbit in $\mathbb{C} \cup \{\infty\}$ through one of the fixed points of K_0: either $G_0/K_0 \cong G_0 \cdot i = U =$ upper half plane, or $G_0/K_0 \cong G_0 \cdot (-i) = \bar{U} =$ lower half plane. The homogeneous holomorphic line bundles $E_n \to U \cong G_0/K_0$ are parametrized by $n \in \mathbb{Z} \cong$ character group of K_0. Since $K_0 \cdot i = \{i\}$, the Zuckerman modules for (U, E_n) are computed by the complex of K_0-finite germs of E_n-valued $(0, p)$-forms at i, with coefficients in the ring of formal power series. The polynomial Dolbeault lemma shows that this complex is acyclic. In degree zero its cohomology reduces to the space of K_0-finite, holomorphic formal germs (i.e., with formal power series coefficients) of sections of E_n around i; both K_0 and $\mathfrak{g}$ operate on such germs by translation. The same discussion applies to the other polarization of G_0/K_0.

Two Borel subgroups of G_0 contain the $\mathbb{R}$-split Cartan subgroup D_0, but they are conjugate and thus correspond to the same G_0-orbit in the flag variety: $\mathbb{R} \cup \{\infty\} = G_0 \cdot 0 \cong G_0/B_0$, with $B_0 =$ lower triangular subgroup. Since $D_0 \cong \{\pm 1\} \times \mathbb{R}$, the datum of a character ε of the two-element group $\{\pm 1\}$ and

[1] See [21] for a heuristic discussion of the Beilinson-Bernstein construction in general and of twisted sheaves of differential operators in particular

of a complex number ζ ($\mathbb{C} \cong$ character group of $\mathbb{R}$!) determines a homogeneous line bundle $E_{\zeta,\varepsilon} \to G_0/D_0$; as in the general case, the bundle drops to G_0/B_0. The formal complex in this situation consists of K_0-finite, $E_{\zeta,\varepsilon}$-valued, relative differential forms for the fibration $G_0/D_0 \to G_0/B_0$, defined in a formal neighborhood of the K_0-orbit through the identity coset. The polynomial Poincaré lemma, applied fibre-by-fibre, shows that cohomology occurs only in degree zero – the space of K_0-finite C^∞ sections of $E_{\zeta,\varepsilon}$ over $G_0/B_0 \cong \mathbb{R} \cup \{\infty\}$.

Still in the case of $G_0 = SL(2, \mathbb{R})$, $K = SO(2, \mathbb{C})$ has three orbits in the flag variety $X = \mathbb{C} \cup \{\infty\}$, namely $\{i\}$, $\{-i\}$, and their common complement. The duality pairs these orbits with the three G_0-orbits U, $\bar{U}$, $\mathbb{R} \cup \{\infty\}$, in the given order. For each $n \in \mathbb{Z}$, the line bundle $E_n \to U$ extends $SL(2, \mathbb{C})$-equivariantly to all of $\mathbb{C} \cup \{\infty\}$. To be consistent with our previous notation, we write F_n for the fibre of E_n at i; then F_n is a "K-homogeneous line bundle" over the one-point space $\{i\}$. The Beilinson-Bernstein sheaf $j_+ \mathcal{O}_{\{i\}}(F_n)$ is supported at i, and has no higher cohomology. Its sections are E_n-valued "algebraic distributions" with support at i – i.e., principal parts around i of rational sections of E_n. Via multiplication and residues such "algebraic distributions" are dual to germs of regular sections of $E_{-n} \otimes \Omega_X$ at i, even to formal germs. The pairing exhibits the Beilinson-Bernstein module and the module of K_0-finite formal germs as dual Harish-Chandra modules – the assertion of the duality theorem in this special case. For the G_0-orbit $\bar{U}$ and the dual K-orbit $\{-i\}$ the situation is entirely analogous.

At points of $X - \{\pm i\}$, K has isotropy subgroup $\{\pm 1\}$, so the duality between orbits carries a line bundle $E_{\zeta,\varepsilon} \to \mathbb{R} \cup \{\infty\}$ to a K-homogeneous bundle $F_\varepsilon \to X - \{\pm i\}$ which depends only on ε, and not on the continuous parameter ζ. The second ingredient of the construction, the twisted sheaf of differential operators $\mathcal{D}$, is specified by ζ. Since X contains $X - \{\pm i\}$ as an open subset, the direct image $j_+ \mathcal{O}_{X - \{\pm i\}}(F_\varepsilon)$ coincides with the direct image in the category of sheaves. Again the higher cohomology vanishes; the space of global sections consists of the algebraic sections of F_ε over $X - \{\pm i\}$. The Lie algebra $\mathfrak{g}$ acts via the inclusion $\mathfrak{g} \hookrightarrow \Gamma \mathcal{D}$, whereas K_0 acts already on the level of the bundle F_ε. Global sections of F_ε can be restricted to the K_0-orbit $\mathbb{R} \cup \{\infty\}$, and their restrictions can then be integrated against K_0-finite smooth sections of $E_{-\zeta,\varepsilon} \otimes \Omega_X \to \mathbb{R} \cup \{\infty\}$. This pairing is $\mathfrak{g}$-equivariant and realizes the duality between the two constructions.

The fact that all K-orbits in X are affine and the resulting vanishing of the higher cohomology groups of the sheaves $j_+ \mathcal{O}_Q(F)$ is an atypical feature of $SL(2, \mathbb{R})$. A vanishing theorem, formally analogous to Cartan's Theorem B, exists also in the general case, but it depends on a positivity conditions for the twisted sheaf of differential operators $\mathcal{D}$ [1]; cohomology may occur in various degrees when this condition fails.

By its very nature, the Beilinson-Bernstein construction leads to a classification of the irreducible Harish-Chandra modules. The equivalence of categories between $\mathcal{U}(\mathfrak{g})$-modules and sheaves of $\mathcal{D}$-modules on X associates irreducible K-equivariant sheaves to irreducible Harish-Chandra modules. Such sheaves can be classified by geometric arguments: they arise as unique irreducible subsheaves (of $\mathcal{D}$-modules) of direct images $j_+ \mathcal{O}_Q(F)$. Whenever the vanishing theorem applies, the global sections of the subsheaf constitute an irreducible Harish-Chandra module or reduce to zero. In this manner irreducible Harish-Chandra modules correspond bijectively to certain Beilinson-Bernstein data $(Q, F, \mathscr{D})$.

Two earlier classification schemes, due to Langlands [15] and Vogan-Zuckerman [22], can be interpreted geometrically in terms of Zuckerman's construction, though the original statements and proofs are non-geometric. In effect, Langlands classifies irreducible Harish-Chandra modules as quotients of standard modules attached to "maximally real" polarizations, whereas Vogan-Zuckerman work with quotients of standard modules that correspond to "maximally complex" polarizations. Via the duality theorem, the Beilinson-Bernstein classification can be rephrased as follows: under mild regularity conditions, standard Zuckerman modules corresponding to negative polarizations have unique irreducible quotients; every irreducible Harish-Chandra module arises as such a quotient. In a continuation of this paper we shall show how changes of polarization affect standard modules. This, coupled with the duality theorem, makes it possible to relate the three classifications directly and totally explicitly.

The Beilinson-Bernstein construction – alternatively, a similar idea of Brylinski-Kashiwara [6] – is the crucial ingredient of the proof of the Kazhdan-Lusztig conjectures: the equivalence of categories between $\mathscr{U}(\mathfrak{g})$-modules and sheaves of $\mathscr{D}$-modules, followed by the equivalence of categories between sheaves of $\mathscr{D}$-modules and perverse sheaves (the "Riemann-Hilbert correspondence" [5]) translates the decomposition problem for Verma modules into a combinatorial problem, which has already been solved by Kazhdan and Lusztig [14]. The same line of reasoning was carried over to the setting of Harish-Chandra modules by Lusztig-Vogan [16] and Vogan [23]. The first of the two papers treats the combinatorial aspects, the second identifies the standard modules of Langlands' classification, in the case of integral infinitesimal characters, with their Beilinson-Bernstein counterparts. In effect, this last step is a special case of our results on the connection between the different classifications.

Via the duality theorem a number of known, but seemingly subtle results on the Zuckerman modules become consequences of quite general, or even obvious properties of the Beilinson-Bernstein construction. The vanishing theorem of Beilinson-Bernstein and the vanishing of cohomology below degree zero, for example, are far more transparent than the equivalent statements on the Zuckerman side. In the continuation of this paper we shall explore other, less immediate implications of the duality, in particular geometric explanations and proofs of certain irreducibility theorems.

Our main result was announced in [21]. It is related to work of Bernstein, who has given a $\mathscr{D}$-module interpretation of Zuckerman's functor. He has recently informed us that he now sees how this may be used to prove the duality theorem.

The paper is organized as follows. Section two recalls details of the Beilinson-Bernstein construction; we then express the cohomology of the sheaves $j_+\mathcal{O}_Q(F)$ in terms of a particular complex. We do the same for the Zuckerman modules in section three. The proof of the duality theorem is completed in section four, where we set up a duality between the two complexes. An appendix, addressed to non-experts on the theory of $\mathscr{D}$-modules, summarizes some technical results for which there is no ready reference. To simplify the exposition, the main body of the paper establishes the duality in the setting of a connected group with finite center; a second appendix shows how both of these restrictions can be removed.

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§ 2. Localization of Harish-Chandra modules

In this section we describe first, in a sketchy way, the results of A. Beilinson and J. Bernstein on the localization of $\mathfrak{g}$-modules [1]. For more details see [18].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the group of inner automorphisms of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{b}$ be a fixed Borel subalgebra in $\mathfrak{g}$ and B the corresponding Borel subgroup in G. Then the flag variety $X=G/B$ of G can be identified with the *variety of Borel subalgebras* of $\mathfrak{g}$. The group G acts naturally on the trivial vector bundle $X\times\mathfrak{g}\to X$, and the vector bundle $\mathscr{B}$ of Borel subalgebras is a homogeneous vector subbundle of it. For each $x\in X$, we denote the corresponding Borel subalgebra of $\mathfrak{g}$ by $\mathfrak{b}_x$, and the nilpotent radical of $\mathfrak{b}_x$ by $\mathfrak{n}_x$. Hence, we have the homogeneous vector subbundle $\mathscr{N}$ of $\mathscr{B}$ of nilpotent radicals.

Let $\mathscr{H}=\mathscr{B}/\mathscr{N}$. Then $\mathscr{H}$ is a homogeneous vector bundle over X with fiber $\mathfrak{b}_x/\mathfrak{n}_x$ over $x\in X$. The group B acts trivially on $\mathfrak{b}/\mathfrak{n}$, hence $\mathscr{H}$ is the trivial vector bundle over X with fiber $\mathfrak{h}\cong\mathfrak{b}/\mathfrak{n}$. We call the abelian Lie algebra $\mathfrak{h}$ the (abstract) *Cartan algebra* for $\mathfrak{g}$.

Let $\mathcal{O}_X$ be the structure sheaf of the algebraic variety X, i.e. the sheaf of regular functions on X. Let $\mathfrak{g}^0=\mathcal{O}_X\otimes_{\mathbb{C}}\mathfrak{g}$ be the sheaf of local sections of the trivial bundle $X\times\mathfrak{g}$. Denote by $\mathfrak{b}^0$ and $\mathfrak{n}^0$ the corresponding subsheaves of local sections of $\mathscr{B}$ and $\mathscr{N}$, respectively. If we denote by τ the natural homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra of vector fields on X, we can define a structure of a sheaf of complex Lie algebras on $\mathfrak{g}^0$ by putting

$$[f\otimes\xi, g\otimes\eta]=f\tau(\xi)\,g\otimes\eta-g\tau(\eta)f\otimes\xi+fg\otimes[\xi,\eta]$$

for $f, g\in\mathcal{O}_X$ and $\xi,\eta\in\mathfrak{g}$. Then $\mathfrak{b}^0$ and $\mathfrak{n}^0$ become sheaves of Lie subalgebras of $\mathfrak{g}^0$. In fact, the homogeneity of $\mathscr{B}$ and $\mathscr{N}$ implies that $\mathfrak{b}^0$ and $\mathfrak{n}^0$ are sheaves of ideals in $\mathfrak{g}^0$. We extend τ to a homomorphism from $\mathfrak{g}^0$ into the sheaf of Lie algebras of local vector fields on X. The kernel of this extension, which we also denote by τ, coincides with $\mathfrak{b}^0$. The quotient sheaf $\mathfrak{h}^0=\mathfrak{b}^0/\mathfrak{n}^0$ is the sheaf of local sections of $\mathscr{H}$ and is therefore equal to the sheaf of abelian Lie algebras $\mathcal{O}_X\otimes_{\mathbb{C}}\mathfrak{h}$.

Similarly, if we denote by $\mathscr{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, we can define a multiplication in the sheaf $\mathscr{U}^0=\mathcal{O}_X\otimes_{\mathbb{C}}\mathscr{U}(\mathfrak{g})$ by

$$(f\otimes\xi)(g\otimes\eta)=f\tau(\xi)\,g\otimes\eta+fg\otimes\xi\eta,$$

where $f, g \in \mathcal{O}_X$ and $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$. In this way $\mathcal{U}^0$ becomes a sheaf of complex associative algebras on X. Then $\mathfrak{g}^0$ is a subsheaf of $\mathcal{U}^0$, and the natural commutator in $\mathcal{U}^0$ induces the bracket operation on $\mathfrak{g}^0$. It follows from the previous remarks that the sheaf of right ideals $\mathfrak{n}^0 \mathcal{U}^0$ generated by $\mathfrak{n}^0$ in $\mathcal{U}^0$ is a sheaf of two-sided ideals in $\mathcal{U}^0$. The quotient $\mathcal{D}_{\mathfrak{h}} = \mathcal{U}^0 / \mathfrak{n}^0 \mathcal{U}^0$ is therefore a sheaf of complex associative algebras on X.

Since $\mathfrak{h}^0$ is a sheaf of Lie subalgebras of $\mathcal{D}_{\mathfrak{h}}$ there exists a natural homomorphism of the enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ into $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$. One can check that its image is equal to the G-invariants of $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$. On the other hand, the natural homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$ maps the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ into the G-invariants of $\Gamma(X, \mathcal{D}_{\mathfrak{h}})$. Finally, there is the canonical Harish-Chandra homomorphism $\gamma: \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$, defined as follows: For any $x \in X$, the sum of the subalgebra $\mathcal{U}(\mathfrak{b}_x)$ and the right ideal $\mathfrak{n}_x \mathcal{U}(\mathfrak{g})$ contains the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$, so $\mathcal{Z}(\mathfrak{g})$ projects naturally into

$$\mathcal{U}(\mathfrak{b}_x)/(\mathfrak{n}_x \mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{b}_x)) = \mathcal{U}(\mathfrak{b}_x)/\mathfrak{n}_x \mathcal{U}(\mathfrak{b}_x) = \mathcal{U}(\mathfrak{b}_x/\mathfrak{n}_x);$$

the composition of the projection with the natural isomorphism $\mathcal{U}(\mathfrak{b}_x/\mathfrak{n}_x) \cong \mathcal{U}(\mathfrak{h})$ is independent of x and, by definition, equal to γ. A simple argument shows that the diagram

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\ \gamma\ } & \mathcal{U}(\mathfrak{h}) \\ & \searrow \quad \swarrow & \\ & \mathcal{D}_{\mathfrak{h}} & \end{array}$$

is commutative.

Let $x \in X$. Fix a Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$. Let R be root system of $\mathfrak{g}$ in $\mathfrak{c}^*$ and for $\alpha \in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root subspace of $\mathfrak{g}$. We order R so that the set R^+ of positive roots corresponds to $\mathfrak{n}_x$,

$$R^+ = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_x\}$$

(this is the ordering opposite to the one used by Beilinson and Bernstein [1]). The canonical isomorphism $\mathfrak{c} \to \mathfrak{b}_x/\mathfrak{n}_x \to \mathfrak{h}$ induces an isomorphism of the triple $(\mathfrak{c}^*, R, R^+)$ onto the triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ – the *Cartan triple* of $\mathfrak{g}$. We call the inverse isomorphism of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ onto $(\mathfrak{c}^*, R, R^+)$ a *specialization* at x.

Let W be the Weyl group of Σ. Let ρ be the half-sum of all positive roots in Σ. The enveloping algebra $\mathcal{U}(\mathfrak{h})$ of $\mathfrak{h}$ is naturally isomorphic to the algebra of polynomials on $\mathfrak{h}^*$, and therefore any $\lambda \in \mathfrak{h}^*$ determines a homomorphism of $\mathcal{U}(\mathfrak{h})$ into $\mathbb{C}$. Let I_λ be the kernel of the homomorphism $\sigma_\lambda: \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$ determined by $\lambda + \rho$. Then $\gamma^{-1}(I_\lambda)$ is a maximal ideal in $\mathcal{Z}(\mathfrak{g})$, and, by an old result of Harish-Chandra, for $\lambda, \mu \in \mathfrak{h}^*$,

$$\gamma^{-1}(I_\lambda) = \gamma^{-1}(I_\mu) \text{ if and only if } w\lambda = \mu \text{ for some } w \in W.$$

For any $\lambda \in \mathfrak{h}^*$, the sheaf $I_\lambda \mathcal{D}_{\mathfrak{h}}$ is a sheaf of two-sided ideals in $\mathcal{D}_{\mathfrak{h}}$; therefore $\mathcal{D}_\lambda = \mathcal{D}_{\mathfrak{h}}/I_\lambda \mathcal{D}_{\mathfrak{h}}$ is a sheaf of complex associative algebras on X. In the case when

$\lambda=-\rho$, we have $I_{-\rho}=\mathfrak{h}\mathscr{U}(\mathfrak{h})$, hence $\mathscr{D}_{-\rho}=\mathscr{U}^0/\mathfrak{b}^0\mathscr{U}^0$, i.e. it is the sheaf of local differential operators on X. In general $\mathscr{D}_\lambda$, $\lambda\in\mathfrak{h}^*$, are *twisted sheaves of differential operators* on X. In the parametrization of homogeneous twisted sheaves of differential operators which we use in Appendix A, we have

$$\mathscr{D}_\lambda=\mathscr{D}_{X;\lambda+\rho}, \qquad \lambda\in\mathfrak{h}^*.$$

The shift by $-\rho$ in this parametrization, which is unnatural in general, reflects the Weyl group symmetry of the global sections of $\mathscr{D}_\lambda$ which we shall now discuss.

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ and $\lambda\in\theta$. Let $J_\theta=\gamma^{-1}(I_\lambda)$ be the maximal ideal in $\mathscr{Z}(\mathfrak{g})$ determined by θ (it is independent of the choice of λ, by the previously mentioned result of Harish-Chandra). Then the elements of $\mathscr{U}_\theta=\mathscr{U}(\mathfrak{g})/J_\theta\mathscr{U}(\mathfrak{g})$ determine global sections of $\mathscr{D}_\lambda$; in fact, we have

$$\Gamma(X,\mathscr{D}_\lambda)=\mathscr{U}_\theta \qquad \text{for any } \lambda\in\theta$$

([1]; for a simple proof of this statement see [19]).

Any $\mathfrak{g}$-module V with infinitesimal character $\chi_\lambda=\sigma_\lambda\circ\gamma$, $\lambda\in\theta$, can be considered as a module over $\mathscr{U}_\theta$, thus

$$\Delta_\lambda(V)=\mathscr{D}_\lambda\otimes_{\mathscr{U}_\theta}V, \qquad V\in\mathscr{M}(\mathscr{U}_\theta),$$

defines a covariant functor Δ_λ from

$\mathscr{M}(\mathscr{U}_\theta)=$ the category of $\mathfrak{g}$-modules with infinitesimal character χ_λ

into

$\mathscr{M}(\mathscr{D}_\lambda)=$ the category of quasi-coherent $\mathscr{D}_\lambda$-modules on X.

The functor Δ_λ is called the *localization functor*.

For any quasi-coherent $\mathscr{D}_\lambda$-module $\mathscr{V}$ on X, the cohomology groups $H^i(X,\mathscr{V})$, $0\leqq i\leqq \dim X$, are $\mathfrak{g}$-modules with infinitesimal character χ_λ; i.e. they define covariant functors going in the opposite direction.

For any root $\alpha\in\Sigma$ we denote by $\alpha^\vee$ its dual root in the dual root system $\Sigma^\vee$ in $\mathfrak{h}$. We say that $\lambda\in\mathfrak{h}^*$ is *regular* if $\alpha^\vee(\lambda)$ is non-zero for any $\sigma\in\Sigma$ and that λ is *antidominant* if $\alpha^\vee(\lambda)$ is not a strictly positive integer for any $\alpha\in\Sigma^+$.

Now we can state the results of Beilinson and Bernstein [1]. First we have

2.1. Theorem (Beilinson, Bernstein). *Let $\mathscr{V}$ be a quasi-coherent $\mathscr{D}_\lambda$-module on the flag variety X. Then*

(i) *if λ is antidominant, all cohomology groups $H^i(X,\mathscr{V})$, $i>0$, vanish;*

(ii) *if λ is antidominant and regular, $\mathscr{V}$ is generated by its global sections.*

This result can be viewed as a vast generalization of the classical Borel-Weil theorem. As one consequence, the localization functor is an equivalence of categories, if $\lambda\in\mathfrak{h}^*$ is antidominant and regular, and the functor Γ is its inverse. If λ is singular and antidominant the situation is more complicated (as can be seen already from the Borel-Weil result). Still, the equivalence of categories statement remains true if one replaces $\mathscr{M}(\mathscr{D}_\lambda)$ with its quotient by the full subcategory consisting of modules with vanishing cohomology. In particular, we have the following simple result [19]:

2.2. Lemma. *Let $\lambda \in \mathfrak{h}^*$ be antidominant.*

(i) *If $\mathscr{V}$ is an irreducible $\mathscr{D}_\lambda$-module, $\Gamma(X, \mathscr{V})$ is an irreducible $\mathfrak{g}$-module or zero.*

(ii) *For any irreducible $\mathfrak{g}$-module V with infinitesimal character χ_λ, there exists a unique irreducible $\mathscr{D}_\lambda$-module $\mathscr{V}$ such that $V = \Gamma(X, \mathscr{V})$.*

We shall use the localization functor in a more restricted category of modules. Let K be a connected complex linear algebraic group and $\varphi: K \to \mathrm{Aut}(\mathfrak{g})$ a morphism of algebraic groups such that the differential of φ is an injection of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{g}$. Then we can identify the Lie algebra $\mathfrak{k}$ with its image in $\mathfrak{g}$. The group K acts naturally on the variety X. We say that the pair $(\mathfrak{g}, K)$ is a *Harish-Chandra pair* if the following condition is satisfied:

(HC) The group K acts on the variety X of Borel subalgebras of $\mathfrak{g}$ with finitely many orbits.

By making the assumption that K is connected we avoid some minor technical difficulties. We shall show in Appendix B how to remove this restriction.

Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. We say that a representation π of K on a complex vector space V is *algebraic* if

(A 1) the space V is a union of finite-dimensional K-invariant subspaces,

(A 2) for each finite-dimensional K-invariant subspace U of V, the action of K induces a morphism of the algebraic group K into the algebraic group $GL(U)$.

The category $\mathscr{M}_{fg}(\mathscr{U}_\theta, K)$ is the full subcategory of $\mathscr{M}(\mathscr{U}_\theta)$ consisting of finitely generated modules on which K acts by an algebraic representation, such that the actions of $\mathfrak{g}$ and K are compatible; concretely:

(CM) the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ agrees with the action of $\mathfrak{k}$ which is the differential of the action of K.

We call the objects of $\mathscr{M}_{fg}(\mathscr{U}_\theta, K)$ *Harish-Chandra modules* (with infinitesimal character χ_λ).

Similarly, $\mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$ is the full subcategory of $\mathscr{M}(\mathscr{D}_\lambda)$ consisting of all coherent $\mathscr{D}_\lambda$-modules on X with an algebraic action of K ([20], Ch. I, § 3), such that the action of $\mathscr{D}_\lambda$ is K-equivariant and compatible with the action of K, i.e.,

(CD) the action of $\mathfrak{k}$ as subalgebra of $\mathfrak{g} \subset \mathscr{U}_\theta = \Gamma(X, \mathscr{D}_\lambda)$ agrees with the differential of the action of K.

We call the objects of $\mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$ *Harish-Chandra sheaves* on X.

The localization functor Δ_λ maps Harish-Chandra modules into Harish-Chandra sheaves. Conversely the cohomology groups of Harish-Chandra sheaves are Harish-Chandra modules. The finiteness condition (HC) puts severe restrictions on the structure of Harish-Chandra sheaves:

2.3. Lemma (Beilinson, Bernstein [1]). *Any $\mathscr{V} \in \mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$ is a holonomic $\mathscr{D}_\lambda$-module. In particular, $\mathscr{V}$ is of finite length.*

The preceding results lead to a classification of irreducible Harish-Chandra sheaves. If $\mathscr{V}$ is an irreducible Harish-Chandra sheaf in $\mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$, its support $\operatorname{supp} \mathscr{V}$ is an irreducible subvariety of X. Also, $\operatorname{supp} \mathscr{V}$ is K-invariant, hence a union of K-orbits in X. Thus $\operatorname{supp} \mathscr{V}$ is the closure of a K-orbit Q in X.

Let $i: Q \to X$ be the natural immersion and $X' = X \setminus (\bar{Q} \setminus Q)$. Then X' is an open subvariety of X, Q is a closed smooth subvariety of X' and $\mathscr{V} | X'$ is a holonomic $\mathcal{D}_\lambda | X'$-module on X'. As explained in Appendix A, $\mathcal{D}_\lambda$ "induces" a K-homogeneous twisted sheaf of differential operators $(\mathcal{D}_\lambda)^i$ on Q, and the irreducible $\mathcal{D}_\lambda | X'$-module $\mathcal{V} | X'$ is isomorphic to the 0^{th} direct image of a holonomic $(\mathcal{D}_\lambda)^i$-module τ on Q under the immersion of Q into X'. Since Q is a K-orbit and $\mathcal{V}$ a $(\mathcal{D}_\lambda, K)$-module, τ is an irreducible $((\mathcal{D}_\lambda)^i, K)$-connection on Q. On the other hand, the 0^{th} direct image $R^0 i_+(\tau)$ is a holonomic $\mathcal{D}_\lambda$-module on X which is a Harish-Chandra sheaf. We denote it by $\mathcal{I}(Q, \tau)$, and call it the *standard Harish-Chandra sheaf* for the data (Q, τ). We can view $\mathcal{I}(Q, \tau)$ as the $\mathcal{D}_\lambda$-module obtained by applying to τ the 0^{th} direct image for closed immersion of Q into X', followed by the 0^{th} direct image for the open immersion of X' into X. By the very nature of the second step, every nonzero $\mathcal{D}_\lambda$-submodule of $\mathcal{I}(Q, \tau)$ restricts to the same $\mathcal{D}_\lambda | X'$-module on X', which is isomorphic to $\mathcal{V} | X'$. Therefore $\mathcal{I}(Q, \tau)$ has a unique irreducible Harish-Chandra subsheaf, which we denote by $\mathcal{L}(Q, \tau)$. By construction it is isomorphic to $\mathcal{V}$.

This leads to the classification of irreducible Harish-Chandra sheaves [1]:

2.4. Theorem (Beilinson, Bernstein). (i) *$\mathcal{L}(Q, \tau) = \mathcal{L}(Q', \tau')$ if and only if $(Q, \tau) = (Q', \tau')$.*

(ii) *Any irreducible Harish-Chandra sheaf is isomorphic to some $\mathcal{L}(Q, \tau)$.*

From 2.2 it follows that, for a fixed antidominant $\lambda \in \mathfrak{h}^*$, the module $\Gamma(X, \mathcal{L}(Q, \tau))$ is either an irreducible Harish-Chandra module or zero, and all irreducible Harish-Chandra modules in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ are obtained in this way. Any irreducible Harish-Chandra module in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ is attached to a unique set of data (Q, τ).

Let $\lambda \in \mathfrak{h}^*$, Q a K-orbit in X and $x \in Q$. Then $\lambda + \rho$ determines a linear form on $\mathfrak{b}_x$. The irreducible $((\mathcal{D}_\lambda)^i, K)$-connections τ on Q are parametrized by the irreducible finite-dimensional algebraic representations of the stabilizer S_x of x in K with the property that their differential is a direct sum of copies of the restriction of $\lambda + \rho$ to $\mathfrak{k} \cap \mathfrak{b}_x$. To describe them, we must first understand the structure of S_x. Let U_x be the unipotent radical of S_x. Then U_x is a connected closed subgroup of S_x ([13], Chap. 10). The Lie algebra of U_x is a subalgebra of $\mathfrak{n}_x$, hence the irreducible representations determining irreducible $((\mathcal{D}_\lambda)^i, K)$-connections on Q are trivial on U_x. On the other hand, if T is any maximal closed reductive subgroup of S_x, S_x is a semidirect product of T with U_x ([13], 14.2). The Lie algebra $\mathfrak{t}$ of T is a reductive subalgebra of the Borel subalgebra $\mathfrak{b}_x$, hence an abelian subalgebra of some Cartan subalgebra of $\mathfrak{g}$. The canonical isomorphism of $\mathfrak{b}_x/\mathfrak{n}_x$ onto $\mathfrak{h}$ identifies $\mathfrak{t}$ with a subalgebra of $\mathfrak{h}$, hence $\lambda + \rho$ defines a linear form on $\mathfrak{t}$ by specialization and restriction. It follows that the irreducible $((\mathcal{D}_\lambda)^i, K)$-connections are parametrized by irreducible finite-dimensional representations of T whose differentials are direct sums of copies of this form.

Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ and R^+ a set of positive roots in the root system R of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$. We call such pair $(\mathfrak{c}, R^+)$ an *ordered Cartan subalgebra.* Let Y be the set of all ordered Cartan subalgebras of $\mathfrak{g}$. The group G acts on Y by conjugation, and this action is transitive. The stabilizer of a point $y \in Y$ is the corresponding Cartan subgroup of G. Therefore Y has a natural structure of an algebraic variety, which we call the *variety of ordered Cartan* *subalgebras*. Since Cartan subgroups are reductive, a result of Mumford ([20], Theorem 1.1) implies that Y is an affine variety. Each ordered Cartan subalgebra $(\mathfrak{c}, R^+)$ determines a Borel subalgebra $\mathfrak{b}$ spanned by $\mathfrak{c}$ and the root subspaces of $\mathfrak{g}$ corresponding to the elements of R^+. We thus have a natural projection p from the variety of ordered Cartan subalgebras Y onto the flag variety X of $\mathfrak{g}$. Then p is a morphism of algebraic varieties, in fact an affine morphism since Y itself is affine.

The group K acts on Y by conjugation. Let y be a point of Y and $\mathfrak{h}_y$ the corresponding Cartan subalgebra. Then the stabilizer T_y of y in K is a reductive subgroup of K with Lie algebra $\mathfrak{t}_y$ equal to $\mathfrak{k} \cap \mathfrak{h}_y$. By the previously mentioned result of Mumford, the K-orbit of y is an affine variety. We conclude that all K-orbits in Y are affinely imbedded in Y.

Let Q be a K-orbit in X. We say that a K-orbit in Y *lies over* Q if p maps it onto Q. We want to study in more details the structure of a particular type of K-orbits in Y lying over Q. Let $x \in Q$, and B_x the corresponding Borel subgroup of G. Then, by the previous discussion, the stabilizer S_x of x in K is the semidirect product of a maximal reductive subgroup T and the unipotent radical U_x. The integer $s = \dim U_x$ depends only on Q. As a closed reductive subgroup of B_x, $\varphi(T)$ is contained in a Cartan subgroup $H_y \subset B_x$ for some $y \in Y$. Therefore T stabilizes y; in other words $T \subset T_y$. But T is a maximal closed reductive subgroup of S_x, so $T = T_y$. The K-orbit $\tilde{Q}$ of y in Y, which lies over Q, has the following property:

(O) The canonical projection $\pi: \tilde{Q} \to Q$ is an affine morphism with fibres isomorphic to $\mathbb{C}^s$.

We call such orbit $\tilde{Q}$ a *standard orbit* lying over Q. It is clear that (O) characterizes completely standard orbits lying over Q and they are all mutually isomorphic.

In the following we fix a K-orbit Q in X and a standard orbit $\tilde{Q}$ lying over Q. Also, we fix $x \in Q$ and $y \in \tilde{Q}$ such that $p(y) = x$. We denote by $\pi: \tilde{Q} \to Q$ the canonical projection, and by $i: Q \to X$ and $j: \tilde{Q} \to Y$ the canonical injections. We then have the following commutative diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\ j\ } & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow{\ i\ } & X \end{array}$$

in which the morphisms p, π and j are affine.

Let $\lambda \in \mathfrak{h}^*$. The natural isomorphism of $\mathfrak{b}_x/\mathfrak{n}_x$ onto $\mathfrak{h}$ defines Lie algebra morphisms of $\mathfrak{b}_x \cap \mathfrak{k}$ into $\mathfrak{h}$ and $\mathfrak{t}_y$ into $\mathfrak{h}$. From the previous discussion it follows that there is a bijection between the irreducible finite-dimensional algebraic representations of T_y whose differential is a direct sum of copies of $\lambda + \rho$ specialized at x and then restricted to $\mathfrak{t}_y$, and the irreducible finite-dimensional algebraic representations of S_x whose differential is a direct sum of copies of the representation of $\mathfrak{b}_x \cap \mathfrak{k}$ defined analogously by $\lambda + \rho$. This bijection induces a bijection of corresponding K-homogeneous connections on $\tilde{Q}$ and Q. If τ is such a connection on Q we denote by $\tilde{\tau} = \pi^+(\tau)$ the corresponding connection on $\tilde{Q}$. The connection $\tilde{\tau}$ is a left $((\mathscr{D}_\lambda)^i)^\pi$-module, and

$$((\mathscr{D}_\lambda)^i)^\pi = (\mathscr{D}_{X,\lambda+\rho})^{i\circ\pi} = (\mathscr{D}_{X,\lambda+\rho})^{p\circ j} = ((\mathscr{D}_{X,\lambda+\rho})^p)^j = (\mathscr{D}_{Y,\lambda+\rho})^j,$$

hence we can view $\tilde{\tau}$ as a left $(\mathscr{D}_{Y,\lambda+\rho})^j$-module.

The inverse of the map $\tau \to \tilde{\tau}$ is given by the following result, which is critical for our purposes.

2.5. Lemma. *We have*

$$R^q \pi_+(\tilde{\tau}) = 0 \quad \textit{for } q \neq -s,$$
$$R^{-s} \pi_+(\tilde{\tau}) = \tau.$$

Proof. Since $\tilde{\tau}$ is a $((\mathscr{D}_{Y,\lambda+\rho})^j, K)$-connection, its direct images are $((\mathscr{D}_\lambda)^i, K)$-modules, and in particular K-homogeneous $\mathcal{O}_Q$-modules. Hence they are completely determined by their geometric fibres at a point x considered as modules for the stabilizer S_x, and all their higher geometric fibres vanish. Let $F = \pi^{-1}(x)$, π' the projection of F into $\{x\}$, and $i_x\colon \{x\} \to X$ and $i_F\colon F \to \tilde{Q}$ the natural immersions. Then F is a smooth closed subvariety of $\tilde{Q}$ isomorphic to $\mathbb{C}^s$. Moreover, it is the orbit of y under S_x and under its unipotent radical U_x. Since $\tilde{\tau}$ is a K-homogeneous connection, the inverse image $i_F^+(\tilde{\tau})$ is a S_x-homogeneous connection and all higher inverse images of $\tilde{\tau}$ vanish. By base change ([5], 8.4), we have

$$T_x(R^q \pi_+(\tilde{\tau})) = i_x^+(R^q \pi_+(\tilde{\tau})) = R^q \pi'_+(i_F^+(\tilde{\tau})).$$

As U_x-homogeneous connection, $i_F^+(\tilde{\tau})$ is isomorphic to a direct sum of a number of copies of $\mathcal{O}_F$, since the stabilizer of y in U_x is trivial. From the description of the direct image functor for submersions in terms of the de Rham complex in A.3.3, and using the isomorphism of F with $\mathbb{C}^s$, we see that $R^q \pi'_+(i_F^+(\tilde{\tau}))$ is the $(-q)^{\text{th}}$ homology of the Koszul complex associated to the natural action $\partial_1, \ldots, \partial_s$ on a direct sum of $\mathbb{C}[X_1, \ldots, X_s]$. By ([7], Chap. X, §9, no. 6, Remarque 4) it follows that $R^q \pi'_+(i_F^+(\tilde{\tau})) = 0$ for $q \neq -s$ and $R^{-s}\pi'_+(i_F^+(\tilde{\tau}))$ is the space of constant global sections of $i_F^+(\tilde{\tau})$. Hence, as S_x-module, $R^{-s}\pi'_+(i_F^+(\tilde{\tau}))$ is isomorphic to the geometric fibre of τ at x. The lemma follows. □

As we have seen, the irreducible $((\mathscr{D}_\lambda)^i, K)$-connections on Q are parametrized by irreducible finite-dimensional algebraic representations of T_y whose differential is a direct sum of copies of the specialization and restriction of $\lambda + \rho$ to $\mathfrak{t}_y$. The standard Harish-Chandra sheaf $\mathscr{I}(Q, \tau) = R^0 i_+(\tau)$ corresponding to the data (Q, τ) is a special case of the following expression for the direct images of the connection τ:

$$\begin{aligned} R^q i_+(\tau) &= (R^q i_+)(R^{-s}\pi_+)(\tilde{\tau}) = R^{q-s}(i\circ\pi)_+(\tilde{\tau}) \\ &= R^{q-s}(p\circ j)_+(\tilde{\tau}) = R^{q-s}p_+(R^0 j_+(\tilde{\tau})), \qquad q \in \mathbb{Z}; \end{aligned}$$

here we have used the spectral sequence for the composition of the direct images, as well as the fact that the morphisms p, π and j are affine and that i and j are immersions.

Using the relative de Rham complex (A.3.3) we can express this in the following form. First, we transfer our situation from left $\mathscr{D}$-modules to right $\mathscr{D}$-modules.

The left $\mathscr{D}_\lambda$-module $R^q p_+(R^0 j_+(\tilde{\tau}))$ is a right $\mathscr{D}_{-\lambda}$-module. In the notation of A.1

$$(\mathscr{D}_{-\lambda})^p = (\mathscr{D}_{X,-\lambda+\rho})^p = \mathscr{D}_{Y,-\lambda+\rho}.$$

Moreover, $R^0 j_+(\tilde{\tau})$ is a left $\mathscr{D}_{Y,\lambda+\rho}$-module, hence a right $\mathscr{D}_{Y,-\lambda-\rho}$-module. It follows that, if we denote by $\Omega_{Y|X}$ the invertible $\mathcal{O}_Y$-module of relative differential forms of top degree for the projection of Y to X, $R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}$ is a right $\mathscr{D}_{Y,-\lambda+\rho}$-module. As explained in A.3.3, this implies

$$R^q p_+(R^0 j_+(\tilde{\tau})) = H^q(p_*(C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))),$$

as left $\mathscr{D}_\lambda$-module. Then

$$H^{q-s}(p_*(C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))) = R^q i_+(\tau) \qquad \text{for } q \in \mathbb{Z};$$

in particular

$$H^{-s}(p_*(C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))) = \mathscr{I}(Q, \tau).$$

This gives the expression for the standard Harish-Chandra sheaves we alluded to before. We can use it to calculate the cohomology of Harish-Chandra sheaves. Since the morphism p and the variety Y are affine, the components of the direct image of the relative de Rham complex are $\Gamma(X, .)$-acyclic. We therefore have the spectral sequence

$$\begin{aligned} &H^q(X, H^r(p_*(C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X})))) \\ &\quad \Rightarrow H^{q+r}(\Gamma(X, p_*(C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X})))) \\ &\quad = H^{q+r}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))) \end{aligned}$$

([12], II.2.4). This leads to the following result:

2.6. Proposition. *There exists a first quadrant spectral sequence*

$$H^q(X, R^r i_+(\tau)) \Rightarrow H^{q+r-s}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))).$$

Since $E_2^{0,0} = E_\infty^{0,0}$, we conclude:

2.7. Corollary.

$$\Gamma(X, \mathscr{I}(Q, \tau)) = H^{-s}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))).$$

In two important special cases the spectral sequence 2.6 collapses. If λ is antidominant, all higher cohomology groups vanish by 2.1, and we have

$$\Gamma(X, R^q i_+(\tau)) = H^{q-s}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))).$$

If i is an affine imbedding, the higher direct images vanish and we have

$$H^q(X, R^0 i_+(\tau)) = H^{q-s}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))).$$

As we shall see in Sect. 4, all K-orbits are affinely imbedded if $(\mathfrak{g}, K)$ is a Harish-Chandra pair with the additional property that the Lie algebra $\mathfrak{k}$ of K is a fixed point set of an involution. In particular, the preceding identity applies in this case.

On the other hand, we remarked before that $\tilde{Q}$ is an affine variety and therefore affinely imbedded into Y. This enables us to rewrite the right-hand side of 2.6 in a simpler way. The connection $\tilde{\tau}$ is a left $(\mathscr{D}_{Y,\lambda+\rho})^j$-module, hence, $\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})$ is a left $(\mathscr{D}_{Y,\lambda-\rho})^j$-module, and therefore a right $(\mathscr{D}_{Y,-\lambda+\rho})^j$-module. The dual $(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))^\vee$ of $\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})$, which we denote by $\mathscr{U}$, is a homogeneous $\mathcal{O}_{\tilde{Q}}$-module. By differentiation of the K-action $\mathscr{U}$ becomes a connection for $\mathscr{D}_{\mathscr{U}}=(\mathscr{D}_{Y,-\lambda+\rho})^j$. From A.3.3 we know that, in our situation, the shifted relative de Rham complex can be written as

$$C_{Y|X}(R^0 j_+(\tilde{\tau})\otimes_{\mathcal{O}_Y}\Omega_{Y|X})=(R^0 j_+(\tilde{\tau})\otimes_{\mathcal{O}_Y}\Omega_{Y|X})\otimes_{\mathcal{O}_Y}\wedge\mathscr{T}_{Y|X}$$
$$=(R^0 j_+(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})))\otimes_{\mathcal{O}_Y}\wedge\mathscr{T}_{Y|X}.$$

Moreover, the right action of $\mathscr{U}(\mathfrak{g})$ described in A.3.3, composed with the principal antiautomorphism, defines a left action of $\mathscr{U}(\mathfrak{g})$ on the complex. In addition, since the varieties Y and Q are affine, we get

$$\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau})\otimes_{\mathcal{O}_Y}\Omega_{Y|X}))=\Gamma(Y,(R^0 j_+(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))\otimes_{\mathcal{O}_Y}\wedge\mathscr{T}_{Y|X})$$
$$=\Gamma(Y, R^0 j_+(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})))\otimes_{\Gamma(Y,\mathcal{O}_Y)}\Gamma(Y,\wedge\mathscr{T}_{Y|X})$$
$$=\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))\otimes_{\Gamma(\tilde{Q},\mathscr{D}_{\mathscr{U}})}\Gamma(\tilde{Q},\otimes_{\tilde{Q}\to Y,-\lambda+\rho})\mathscr{D}_{\Gamma(Y,\mathcal{O}_Y)}\Gamma(Y,\wedge\mathscr{T}_{Y|X}).$$

In terms of the notation

$$R_{\tilde{Q}\to Y|X,\mu}=\Gamma(\tilde{Q},\mathscr{D}_{\tilde{Q}\to Y,\mu})\otimes_{\Gamma(Y,\mathcal{O}_Y)}\Gamma(Y,\wedge\mathscr{T}_{Y|X}),$$

this becomes:

2.8. Lemma. *As left* $\mathscr{U}(\mathfrak{g})$*-module,*

$$\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau})\otimes_{\mathcal{O}_Y}\Omega_{Y|X}))=\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))\otimes_{\Gamma(\tilde{Q},\mathscr{D}_{\mathscr{U}})}R_{\tilde{Q}\to Y|X,-\lambda+\rho}.$$

§ 3. Standard Zuckerman modules

Let $\mathfrak{g}$ be a complex Lie algebra, T a complex reductive linear algebraic group and $\varphi: T\to \operatorname{Aut}(\mathfrak{g})$ a morphism of algebraic groups such that its differential, which is a Lie algebra morphism from the Lie algebra $\mathfrak{t}$ of T into the derivations of $\mathfrak{g}$, factors through an injective morphism of $\mathfrak{t}$ into $\mathfrak{g}$. Then we can identify the Lie algebra $\mathfrak{t}$ with its image in $\mathfrak{g}$. Also, $\mathfrak{t}$ is reductive in $\mathfrak{g}$.

We say that (π, V) is a $(\mathfrak{g}, T)$-module if it is simultaneously a module for $\mathfrak{g}$ and T, such that

(Z1) T acts by an algebraic representation; cf. Sect. 2,

(Z2) the action of $\mathfrak{t}$ as a subalgebra of $\mathfrak{g}$ agrees with the action of $\mathfrak{t}$ which is the differential of the action of T, and $t\cdot\xi\cdot t^{-1}\cdot v=(\operatorname{Ad}\varphi(t)\,\xi)\cdot v$ for $t\in T$, $\xi\in\mathfrak{g}$ and $v\in V$.

A morphism of $(\mathfrak{g}, T)$-modules is a linear map which preserves the $\mathfrak{g}$- and T-module structure. We denote by $\operatorname{Hom}_{\mathfrak{g},T}(U, V)$ the linear space of all $(\mathfrak{g}, T)$-

morphisms between the $(\mathfrak{g}, T)$-modules U and V. The $(\mathfrak{g}, T)$-modules form an abelian category, which we denote by $\mathcal{M}(\mathfrak{g}, T)$.

Let $(\mathfrak{k}, T)$ and $(\mathfrak{g}, T)$ be two pairs such that the Lie algebra $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ containing the Lie algebra $\mathfrak{t}$ of T. We then can define two exact covariant functors $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}$ and $\mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}$ from the category $\mathcal{M}(\mathfrak{k}, T)$ into the category $\mathcal{M}(\mathfrak{g}, T)$ as follows.

If U is a $(\mathfrak{k}, T)$-module, the linear space of $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(U)$ is $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} U$, where $\mathcal{U}(\mathfrak{g})$ is considered as right $\mathcal{U}(\mathfrak{k})$-module via right multiplication. The action of $\mathfrak{g}$ is given as left multiplication on the first factor, and the action of T as the tensor product of the adjoint action on $\mathcal{U}(\mathfrak{g})$ and the action on U. This procedure gives us a covariant functor, and the Poincaré-Birkhoff-Witt theorem implies that it is exact.

For any T-module V, we denote by $V_{[T]}$ its largest algebraic submodule. Let $\mathrm{Hom}_{\mathfrak{k}}(\mathcal{U}(\mathfrak{g}), U)$ be the space of all $\mathfrak{k}$-morphisms from $\mathcal{U}(\mathfrak{g})$, considered as $\mathfrak{k}$-module via left multiplication, to U. The adjoint action of T on $\mathcal{U}(\mathfrak{g})$ and the action on U induce a natural action of T on the space $\mathrm{Hom}_{\mathfrak{k}}(\mathcal{U}(\mathfrak{g}), U)$. The linear space of $\mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}(U)$ is $\mathrm{Hom}_{\mathfrak{k}}(\mathcal{U}(\mathfrak{g}), U)_{[T]}$. The action of $\mathfrak{g}$ is given by right multiplication on $\mathcal{U}(\mathfrak{g})$. This construction gives a covariant functor, and its exactness follows again from the Poincaré-Brikhoff-Witt theorem.

We have the following two forms of Frobenius reciprocity:

$$\mathrm{Hom}_{\mathfrak{g},T}(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(U), V)=\mathrm{Hom}_{\mathfrak{k},T}(U,V) \tag{3.1}$$

$$\mathrm{Hom}_{\mathfrak{g},T}(V, \mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}(U))=\mathrm{Hom}_{\mathfrak{k},T}(V, U) \tag{3.2}$$

for any $(\mathfrak{g}, T)$-module V; therefore $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}$ is the left adjoint, and $\mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}$ the right adjoint of the forgetful functor from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{k}, T)$.

Let V be a $(\mathfrak{g}, T)$-module. Both $\mathfrak{g}$ and T act on the linear dual V^* of V by the contragredient action. One checks that $V^\vee = V^*_{[T]}$ is also $\mathfrak{g}$-invariant, and in fact a $(\mathfrak{g}, T)$-module. We call $V^\vee$ the contragredient of V. The functor $V \rightsquigarrow V^\vee$ from the category $\mathcal{M}(\mathfrak{g}, T)$ into itself is exact and contravariant. it relates the functors $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}$ and $\mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}$:

3.1. Lemma. *Let U be a $(\mathfrak{k}, T)$-module. Then*

$$(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(U))^\vee = \mathrm{pro}_{\mathfrak{k},T}^{\mathfrak{g},T}(U^\vee).$$

Proof. This is a minor modification of ([10], 5.5.5). □

Now let T be a closed reductive subgroup of another reductive algebraic group K, and $\varphi_T\colon T \to \mathrm{Aut}(\mathfrak{g})$ and $\varphi_K\colon K \to \mathrm{Aut}(\mathfrak{g})$ morphisms of algebraic groups which induce inclusions of Lie algebras, such that $\varphi_K | T = \varphi_T$. We then have the natural forgetful functor $\mathrm{For}\colon \mathcal{M}(\mathfrak{g}, K) \rightsquigarrow \mathcal{M}(\mathfrak{g}, T)$. In his lectures at the Institute for Advanced Study, Princeton, in the fall of 1977, G. Zuckerman introduced a covariant functor $\Gamma_{K,T}$ from the category $\mathcal{M}(\mathfrak{g}, T)$ into the category $\mathcal{M}(\mathfrak{g}, K)$ which is the right adjoint of For (for a detailed discussion of Zuckerman's construction and related results see Chap. 6 of D. Vogan's book [22]). The two forgetful functors $F_T\colon \mathcal{M}(\mathfrak{g}, T) \rightsquigarrow \mathcal{M}(\mathfrak{k}, T)$ and $F_K\colon \mathcal{M}(\mathfrak{g}, K) \rightsquigarrow \mathcal{M}(\mathfrak{k}, K)$ fit into a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(\mathfrak{g}, T) & \xrightarrow{F_T} & \mathcal{M}(\mathfrak{k}, T) \\ \downarrow{\scriptstyle \Gamma_{K,T}} & & \downarrow{\scriptstyle \Gamma_{K,T}} \\ \mathcal{M}(\mathfrak{g}, K) & \xrightarrow{F_K} & \mathcal{M}(\mathfrak{k}, K). \end{array}$$

Moreover, $\Gamma_{K,T}$ is left exact, and its right derived functors $R^i\Gamma_{K,T}$, $0 \leqq i \leqq \dim(K/T)$, satisfy the analogous commutativity property with respect to F_T and F_K. This justifies our use of the same symbol $R^i\Gamma_{K,T}$ in both instances.

The functors $R^i\Gamma_{K,T}: \mathcal{M}(\mathfrak{g}, T) \rightsquigarrow \mathcal{M}(\mathfrak{g}, K)$ play an important role in representation theory. In the remainder of this section, we calculate these functors by means of a particular resolution, for certain T, K and certain modules in $\mathcal{M}(\mathfrak{g}, T)$.

Let $\mathfrak{c}$ be an abelian subalgebra of $\mathfrak{g}$ and T a closed reductive subgroup of K which centralizes $\mathfrak{c}$, with Lie algebra $\mathfrak{t}$ equal to $\mathfrak{c} \cap \mathfrak{k}$; in particular, $\mathfrak{c} \supset \mathfrak{t}$.

3.2. Lemma. *For any* $(\mathfrak{c}, T)$*-module* U, $\mathrm{pro}_{\mathfrak{c},T}^{\mathfrak{g},T}(U)$ *is* $\Gamma_{K,T}$*-acyclic.*

By the previous remark, it suffices to show that it is $(\mathfrak{k}, T)$-injective. Let V be a $(\mathfrak{k}, T)$-module. Then by (3.1) and (3.2)

$$\mathrm{Hom}_{\mathfrak{k},T}(V, \mathrm{pro}_{\mathfrak{c},T}^{\mathfrak{g},T}(U)) = \mathrm{Hom}_{\mathfrak{g},T}(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V), \mathrm{pro}_{\mathfrak{c},T}^{\mathfrak{g},T}(U)) = \mathrm{Hom}_{\mathfrak{c},T}(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V), U).$$

The $(\mathfrak{k}, T)$-injectivity of $\mathrm{pro}_{\mathfrak{c},T}^{\mathfrak{g},T}(U)$ thus reduces to the following statement:

3.3. Lemma. *Let* V *be a* $(\mathfrak{k}, T)$*-module. Then* $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V)$ *is* $(\mathfrak{c}, T)$*-projective.*

Proof. Let $\mathfrak{a}$ be a complement of $\mathfrak{t}$ in $\mathfrak{c}$. Because T is reductive and $\mathfrak{a} \cap \mathfrak{k} = \{0\}$, we can find a T-invariant subspace $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s} \oplus \mathfrak{k}$, as T-module. Let λ be the symmetrization map from the symmetric algebra $S(\mathfrak{g})$ into $\mathcal{U}(\mathfrak{g})$ ([8], Chap. I, 2.6). We can define a linear map

$$\omega: \mathcal{U}(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{s}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{k}) \to \mathcal{U}(\mathfrak{g})$$

by the rule

$$\omega(\xi \otimes \eta \otimes \zeta) = \xi \lambda(\eta) \zeta,$$

for $\xi \in \mathcal{U}(\mathfrak{a})$, $\eta \in S(\mathfrak{s})$ and $\zeta \in \mathcal{U}(\mathfrak{k})$. According to Poincaré-Birkhoff-Witt, this map is an isomorphism of T-modules. It is also an isomorphism of left $\mathcal{U}(\mathfrak{a})$-modules for left multiplication, and right $\mathcal{U}(\mathfrak{k})$-modules for right multiplication. Therefore $\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V) = \mathcal{U}(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{s}) \otimes_{\mathbb{C}} V$ both as an $\mathfrak{a}$- and T-module.

Let W be a $(\mathfrak{c}, T)$-module. Then

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{c},T}(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V), W) &= \mathrm{Hom}_{\mathfrak{a}}(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V), W) \cap \mathrm{Hom}_T(\mathrm{ind}_{\mathfrak{k},T}^{\mathfrak{g},T}(V), W) \\ &= \mathrm{Hom}_T(S(\mathfrak{s}) \otimes_{\mathbb{C}} V, W). \end{aligned}$$

Since T is reductive, the functor $W \rightsquigarrow \mathrm{Hom}_T(S(\mathfrak{s}) \otimes_{\mathbb{C}} V, W)$ is exact. This implies the conclusion of the lemma. □

We now consider a pair $(\mathfrak{g}, K)$ with $\mathfrak{g}$ semisimple. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$, and T the centralizer of $\mathfrak{c}$ in K. Then T is a reductive subgroup of K. The choice of a set of positive roots R^+ in the root system R of $(\mathfrak{g}, \mathfrak{c})$

determines a Borel subalgebra $\mathfrak{b}$, spanned by $\mathfrak{c}$ and the root subspaces corresponding to the positive roots. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. Then T normalizes $\mathfrak{b}$ and $\mathfrak{n}$.

Denote by $N(\mathfrak{n})$ the standard complex of $\mathfrak{n}$,

$$N^{-q}(\mathfrak{n}) = \mathscr{U}(\mathfrak{n}) \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}, \qquad 0 \leqq q \leqq \dim \mathfrak{n},$$

with differential

$$d(\xi \otimes \eta_1 \wedge \ldots \wedge \eta_q) = \sum_{i=1}^{q} (-1)^{i+1} \xi \eta_i \otimes \eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_q$$
$$+ \sum_{1 \leqq i < j \leqq q} (-1)^{i+j} \xi \otimes [\eta_i, \eta_j] \wedge \eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_q,$$

for $\xi \in \mathscr{U}(\mathfrak{n})$ and $\eta_1, \eta_2, \ldots, \eta_q \in \mathfrak{n}$. This is a right resolution of the trival $\mathfrak{n}$-module $\mathbb{C}$ in the category of $\mathfrak{n}$-modules ([9], Chap. XIII, Theorem 7.1). Let U be a $(\mathfrak{b}, T)$-module in which $\mathfrak{n}$ acts trivially. The complex $(N(\mathfrak{n}) \otimes_{\mathbb{C}} U, d \otimes 1)$ is then a left resolution of the $\mathfrak{n}$-module U. Because $\mathfrak{n}$ and U are $(\mathfrak{c}, T)$-modules, the spaces $N(\mathfrak{n}) \otimes_{\mathbb{C}} U$ are $(\mathfrak{c}, T)$-modules and, as can be checked, the differentials commute with the $(\mathfrak{c}, T)$-action. Therefore $N(\mathfrak{n}) \otimes_{\mathbb{C}} U$ is a complex of $(\mathfrak{c}, T)$-modules. For each q, $0 \leqq q \leqq \dim \mathfrak{n}$, there is a natural map from $N^q(\mathfrak{n}) \otimes_{\mathbb{C}} U$ into $\mathscr{U}(\mathfrak{b}) \otimes_{\mathscr{U}(\mathfrak{c})} (\wedge^q \mathfrak{n} \otimes_{\mathbb{C}} U)$, and by the Poincaré-Birkhoff-Witt theorem this map is an isomorphism of vector spaces. Also, the $\mathfrak{c}$-action on $N(\mathfrak{n}) \otimes_{\mathbb{C}} U$ corresponds to the action by left multiplication on the first factor of $\mathscr{U}(\mathfrak{b}) \otimes_{\mathscr{U}(\mathfrak{c})} (\wedge \mathfrak{n} \otimes_{\mathbb{C}} U)$. Similarly, this isomorphism preserves the T-module structures. Thus we can interpret our complex as the complex $\mathrm{ind}_{\mathfrak{c}, T}^{\mathfrak{b}, T}(\wedge \mathfrak{n} \otimes_{\mathbb{C}} U)$, with differential $d \otimes 1$. The differential commutes with the $(\mathfrak{b}, T)$-action. We have produced a left resolution of the $(\mathfrak{b}, T)$-module U. Tensoring from the left by $\mathscr{U}(\mathfrak{g})$, considered as right $\mathscr{U}(\mathfrak{b})$-module, we obtain a left resolution of $\mathrm{ind}_{\mathfrak{b}, T}^{\mathfrak{g}, T}(U)$ in $\mathscr{M}(\mathfrak{g}, T)$ by the complex $\mathrm{ind}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(\wedge \mathfrak{n} \otimes_{\mathbb{C}} U)$; here we have once more used the Poincaré-Birkhoff-Witt Theorem. According to 3.1, the contragredient of our complex $\mathrm{ind}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(\wedge \mathfrak{n} \otimes_{\mathbb{C}} U)$ is $\mathrm{pro}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(\wedge \mathfrak{n}^* \otimes_{\mathbb{C}} U^\vee)$, which resolves $\mathrm{pro}_{\mathfrak{b}, T}^{\mathfrak{g}, T}(U^\vee)$ on the right.

This proves the following result (compare [11], (5.2)).

3.4. Lemma. *Let U be a finite-dimensional $(\mathfrak{b}, T)$-module, in which $\mathfrak{n}$ acts trivially. There is a right resolution of $\mathrm{pro}_{\mathfrak{b}, T}^{\mathfrak{g}, T}(U)$ in $\mathscr{M}(\mathfrak{g}, T)$,*

$$0 \to \mathrm{pro}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(U) \to \mathrm{pro}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(\mathfrak{n}^* \otimes_{\mathbb{C}} U) \to \ldots \to \mathrm{pro}_{\mathfrak{c}, T}^{\mathfrak{g}, T}(\wedge^q \mathfrak{n}^* \otimes_{\mathbb{C}} U) \to \ldots,$$

with differential

$$dR(\xi)(\eta_1 \wedge \eta_2 \wedge \ldots \wedge \eta_q \otimes f) = \sum_{i=1}^{q} (-1)^i R(\eta_i \xi)(\eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_q \otimes f)$$
$$+ \sum_{1 \leqq i < j \leqq q} (-1)^{i+j} R(\xi)([\eta_i, \eta_j] \wedge \eta_1 \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_q \otimes f),$$

for $\xi \in \mathscr{U}(\mathfrak{g})$, $\eta_1, \eta_2, \ldots, \eta_q \in \mathfrak{n}$ and $f \in U^\vee$.

Any finite-dimensional $(\mathfrak{c}, T)$-module U can be interpreted as $(\mathfrak{b}, T)$-module with trivial $\mathfrak{n}$-action. The $(\mathfrak{g}, K)$-module

$$I^q(\mathfrak{c}, R^+, U) = (R^q \Gamma_{K,T})(\mathrm{pro}_{\mathfrak{b},T}^{\mathfrak{g},T}(U)),$$

is the q^{th}-*standard Zuckerman module* for the data $(\mathfrak{c}, R^+, U)$.

Combining 3.2 and 3.4, we get:

3.5. Proposition. *The standard Zuckerman modules* $I^q(\mathfrak{c}, R^+, U)$, $0 \leqq q \leqq \dim(K/T)$, *are the cohomology modules of the complex*

$$\Gamma_{K,T}(\mathrm{pro}_{\mathfrak{c},T}^{\mathfrak{g},T}(\wedge \mathfrak{n}^* \otimes_{\mathbb{C}} U)).$$

We return to the setting of Sect. 2, in particular we again assume that K is connected. Let y be the point in the variety of ordered Cartan subalgebras Y determined by the data $(\mathfrak{c}, R^+)$, and $\tilde{Q}$ its K-orbit. The triple $(\mathfrak{c}, R, R^+)$ is a specialization of $(\mathfrak{h}, \Sigma, \Sigma^+)$. For $\mu \in \mathfrak{h}^*$, we denote by $\mathbb{C}_\mu$ the one-dimensional $\mathfrak{c}$-module determined by the specialization of the linear form μ to $\mathfrak{c}$. We consider a finite dimensional $(\mathfrak{c}, T)$-module U which is, as $\mathfrak{c}$-module, a direct sum of copies of $\mathbb{C}_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Let $\mathscr{U}$ be the K-homogeneous locally free $\mathcal{O}_{\tilde{Q}}$-module with geometric fibre U at y. As explained in Appendix A, the linear form λ on $\mathfrak{h}$ determines a homogeneous twisted sheaf of differential operators $\mathscr{D}_{Y,\lambda}$ on Y, and $\mathscr{D}_{\mathscr{U}} = (\mathscr{D}_{Y,\lambda})^j$ is the K-homogeneous twisted sheaf of differential operators on $\tilde{Q}$ which corresponds to the differential of the K-action on $\mathscr{U}$. At the end of Sect. 2, we introduced a complex $R_{\tilde{Q} \to Y|X,\lambda}$ of left $\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{U}})$- and right $\mathscr{U}(\mathfrak{g})$-modules. Therefore we can consider the complex

$$\mathrm{Hom}_{\Gamma(\tilde{Q},\mathscr{D}_{\mathscr{U}})}(R_{\tilde{Q} \to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathscr{U}))$$

of left $\mathscr{U}(\mathfrak{g})$-modules. Since $\tilde{Q}$ is an affine variety, localization defines an equivalence of the category of $\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{U}})$-modules with the category $\mathscr{M}(\mathscr{D}_{\mathscr{U}})$. In addition, if we denote by $i_y\colon \{y\} \to Y$ the natural injection, we have the functor i_y^+ from the category $\mathscr{M}(\mathscr{D}_{\mathscr{U}})$ into the category of vector spaces, which associates to a $\mathscr{D}_{\mathscr{U}}$-module $\mathscr{F}$ its geometric fibre $T_y(\mathscr{F})$ at y. The composition of these two functors defines a right exact functor ψ_y from the category of $\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{U}})$-modules into the category of vector spaces.

Now we calculate the action of this functor on our complex. We start with a discussion of the action of ψ_y on $R_{\tilde{Q} \to Y|X,\mu}$. The localization of $R_{\tilde{Q} \to Y|X,\mu}$ is equal to $\mathscr{D}_{\tilde{Q} \to Y,\mu} \otimes_{j^{-1}(\mathcal{O}_Y)} j^{-1}(\wedge \mathscr{T}_{Y|X})$. Therefore, in the notation of A.3.3,

$$\begin{aligned}\psi_y(R_{\tilde{Q} \to Y|X,\mu}) &= i_y^+(\mathscr{D}_{\tilde{Q} \to Y,y}) \otimes_{\mathcal{O}_{Y,y}} \wedge \mathscr{T}_{Y|X,y} \\ &= \mathscr{D}_{\{y\} \to Y,\mu} \otimes_{\mathcal{O}_{Y,y}} \wedge \mathscr{T}_{Y|X,y} = T_y(C_{Y|X}(\mathscr{D}_{Y,\mu})).\end{aligned}$$

Hence $\psi_y(R_{\tilde{Q} \to Y|X,\mu})$ is a left resolution of the right $\mathscr{U}(\mathfrak{g})$-module

$$T_y(\mathscr{D}_{Y \to X,\mu}) = (p \circ i_y)^*(\mathscr{D}_{X,\mu}) = T_x(\mathscr{D}_{X,\mu}) = \mathbb{C}_{-\mu} \otimes_{\mathscr{U}(\mathfrak{b})} \mathscr{U}(\mathfrak{g}).$$

In addition, we have the following result:

3.6. Lemma. *Let* $\mu \in \mathfrak{h}^*$. *Then*

$$\psi_y(R\bar{Q}^{q}_{\to Y|X,\mu}) = (\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g})$$

as right $\mathscr{U}(\mathfrak{g})$*-module.*

Proof. As we remarked above

$$\psi_y(R\bar{Q}^{q}_{\to Y|X,\mu}) = \mathscr{D}_{\{y\}\to Y,\mu} \otimes_{\mathscr{O}_{Y,y}} \wedge^q \mathscr{T}_{Y|X,y},$$

which contains

$$\mathbb{C} \otimes_{\mathscr{O}_{Y,y}} \wedge^q \mathscr{T}_{Y|X,y} = \wedge^q T_{Y|X,y}(Y) = \wedge^q \mathfrak{n}$$

as a linear subspace (here we denoted by $T_{Y|X,y}(Y)$ the kernel of the differential of the projection $p\colon Y \to X$). Using this natural identification, we can define a bilinear map $\varphi\colon \wedge^q \mathfrak{n} \times \mathscr{U}(\mathfrak{g}) \to \psi_y(R\bar{Q}^{q}_{\to Y|X,\lambda})$ by

$$\varphi(s, \zeta) = (1 \otimes s)\cdot\zeta, \qquad s \in \wedge^q \mathfrak{n},\ \zeta \in \mathscr{U}(\mathfrak{g}).$$

For $\xi \in \mathfrak{c}$, $s \in \wedge^q \mathfrak{n}$ and $\zeta \in \mathscr{U}(\mathfrak{g})$, we have

$$\begin{aligned}\varphi(s, \xi\zeta) &= (1 \otimes s)\cdot\xi\zeta = ((1 \otimes s)\cdot\xi)\cdot\zeta \\ &= (\mu(\xi)(1 \otimes s) - 1 \otimes (\mathrm{ad}\,\xi)\,s)\cdot\zeta = \varphi(\mu(\xi)\,s + s\cdot\xi, \zeta).\end{aligned}$$

Hence φ defines a linear map from $(\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g})$, which we denote by the same symbol. This map is a homomorphism of right $\mathscr{U}(\mathfrak{g})$-modules. The filtration of $\mathscr{D}_{\{y\}\to Y,\mu}$, by degree of differential operators, induces a filtration on $\psi_y(R\bar{Q}^{q}_{\to Y|X,\lambda})$. If we denote by $T_y(Y)$ the tangent space of Y at y, we have

$$Gr\,\psi_y(R\bar{Q}^{q}_{\to Y|X,\lambda}) = S(T_y(Y)) \otimes_{\mathbb{C}} \wedge^q T_{Y|X,y}(Y).$$

We equip $\mathscr{U}(\mathfrak{g})$ with a filtration $F\mathscr{U}(\mathfrak{g})$ such that $F_r\mathscr{U}(\mathfrak{g})$, $r \in \mathbb{Z}_+$, is the left $\mathscr{U}(\mathfrak{c})$-submodule of $\mathscr{U}(\mathfrak{g})$ generated by $\mathscr{U}_r(\mathfrak{g})$ ($= r^{\text{th}}$ subspace of $\mathscr{U}(\mathfrak{g})$ with respect to the standard filtration). By the Poincaré-Brikhoff-Witt theorem, this induces a filtration on $(\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g})$ with

$$Gr(\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}) = (\mathbb{C}_{-\mu} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathbb{C}} S(\mathfrak{g}/\mathfrak{c}).$$

Then φ is compatible with these filtrations and induces the natural linear isomorphism of the graded spaces. The lemma follows. □

According to the preceding discussion, the functor ψ_y induces a morphism of the complex $\mathrm{Hom}_{\Gamma(\tilde{Q},\mathscr{D}_{\mathscr{U}})}(R_{\tilde{Q}\to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathscr{U}))$ into the complex $\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}), U)$. Since the action of T on $(\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g})$ is algebraic, the complex of $(\mathfrak{g}, T)$-modules

$$\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}), U)_{[T]}$$

is a right resolution of $\mathrm{pro}^{\mathfrak{g},T}_{\mathfrak{b},T}(U)$. In addition,

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}), U)_{[T]} &= \mathrm{Hom}_{\mathfrak{c}}(\mathscr{U}(\mathfrak{g}), \wedge \mathfrak{n}^* \otimes_{\mathbb{C}} U)_{[T]} \\ &= \mathrm{pro}^{\mathfrak{g},T}_{\mathfrak{c},T}(\wedge \mathfrak{n}^* \otimes_{\mathbb{C}} U),\end{aligned}$$

and the terms in our resolution are equal to those of the resolution introduced in 3.4. One can check that the differentials agree up to sign, but this is not necessary for our purposes.

To sum up, ψ_y defines a morphism of complexes of $(\mathfrak{g}, K)$-modules

$$\Psi: \mathrm{Hom}_{\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})}(R_{\tilde{Q}\to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathcal{U}))_{[K]} \to \Gamma_{K,T}(\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge \mathfrak{n}) \otimes_{\mathcal{U}(\mathfrak{c})} \mathcal{U}(\mathfrak{g}), U)_{[T]}),$$

and the cohomology of the latter complex calculates standard Zuckerman modules.

3.7. Lemma. *The morphism Ψ is an isomorphism.*

From Appendix A we know that $\mathcal{D}_{\tilde{Q}\to Y,\lambda}$ has a natural filtration according to normal degree. The filtrants $F_r \mathcal{D}_{\tilde{Q}\to Y,\lambda}$, $r\in\mathbb{Z}_+$, are K-homogeneous and invariant under the right $\mathfrak{k}$-action, and

$$Gr\,\mathcal{D}_{\tilde{Q}\to Y,\lambda} = \mathcal{D}_{\mathcal{U}} \otimes_{\mathcal{O}_{\tilde{Q}}} S(\mathcal{N}_{Y|\tilde{Q}}).$$

Since $\tilde{Q}$ is affine, this filtration induces a filtration $F_r \Gamma(\tilde{Q}, \mathcal{D}_{\tilde{Q}\to Y,\lambda})$, $r\in\mathbb{Z}_+$, on the global sections of $\mathcal{D}_{\tilde{Q}\to Y,\lambda}$. We note that

$$\begin{aligned} Gr_r \Gamma(\tilde{Q}, \mathcal{D}_{\tilde{Q}\to Y,\lambda}) &= \Gamma(\tilde{Q}, Gr_r \mathcal{D}_{\tilde{Q}\to Y,\lambda}) \\ &= \Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}} \otimes_{\mathcal{O}_{\tilde{Q}}} S^r(\mathcal{N}_{Y|\tilde{Q}})) \\ &= \Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}}) \otimes_{\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})} \Gamma(\tilde{Q}, S^r(\mathcal{N}_{Y|\tilde{Q}})). \end{aligned}$$

For any $q\in\mathbb{Z}_+$, $\wedge^q \mathcal{T}_{Y|X}$ is a locally free $\mathcal{O}_Y$-module of finite rank, hence $\Gamma(Y, \wedge^q \mathcal{T}_{Y|X})$ is a projective $\Gamma(Y, \mathcal{O}_Y)$-module. It follows that

$$F_r R^{-q}_{\tilde{Q}\to Y|X,\lambda} = F_r \Gamma(\tilde{Q}, \mathcal{D}_{\tilde{Q}\to Y,\lambda}) \otimes_{\Gamma(Y,\mathcal{O}_Y)} \Gamma(Y, \wedge^q \mathcal{T}_{Y|X}),$$

for $r\in\mathbb{Z}_+$, defines a filtration of $R^{-q}_{\tilde{Q}\to Y|X,\lambda}$ by left $\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})$- and right $\mathcal{U}(\mathfrak{k})$-modules, with

$$Gr_r R^{-q}_{\tilde{Q}\to Y|X,\lambda} = \Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}}) \otimes_{\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})} \Gamma(\tilde{Q}, S^r(\mathcal{N}_{Y|\tilde{Q}}) \otimes_{\mathcal{O}_{\tilde{Q}}} \wedge^q j^*(\mathcal{T}_{Y|X})).$$

Again, $S^r(\mathcal{N}_{Y|\tilde{Q}}) \otimes_{\mathcal{O}_{\tilde{Q}}} \wedge^q j^*(\mathcal{T}_{Y|X})$ is a locally free $\mathcal{O}_{\tilde{Q}}$-module of finite rank, hence $\Gamma(\tilde{Q}, S^r(\mathcal{N}_{Y|\tilde{Q}}) \otimes_{\mathcal{O}_{\tilde{Q}}} \wedge^q j^*(\mathcal{T}_{Y|X}))$ is a projective $\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})$-module and $Gr_r R^{-q}_{\tilde{Q}\to Y|X,\lambda}$ is a projective $\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})$-module. By restriction

$$V_q = \mathrm{Hom}_{\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})}(R^{-q}_{\tilde{Q}\to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathcal{U}))_{[K]}$$

maps into

$$V_{q,r} = \mathrm{Hom}_{\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})}(F_r R^{-q}_{\tilde{Q}\to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathcal{U})).$$

According to the preceding discussion, there is exact sequence

$$0 \to \mathrm{Hom}_{\Gamma(\tilde{Q}, \mathcal{D}_{\mathcal{U}})}(\mathrm{Gr}_r R^{-q}_{\tilde{Q}\to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathcal{U})) \to V_{q,r} \to V_{q,r-1} \to 0.$$

In particular, restriction from $V_{q,r}$ into $V_{q,r-1}$ is surjective. Also,

$$\begin{aligned}&\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\mathcal{U}})}(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}, \Gamma(\tilde{Q},\mathcal{U}))\\&=\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{O}_{\tilde{Q}})}(\Gamma(\tilde{Q}, S^r(\mathcal{N}_{Y|\tilde{Q}})\otimes_{\mathcal{O}_{\tilde{Q}}}\wedge^q j^*(\mathcal{T}_{Y|X})), \Gamma(\tilde{Q},\mathcal{U}))\\&=\Gamma(\tilde{Q}, \mathcal{H}om_{\mathcal{O}_{\tilde{Q}}}(S^r(\mathcal{N}_{Y|\tilde{Q}})\otimes_{\mathcal{O}_{\tilde{Q}}}\wedge^q j^*(\mathcal{T}_{Y|X}), \mathcal{U}))\\&=\Gamma(\tilde{Q}, (S^r(\mathcal{N}_{Y|\tilde{Q}})\otimes_{\mathcal{O}_{\tilde{Q}}}\wedge^q j^*(\mathcal{T}_{Y|X}))^\vee\otimes_{\mathcal{O}_{\tilde{Q}}}\mathcal{U}),\end{aligned}$$

where $\mathcal{L}^\vee$ denotes the dual of a locally free $\mathcal{O}_{\tilde{Q}}$-module $\mathcal{L}$. Hence $\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\mathcal{U}})}(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}, \Gamma(\tilde{Q},\mathcal{U}))$ is the module of global sections of a K-homogeneous $\mathcal{O}_{\tilde{Q}}$-module and the K-action is algebraic. By induction on r we see that each $V_{q,r}$ is an algebraic K-module. The modules $\{V_{q,r} \mid r\in\mathbb{Z}_+\}$ form a projective system in the category of algebraic K-modules, and V_q is its limit.

Since $F_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}$ and $Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}$ are the spaces of global sections of K-homogeneous $\mathcal{O}_{\tilde{Q}}$-modules, the functor ψ_y induces an exact sequence

$$0\to\psi_y(F_{r-1}R_{\tilde{Q}\to Y|X,\lambda}^{-q})\to\psi_y(F_r R_{\tilde{Q}\to Y|X,\lambda}^{-q})\to\psi_y(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q})\to 0$$

of algebraic T-modules. We put

$$U_{q,r}=\mathrm{Hom}_{\mathbb{C}}(\psi_y(F_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}), U).$$

Since T is reductive, there is exact sequence

$$0\to\mathrm{Hom}_{\mathbb{C}}(\psi_y(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}), U)_{[T]}\to(U_{q,r})_{[T]}\to(U_{q,r-1})_{[T]}\to 0$$

of $(\mathfrak{k}, T)$-modules. We argue as in the proof of 3.7, to show that

$$\psi_y(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q})=(\mathbb{C}_{-\lambda}\otimes_{\mathbb{C}}S^r(\mathfrak{g}/(\mathfrak{k}+\mathfrak{c}))\otimes_{\mathbb{C}}\wedge^q\mathfrak{n})\otimes_{\mathcal{U}(\mathfrak{t})}\mathcal{U}(\mathfrak{k});$$

here we view $S^r(\mathfrak{g}/(\mathfrak{k}+\mathfrak{c}))$ as the geometric fibre of $S^r(\mathcal{N}_{Y|\tilde{Q}})$ at y. This implies

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(\psi_y(Gr_r R_{\tilde{Q}\to Y|X,\lambda}^{-q}), U)_{[T]}&=\mathrm{Hom}_{\mathfrak{t}}(\mathcal{U}(\mathfrak{k}), (S^r(\mathfrak{g}/(\mathfrak{k}+\mathfrak{c}))\otimes_{\mathbb{C}}\wedge^q\mathfrak{n})^*\otimes_{\mathbb{C}}U)_{[T]}\\&=\mathrm{pro}_{\mathfrak{t},T}^{\mathfrak{k},T}((S^r(\mathfrak{g}/(\mathfrak{k}+\mathfrak{c}))\otimes_{\mathbb{C}}\wedge^q\mathfrak{n})^*\otimes_{\mathbb{C}}U).\end{aligned}$$

Let V be an algebraic T-module and $\mathcal{V}$ the corresponding K-homogeneous $\mathcal{O}_{\tilde{Q}}$-module on $\tilde{Q}$. We can interpret the $(\mathfrak{t}, T)$-module

$$\hat{\mathcal{V}}_y=\mathrm{Hom}_{\mathfrak{t}}(U(\mathfrak{k}), V)_{[T]}=\mathrm{pro}_{\mathfrak{t},T}^{\mathfrak{k},T}(V),$$

geometrically as a formal completion of $\Gamma(\tilde{Q},\mathcal{V})$ at y.

3.8. Lemma. (i) *The homomorphism $\Gamma(\tilde{Q},\mathcal{V})\to\hat{\mathcal{V}}_y$ induces an isomorphism of K-modules*

$$\Gamma(\tilde{Q},\mathcal{V})=\Gamma_{K,T}(\hat{\mathcal{V}}_y).$$

(ii) $R^i\Gamma_{K,T}(\hat{\mathcal{V}}_y)=0$ *for* $i>0$.

Proof. (i) The canonical homomorphism we described is injective. On the other hand, from Frobenius reciprocity and its algebraic version (3.2) one sees that the K-multiplicities in $\Gamma(\tilde{Q},\mathcal{V})$ and $\Gamma_{K,T}(\hat{\mathcal{V}}_y)$ are finite and equal.

(ii) Since $\hat{\mathcal{V}}_y=\mathrm{pro}_{\mathfrak{t},T}^{\mathfrak{k},T}(V)$ is injective in $\mathcal{M}(\mathfrak{k}, T)$, the assertion is evident. □

There is a commutative diagram of K-modules,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\tilde{Q}, \mathscr{F}^{q,r}) & \longrightarrow & V_{q,r} & \longrightarrow & V_{q,r-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \Psi^q_r & & \downarrow \Psi^q_{r-1} & & \\ 0 & \longrightarrow & \Gamma_{K,T}((\mathscr{F}^{q,r})^\wedge_y) & \longrightarrow & \Gamma_{K,T}((U_{q,r})_{[T]}) & \longrightarrow & \Gamma_{K,T}((U_{q,r-1})_{[T]}) & \longrightarrow & 0, \end{array}$$

with

$$\mathscr{F}^{q,r} = (S^r(\mathscr{N}_{Y|\tilde{Q}}) \otimes_{\mathcal{O}_{\tilde{Q}}} \wedge^q j^*(\mathscr{T}_{Y|X}))^\vee \otimes_{\mathcal{O}_{\tilde{Q}}} \mathscr{U}.$$

We already remarked that the first row is exact. By 3.8 (ii) the second row is also exact. The first vertical arrow is an isomorphism by 3.8 (i), and the second and third are the morphisms induced by the functor ψ_y. By induction on r we see that the Ψ^q_r, $r \in \mathbb{Z}_+$, are isomorphisms. We can therefore identify the projective systems $\{V_{q,r} \mid r \in \mathbb{Z}_+\}$ and $\{\Gamma_{K,T}((U_{q,r})_{[T]}) \mid r \in \mathbb{Z}_+\}$. This gives a projective family of morphisms

$$\Phi^q_r \colon \Gamma_{K,T}(\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}), U)_{[T]}) \to V_{q,r},$$

which factor through V_q and induce

$$\Phi^q \colon \Gamma_{K,T}(\mathrm{Hom}_{\mathbb{C}}((\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \wedge^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g}), U)_{[T]}) \to V_q.$$

It follows that $\Phi^q \circ \Psi^q = 1$, and Φ^q is surjective. On the other hand, any element of the kernel of Φ^q has trivial restrictions to all filtrants $\psi_y(F_r R^{-q}_{\tilde{Q} \to Y|X,\lambda})$, $r \in \mathbb{Z}_+$, of $(\mathbb{C}_{-\lambda} \otimes_{\mathbb{C}} \Lambda^q \mathfrak{n}) \otimes_{\mathscr{U}(\mathfrak{c})} \mathscr{U}(\mathfrak{g})$, hence is equal to zero. This implies that Φ^q is also injective and ends the proof of 3.7.

Finally, 3.7 leads to the following "geometric" version of 3.5.

3.9. Proposition. *Fix $\lambda \in \mathfrak{h}^*$ and let U be a finite-dimensional $(\mathfrak{c}, T)$-module which, as $\mathfrak{c}$-module, is a direct sum of copies of $\mathbb{C}_\lambda$. Let $\mathscr{U}$ be the coherent $\mathcal{O}_{\tilde{Q}}$-module of local sections of the homogeneous vector bundle on $\tilde{Q}$ determined by U. Then the standard Zuckerman modules $I^q(\mathfrak{c}, R^+, U)$, $0 \leq q \leq \dim(K/T)$, are the cohomology modules of the complex*

$$\mathrm{Hom}_{\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{U}})}(R_{\tilde{Q} \to Y|X,\lambda}, \Gamma(\tilde{Q}, \mathscr{U}))_{[K]}.$$

§ 4. The duality theorem

Now we can formulate the results which relate localization to the Zuckerman construction.

In the following $\mathfrak{g}$ will be a complex semisimple Lie algebra, K a connected complex linear algebraic group and $\varphi \colon K \to \mathrm{Aut}(\mathfrak{g})$ a morphism of algebraic groups such that the differential of φ is an injection of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{g}$. We shall identify the Lie algebra $\mathfrak{k}$ with its image in $\mathfrak{g}$. We assume, in addition, that the pair $(\mathfrak{g}, K)$ satisfies the following condition:

(D) The Lie algebra $\mathfrak{k}$ is the fixed point set of an involutive automorphism σ of $\mathfrak{g}$.

Under this condition we can describe the cohomology of standard Harish-Chandra sheaves using standard Zuckerman modules. First we recall a few simple consequences. The algebra $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Therefore, the group K is reductive and the pair $(\mathfrak{g}, K)$ satisfies the assumptions of Sect. 3. In addition, it will be crucial for our purposes to know that K has finitely many orbits on the flag variety X and that these orbits are affinely imbedded; therefore $(\mathfrak{g}, K)$ satisfies also the conditions which were imposed in Sect. 2. The finiteness of the number of orbits is a consequence of a result of Wolf [24], and the affinity of imbeddings is due to Beilinson and Bernstein. We are indebted to Beilinson and Bernstein for communicating their argument to us, which also implies the finiteness statement.

4.1. Proposition (Beilinson, Bernstein). *The group K acts on X with finitely many orbits, and these orbits are affinely imbedded in X.*

We denote the involutive automorphism of G with differential σ by the same letter.

The key step in the proof is the following lemma. First, define an action of G on $X \times X$ by

$$g(x, y) = (gx, \sigma(g)\, y)$$

for $g \in G$, $x, y \in X$.

4.2. Lemma. *The group G acts on $X \times X$ with finitely many orbits. These orbits are affinely imbedded in $X \times X$.*

We claim that 4.1 is a consequence of the lemma. Let Δ be the diagonal in $X \times X$. The lemma implies that the orbit stratification of $X \times X$ induces a stratification of Δ by finitely many irreducible, affinely imbedded subvarieties which are the irreducible components of the intersections of the G-orbits with Δ. These strata are K-invariant, and therefore unions of K-orbits. Let V be one of these subvarieties, $(x, x) \in V$ and Q the K-orbit of (x, x). If we let $\mathfrak{b}_x$ denote the Borel subalgebra of $\mathfrak{g}$ corresponding to x, the tangent space $T_x(X)$ of X at x can be identified with $\mathfrak{g}/\mathfrak{b}_x$. Let p_x be the projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{b}_x$. The tangent space $T_{(x,x)}(X \times X)$ to $X \times X$ at (x, x) can be identified with $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. If the orbit map $f\colon G \to X \times X$ is defined by $f(g) = g(x, x)$, its differential at the identity in G is the linear map $\xi \to (p_x(\xi), p_x(\sigma(\xi)))$ of $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{b}_x \times \mathfrak{g}/\mathfrak{b}_x$. Then the tangent space to V at (x, x) is contained in the intersection of the image of this differential with the diagonal in the tangent space $T_{(x,x)}(X \times X)$, i.e.

$$\begin{aligned} T_{(x,x)}(V) &\subset \{(p_x(\xi), p_x(\xi)) \mid \xi \in \mathfrak{g} \quad \text{such that} \quad p_x(\xi) = p_x(\sigma(\xi))\} \\ &= \{(p_x(\xi), p_x(\xi)) \mid \xi \in \mathfrak{k}\} = T_{(x,x)}(Q). \end{aligned}$$

Consequently the tangent space to V at (x, x) agrees with the tangent space to Q, and Q is open in V. By the irreducibility of V, this implies that V is a K-orbit, and therefore our stratification of the diagonal Δ is the stratification induced via the diagonal map by the K-orbit stratification of X. Proposition 4.1 follows.

To prove 4.2, we fix a point $v \in X$. Let B_v be the Borel subgroup corresponding to v, and put $B = \sigma(B_v)$. Every G-orbit in $X \times X$ intersects $X \times \{v\}$. Let $u \in X$.

Then the intersection of the G-orbit Q through (u, v) with $X \times \{v\}$ is equal to $Bu \times \{v\}$. Because of the Bruhat decomposition ([4], Chap. IV., 14.11), this implies the finiteness of the number of G-orbits in $X \times X$.

To show that the orbit Q is affinely imbedded, we remark first that the Bruhat cell Bu in X is an affine variety ([4], Chap. IV., 14.11). Let $\bar{N}$ be the unipotent radical of a Borel subgroup opposite to B. Then $\sigma(\bar{N})v$ is an open neighborhood of v in X, and the map $\bar{n} \to \sigma(\bar{n})v$ is an isomorphism of $\bar{N}$ onto this neighborhood. The intersection of Q with $X \times \sigma(\bar{N})v$ is equal to the image of the affine variety $Bu \times \bar{N}$ under the map $(x, \bar{n}) \to \bar{n}(x, v)$, which is an immersion. Therefore this set is affine. It follows that we can construct an open cover $(U_i, 1 \leqq i \leqq n)$ of $X \times X$ such that the intersection of Q with each U_i is affine. Since affinity of a morphism is a local property with respect to the target variety, this ends the proof of 4.2.

From 4.1 we see that the pair $(\mathfrak{g}, K)$ satisfies the assumptions of Sect. 2, i.e. it is a Harish-Chandra pair. Moreover, the orbits of K in X are affinely imbedded, hence the spectral sequence 2.6 collapses. Thus we can calculate the cohomology of standard Harish-Chandra sheaves from the complex we described there.

Let $\mathscr{I}(Q, \tau) \in \mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$ be a standard Harish-Chandra sheaf. We fix a standard orbit $\tilde{Q}$ lying above Q, and a point $y \in \tilde{Q}$ lying above $x \in Q$. Then y corresponds to a Cartan subalgebra $\mathfrak{h}_y$ and a system of positive roots Σ_y^+ in $\mathfrak{h}_y^*$. The connection τ on Q corresponds to a connection $\tilde{\tau}$ on $\tilde{Q}$ given by an irreducible finite-dimensional algebraic representation U of $(\mathfrak{h}_y, T_y)$. As $\mathfrak{h}_y$-module U is a direct sum of copies of a linear form on $\mathfrak{h}_y$ which is the specialization of $\lambda + \rho \in \mathfrak{h}^*$. Set $n = \dim X$ and $s = \dim(\mathfrak{k} \cap \mathfrak{n}_x)$. Let Ω_X denote the invertible $\mathcal{O}_X$-module of differential n-forms on X. The geometric fibre $T_x(\Omega_X)$ at x is a one-dimensional $(\mathfrak{h}_y, T_y)$-module, isomorphic to $\wedge^n \mathfrak{n}_x$. The "dual" set of data $(\mathfrak{h}_y, \Sigma_y^+, U^\vee \otimes_{\mathbb{C}} T_x(\Omega_X))$ determines a family of standard Zuckerman modules $I^q(\mathfrak{h}_y, \Sigma_y^+, U^\vee \otimes_{\mathbb{C}} T_x(\Omega_X))$, $0 \leqq q \leqq \dim(K/T)$, as explained in Sect. 3.

We can now state our main result.

4.3. Theorem.

$$H^q(X, \mathscr{I}(Q, \tau))^\vee = I^{s-q}(\mathfrak{h}_y, \Sigma_y^+, U^\vee \otimes_{\mathbb{C}} T_x(\Omega_X)),$$

for all $q \in \mathbb{Z}$.

The proof of 4.3 depends on a simpler duality statement which we discuss first. Let $\mathscr{L}$ be the sheaf of local sections of a homogeneous vector bundle on $\tilde{Q}$, corresponding to a finite-dimensional representation L of T_y. Denote by $\mathscr{L}^\vee$ the sheaf associated to the representation $L^\vee$ of T_y. Then $\mathscr{L} \otimes_{\mathcal{O}_{\tilde{Q}}} \mathscr{L}^\vee$ is the sheaf associated to the representation $L \otimes_{\mathbb{C}} L^\vee$ of T_y. The natural pairing between L and $L^\vee$ induces a $\mathcal{O}_{\tilde{Q}}$-module morphism of $\mathscr{L} \otimes_{\mathcal{O}_{\tilde{Q}}} \mathscr{L}^\vee$ onto $\mathcal{O}_{\tilde{Q}}$, which is compatible with the action of K. Passing to global sections, we obtain a surjective morphism of algebraic K-modules

$$\Gamma(\tilde{Q}, \mathscr{L}) \otimes_{\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})} \Gamma(\tilde{Q}, \mathscr{L}^\vee) \to \Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}}).$$

By differentiation, this is a morphism of $\mathscr{U}(\mathfrak{k})$-modules. Let η be the projection map from the K-module $\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})$ onto its K-invariants, i.e. the constants. Then

η induces a $\mathscr{U}(\mathfrak{k})$-module homomorphism of $\Gamma(\tilde{Q}, \mathscr{L}) \otimes_{\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})} \Gamma(\tilde{Q}, \mathscr{L}^\vee)$ into $\mathbb{C}$. We can view $\mathscr{L}$ as a left $\mathscr{D}_{\mathscr{L}}$-module, and as a right $(\mathscr{D}_{\mathscr{L}})^0$-module. Since the representation of T_y on the top exterior power of the cotangent space of $\tilde{Q}$ at y is trivial, the opposite sheaf of rings $(\mathscr{D}_{\mathscr{L}})^0$ of $\mathscr{D}_{\mathscr{L}}$ is isomorphic to $\mathscr{D}_{\mathscr{L}^\vee}$ by A.2. Moreover, this isomorphism is compatible with the principal antiautomorphism of $\mathscr{U}(\mathfrak{k})$. On the other hand, $\tilde{Q}$ is affine, so $\Gamma(\tilde{Q}, \mathcal{O}_{\tilde{Q}})$ and the differential operators induced by the action of $\mathscr{U}(\mathfrak{k})$ generate the global sections of the sheaf of differential operators on $\mathscr{L}$, resp. $\mathscr{L}^\vee$. Hence, we can form the tensor product $\Gamma(\tilde{Q}, \mathscr{L}) \otimes_{\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{L}^\vee})} \Gamma(\tilde{Q}, \mathscr{L}^\vee)$ and η factors through it. Therefore we can view η as a $\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{L}^\vee})$-invariant pairing on $\Gamma(\tilde{Q}, \mathscr{L}) \times \Gamma(\tilde{Q}, \mathscr{L}^\vee)$.

Denote by $R(K)$ the ring of regular functions on K. By the algebraic version of the Peter-Weyl theorem, the pairing on $R(K) \times R(K)$ defined as multiplication of functions followed by the K-equivariant projection onto the constants is nondegenerate. Moreover, the global sections of $\mathscr{L}$ can be identified with the T_y-invariants $(R(K) \otimes_{\mathbb{C}} L)^{T_y}$ of the module $R(K) \otimes_{\mathbb{C}} L$ where T_y acts on $R(K)$ by right translations and K by left translations, and analogously the global sections of $\mathscr{L}^\vee$ can be identified with $(R(K) \otimes_{\mathbb{C}} L^\vee)^{T_y}$. Hence, our pairing is induced by the pairing on $R(K) \times R(K)$ we just described and the natural pairing of L with $L^\vee$. Since T_y is reductive, we deduce the following result.

4.4. Lemma. *The natural $\Gamma(\tilde{Q}, \mathscr{D}_{\mathscr{L}^\vee})$-invariant pairing on $\Gamma(\tilde{Q}, \mathscr{L}) \times \Gamma(\tilde{Q}, \mathscr{L}^\vee)$ induces an isomorphism between the K-modules $\Gamma(\tilde{Q}, \mathscr{L})^\vee$ and $\Gamma(\tilde{Q}, \mathscr{L}^\vee)$.*

Now we can start the proof of 4.3. As has been remarked the spectral sequence 2.6 collapses because of 4.1. Hence we have

$$H^q(X, \mathscr{I}(Q, \tau)) = H^{q-s}(\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X}))).$$

From 2.8 we know that the right side can be expressed as the cohomology in degree $q-s$ of the complex

$$\Gamma(Y, C_{Y|X}(R^0 j_+(\tilde{\tau}) \otimes_{\mathcal{O}_Y} \Omega_{Y|X})) = \Gamma(\tilde{Q}, \tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})) \otimes_{\Gamma(\tilde{Q}, \mathscr{D}_{\boldsymbol{*}})} R_{\tilde{Q} \to Y|X, -\lambda+\rho}.$$

The duality between $\otimes$ and Hom, together with 4.4, gives a natural pairing of this complex and the complex

$$\mathrm{Hom}_{\Gamma(\tilde{Q}, \mathscr{D}_{\boldsymbol{*}})}(R_{\tilde{Q} \to Y|X, -\lambda+\rho}, \Gamma(\tilde{Q}, (\tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))^\vee))_{[K]}.$$

The latter complex calculates the standard Zuckerman modules $I^q(\mathfrak{h}_y, \Sigma_y^+, U^\vee \otimes_{\mathbb{C}} T_x(\Omega_X))$, as was explained in Sect. 3.

It remains to be shown that the complex

$$\mathrm{Hom}_{\Gamma(\tilde{Q}, \mathscr{D}_{\boldsymbol{*}})}(R_{\tilde{Q} \to Y|X, -\lambda+\rho}, \Gamma(\tilde{Q}, (\tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))^\vee))_{[K]}$$

is the contragredient to the complex

$$\Gamma(\tilde{Q}, \tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X})) \otimes_{\Gamma(\tilde{Q}, \mathscr{D}_{\boldsymbol{*}})} R_{\tilde{Q} \to Y|X, -\lambda+\rho}.$$

If we identify $\Gamma(\tilde{Q}, (\tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))^\vee)$ with the contragredient of $\Gamma(\tilde{Q}, \tilde{\tau} \otimes_{\mathcal{O}_{\tilde{Q}}} j^*(\Omega_{Y|X}))$ by 4.4, we have

$$\begin{aligned}&\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}(R_{\tilde{Q}\to Y|X,-\lambda+\rho},\Gamma(\tilde{Q},(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{\vee}))_{[K]}\\&\quad=\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}(R_{\tilde{Q}\to Y|X,-\lambda+\rho},\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{\vee})_{[K]}\\&\quad=\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}(R_{\tilde{Q}\to Y|X,-\lambda+\rho},\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{*})_{[K]}.\end{aligned}$$

Here we used that the action of K on $R_{\tilde{Q}\to Y|X,-\lambda+\rho}$ is algebraic, hence any element of $\mathrm{Hom}_{\mathbb{C}}(R_{\tilde{Q}\to Y|X,-\lambda+\rho},\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{*})_{[K]}$ takes values in $\Gamma(\tilde{Q},,\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{\vee}$. This implies

$$\begin{aligned}&\mathrm{Hom}_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}(R_{\tilde{Q}\to Y|X,-\lambda+\rho},\Gamma(\tilde{Q},(\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))^{\vee}))_{[K]}\\&\quad=(\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))\otimes_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}R_{\tilde{Q}\to Y|X,-\lambda+\rho})^{*}_{[K]}\\&\quad=(\Gamma(\tilde{Q},\tilde{\tau}\otimes_{\mathcal{O}_{\tilde{Q}}}j^{*}(\Omega_{Y|X}))\otimes_{\Gamma(\tilde{Q},\mathcal{D}_{\lambda})}R_{\tilde{Q}\to Y|X,-\lambda+\rho})^{\vee}.\end{aligned}$$

Therefore our pairing identifies the second complex as the contragredient (in the category of $(\mathfrak{g}, K)$-complexes) of the first complex. Because of the exactness of the contragredient functor, this implies 4.3.

The following two results are simple consequences of 4.3. The first is a result on vanishing of standard Zuckerman modules.

4.5. Corollary. *Let U be a finite-dimensional $(\mathfrak{h}_y, T_y)$-module. The standard Zuckerman modules $I^p(\mathfrak{h}_y, \Sigma_y^+, U)$ vanish for $p>s$.*

Proof. We may suppose that U is irreducible. Thus we can transfer the problem to the cohomology of standard Harish-Chandra sheaves, using 4.3. For these, the cohomology vanishes below zero for trivial reasons. □

The second is a vanishing theorem for cohomology of Harish-Chandra sheaves. It follows immediately from 4.3.

4.6. Corollary.

$$H^p(X, \mathscr{I}(Q,\tau))=0$$

for $p>s$.

Since $\mathfrak{k}\cap\mathfrak{n}_x$ is the Lie algebra of a unipotent subgroup of K,

$$s\leqq 1/2\dim(K/T) \quad\text{and}\quad s\leqq \dim Q.$$

Hence 4.6 gives a stronger vanishing result for the cohomology of standard Harish-Chandra sheaves than the Leray spectral sequence together with the affiness of the orbit imbeddings. The vanishing result 4.5 for standard Zuckerman modules is not entirely trivial either (compare [22], 6.3.21).

Corollaries 4.5 and 4.6 do not depend on the position of $\lambda\in\mathfrak{h}^*$. If we take in the account the position of λ we can get more precise results from the general vanishing theorem ([2, 19]). In this paper we shall restrict ourselves to the simplest case, which corresponds to 2.1 (i). It gives a vanishing theorem for all ordered Cartan subalgebras $(\mathfrak{h}_y, \Sigma_y^+)$ and representations of $(\mathfrak{h}_y, T_y)$ satisfying a positivity condition. More precisely, we have:

4.7. Corollary. *Let U be a finite-dimensional* $(\mathfrak{h}_y, T_y)$*-module such that* $\mathfrak{h}_y$ *acts by a sum of copies of the specialization of* $\lambda \in \mathfrak{h}^*$. *Assume that* $\rho - \lambda$ *is antidominant. Then the standard Zuckerman modules* $I^q(\mathfrak{h}_y, \Sigma_y^+, U)$ *vanish for* $q \neq s$.

In conclusion, we want to discuss how 4.3 applies to real semisimple Lie groups. Let G_0 be a connected semisimple Lie group with finite center and K_0 a maximal compact subgroup of G_0. Denote by $\mathfrak{g}$ the complexified Lie algebra of G_0 and by K the complexification of K_0 ([8], Chap. III, § 6, Def. 4). The Lie algebra $\mathfrak{k}$ of K is naturally identified with the fixed point set of a Cartan involution σ on $\mathfrak{g}$. It follows that the pair $(\mathfrak{g}, K)$ satisfies the condition (D) which was introduced at the beginning of this section.

As we mentioned in the Introduction, there is a natural duality between G_0-orbits and K-orbits in X: for each K-orbit Q there exists a unique G_0-orbit S such that $S \cap Q$ is a K_0-orbit; this relation is symmetric in S and Q [17]. For any $x \in S \cap Q$, we can choose $y \in Y$ such that

(i) $p(y) = x$, i.e. the Borel subalgebra $\mathfrak{b}_x$ is spanned by $\mathfrak{h}_y$ and the root subspaces of Σ_y^+.
(ii) $\mathfrak{h}_y$ is σ-stable and defined over $\mathbb{R}$.

One can check that the K-orbit of y depends only on Q and not on the particular choice of x. Moreover, this K-orbit is a standard orbit over Q, so that 4.3 applies in the present situation.

We fix one such $y \in Y$, and let T, resp. T_0, denote the stabilizers of y in K, resp. K_0. Then the identity component of T is an algebraic torus and T_0 is the compact real form of T. We therefore have a natural bijection between finite-dimensional algebraic representations of T and finite-dimensional continuous representations of T_0. It follows that the K-homogeneous connections τ on Q, and the corresponding Harish-Chandra sheaves $\mathscr{I}(Q, \tau) \in \mathscr{M}_{\mathrm{coh}}(\mathscr{D}_\lambda, K)$, are parametrized by irreducible finite-dimensional continuous representations of T_0, whose differentials are direct sums of copies of the linear form $\lambda + \rho$, specialized to $\mathfrak{h}_y$ and then restricted to the Lie algebra of T_0.

On the other hand, the same data also parametrize standard Zuckerman modules corresponding to S. In more geometric terms, the point $y \in Y$ determines a G_0-orbit $\tilde{S}$ in Y which covers the orbit S in X. The stabilizer of this point is the Cartan subgroup H_0 of G_0 with complexified Lie algebra $\mathfrak{h}_y$. It follows that irreducible finite-dimensional continuous representations of H_0 parametrize irreducible G_0-homogeneous vector bundles on $\tilde{S}$. Standard Zuckerman modules, as we explained in the Introduction, are the formal analogues of the cohomology groups of sheaves of local sections of these vector bundles annihilated by the right action of the polarization $\mathfrak{b}_x$. Moreover, T_0 is the maximal compact subgroup of H_0. Any irreducible finite-dimensional continuous representation of H_0 remains irreducible when restricted to T_0, and its differential is a direct sum of copies of a linear form on $\mathfrak{h}_y$.

Finally, we note that the shift s in the statement of the duality theorem can be interpreted as the difference of dimensions of orbits:

$$\begin{aligned} s &= \dim_{\mathbb{C}}(\mathfrak{k} \cap \mathfrak{n}_x) = \dim_{\mathbb{C}}(\mathfrak{k} \cap \mathfrak{b}_x) - \dim_{\mathbb{C}}(\mathfrak{k} \cap \mathfrak{h}_y) \\ &= \dim_{\mathbb{C}}(\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{h}_y)) - \dim_{\mathbb{C}}(\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{b}_x)) \\ &= \dim_{\mathbb{R}}(Q \cap S) - \dim_{\mathbb{C}} Q. \end{aligned}$$

Appendix A. Some results on $\mathcal{D}$-modules

In this appendix we collect some basic facts on twisted sheaves of differential operators on smooth complex algebraic varieties, and corresponding sheaves of modules, which we use in the paper. These constructions and results are implicit in the work of A. Beilinson and J. Bernstein on localization, and were announced in part in [1] and [2]. Although complete proofs have not appeared yet, the interested reader can extract the arguments from the discussion of the nontwisted case in A. Borel's notes [5]. A complete exposition of the twisted case will appear in [18].

A.1. Twisted sheaves of differential operators

Let X be a smooth complex algebraic variety. Denote by $\mathcal{O}_X$ its structure sheaf. Let $\mathcal{D}_X$ be the sheaf of local differential operators on X.

Denote by i_X the natural homomorphism of the sheaf of rings $\mathcal{O}_X$ into $\mathcal{D}_X$. We can consider the category of pairs $(\mathcal{A}, i_{\mathcal{A}})$ where $\mathcal{A}$ is a sheaf of rings on X and $i_{\mathcal{A}}: \mathcal{O}_X \to \mathcal{A}$ a homomorphism of sheaves of rings. The morphisms are the homomorphisms $\varphi: \mathcal{A} \to \mathcal{B}$ such that $\varphi \circ i_{\mathcal{A}} = i_{\mathcal{B}}$. A pair $(\mathcal{D}, i)$ is called a *twisted sheaf of differential operators* if X has a cover by open sets U such that $(\mathcal{D} \mid U, i \mid U)$ is isomorphic to $(\mathcal{D}_U, i_U)$ [1].

Let $f: Y \to X$ be a morphism of smooth algebraic varieties. Let $\mathcal{D}$ be a twisted sheaf of differential operators on X. Put

$$\mathcal{D}_{Y \to X} = f^*(\mathcal{D}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}.$$

Then $\mathcal{D}_{Y \to X}$ is a right $f^{-1}\mathcal{D}$-module for right multiplication on the second factor. Denote by $\mathcal{D}^f$ the sheaf of differential endomorphisms of the $\mathcal{O}_Y$-module $\mathcal{D}_{Y \to X}$ which are also $f^{-1}\mathcal{D}$-module endomorphisms. Then $\mathcal{D}^f$ is a twisted sheaf of differential operators on Y. If $g: Z \to Y$ is another morphism of smooth algebraic varieties, one has $\mathcal{D}^{f \circ g} = (\mathcal{D}^f)^g$.

Let G be an algebraic group acting on a smooth algebraic variety X. Denote by $\mu: G \times X \to X$ the action morphism, and by $pr_2: G \times X \to X$ the projection to the second variable. The notion of algebraic G-action on an $\mathcal{O}_X$-module $\mathcal{F}$ is defined by giving an $\mathcal{O}_{G \times X}$-module isomorphism of the inverse image $\mu^*(\mathcal{F})$ into $pr_2^*(\mathcal{F})$, subject to certain natural conditions; for details see ([20], 1.6).

Let $\mathcal{D}$ be a twisted sheaf of differential operators on X with an algebraic action γ of G, and $\alpha: \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ a homomorphism of algebras such that

(H1) the multiplication in $\mathcal{D}$ is G-equivariant,

(H2) the differential of the G-action on $\mathcal{D}$ agrees with the action of $\mathfrak{g}$ given by $D \to [\alpha(\xi), D]$ for $\xi \in \mathfrak{g}$ and $D \in \mathcal{D}$,

(H3) if we consider $\mathcal{U}(\mathfrak{g})$ as a G-module for the adjoint action of G, α is a morphism of G-modules.

Then we say that the action of G on $\mathcal{D}$ is compatible with its structure as a twisted sheaf of differential operators. If, in addition, X is a homogeneous space, $\mathcal{D}$ is a *homogeneous twisted sheaf of differential operators* on X.

We shall describe now the parametrization of all homogeneous twisted sheaves of differential operators on a homogeneous space X. Let $x_0 \in X$ and denote by B_{x_0} the stabilizer of x_0, and by $\mathfrak{b}_{x_0}$ its Lie algebra. To each B_{x_0}-invariant form λ on $\mathfrak{b}_{x_0}$ we associate a homogeneous twisted sheaf of differential operators $\mathcal{D}_{X,\lambda}$. First we introduce the structure of a sheaf of algebras on $\mathcal{U}^0 = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ by

$$(f \otimes \xi)(g \otimes \eta) = f\tau(\xi)\, g \otimes \eta + fg \otimes \xi\eta,$$

where $f, g \in \mathcal{O}_X$ and $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ (here τ denotes the natural action of $\mathfrak{g}$ on $\mathcal{O}_X$). Let $\mathfrak{g}^0 = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$, considered as $\mathcal{O}_X$-submodule of $\mathcal{U}^0$. The natural commutator in $\mathcal{U}^0$ induces the structure of a sheaf of Lie algebras on $\mathfrak{g}^0$. The map τ extends to a homomorphism of $\mathfrak{g}^0$ into the sheaf of local vector fields $\mathcal{T}_X$ on X. Denote by $\mathfrak{b}^0$ the kernel of τ. Then $\mathfrak{b}^0$ is a sheaf of ideals in $\mathfrak{g}^0$. The geometric fibre of $\mathfrak{b}^0$ at x_0 is $\mathfrak{b}_{x_0}$. Therefore, to each B_{x_0}-invariant linear form on $\mathfrak{b}_{x_0}$ we can associate a G-equivariant morphism σ_λ of the $\mathcal{O}_X$-module $\mathfrak{b}^0$ into $\mathcal{O}_X$. Let $\varphi_\lambda \colon \mathfrak{b}^0 \to \mathcal{U}^0$ given by $\varphi_\lambda(s) = s - \sigma_\lambda(s)$, $s \in \mathfrak{b}^0$. Then $\operatorname{Im} \varphi_\lambda$ generates a sheaf of two-sided ideals $\mathcal{J}_\lambda$ in $\mathcal{U}^0$. We put

$$\mathcal{D}_{X,\lambda} = \mathcal{U}^0 / \mathcal{J}_\lambda.$$

Then G acts on $\mathcal{D}_{X,\lambda}$, and the natural morphism of $\Gamma(X, \mathcal{U}^0) = \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_{X,\lambda})$ induces a homomorphism of $\mathcal{U}(\mathfrak{g})$ into $\Gamma(X, \mathcal{D}_{X,\lambda})$. One can check that $\mathcal{D}_{X,\lambda}$ is a homogeneous twisted sheaf of differential operators on X.

Let $\mathcal{D}$ be any homogeneous twisted sheaf of differential operators on X. Denote by i_0 the inclusion of x_0 into X. Then $\mathcal{D}^{i_0}$ has a natural structure of a homogeneous twisted sheaf of differential operators on the one-point space $\{x_0\}$, considered as a homogeneous space for B_{x_0}. As such $\mathcal{D}^{i_0}$ is completely determined by a B_{x_0}-invariant linear form μ on $\mathfrak{b}_{x_0}$. Moreover, $\mathcal{D}$ is isomorphic to $\mathcal{D}_{X,\mu}$ as a homogeneous twisted sheaf of differential operators. This shows that the sheaves $\mathcal{D}_{X,\mu}$, $\mu \in (\mathfrak{b}_{x_0}^*)^{B_{x_0}}$, exhaust all G-homogeneous twisted sheaves of differential operators on X.

If $\mathcal{D} = \mathcal{D}_{X,\lambda}$ and $f\colon Y \to X$ is a morphism of smooth algebraic varieties, we put $\mathcal{D}_{Y \to X} = \mathcal{D}_{Y \to X,\lambda}$.

A.2. $\mathcal{D}$-modules

Let $\mathcal{D}$ be a twisted sheaf of differential operators on X. Then the opposite sheaf of rings $\mathcal{D}^0$ is again a twisted sheaf of differential operators on X. We can therefore view left $\mathcal{D}$-modules as right $\mathcal{D}^0$-modules and vice versa. Formally, the category $\mathcal{M}^L(\mathcal{D})$ of quasi-coherent left $\mathcal{D}$-modules on X is isomorphic to the category $\mathcal{M}^R(\mathcal{D}^0)$ of quasi-coherent right $\mathcal{D}^0$-modules on X. Hence one can freely use right and left modules depending on the particular situation.

In the case of a homogeneous space X, if δ is the B_{x_0}-invariant linear form on $\mathfrak{b}_{x_0}$ which is the differential of the representation of B_{x_0} on the top exterior power of the cotangent space at x_0, $(\mathcal{D}_{X,\lambda})^0$ is naturally isomorphic to $\mathcal{D}_{X,-\lambda+\delta}$.

For a category $\mathcal{M}(\mathcal{D})$ of $\mathcal{D}$-modules we denote by $\mathcal{M}_{coh}(\mathcal{D})$ the corresponding subcategory of coherent $\mathcal{D}$-modules. For any coherent $\mathcal{D}$-module $\mathcal{V}$ we can define the *characteristic variety* $Ch(\mathcal{V})$ of $\mathcal{V}$ in the same way as in the nontwisted case [5]. Because its construction is local, the results in the nontwisted case imply:

(a) $Ch(\mathcal{V})$ is a conical subvariety of the cotangent bundle $T^*(X)$,

(b) $\dim Ch(\mathcal{V}) \geqq \dim X$.

If $\dim Ch(\mathcal{V}) = \dim X$ we say that $\mathcal{V}$ is a *holonomic* $\mathcal{D}$-module. Holonomic modules form a subcategory $\mathcal{H}ol(\mathcal{D})$ of $\mathcal{M}_{coh}(\mathcal{D})$.

Modules in $\mathcal{M}_{coh}(\mathcal{D})$ which are coherent as $\mathcal{O}_X$-modules are called *connections.* Connections are locally free as $\mathcal{O}_X$-modules and their characteristic variety is the zero section of $T^*(X)$; in particular they are holonomic.

Assume that G is an algebraic group acting on a smooth algebraic variety X and $\mathcal{D}$ a twisted sheaf of differential operators on X with a compatible action of G. A $(\mathcal{D}, G)$-module $\mathcal{F}$ is an object of $\mathcal{M}(\mathcal{D})$ on which G acts algebraically, so that

(i) the action map from $\mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F}$ into $\mathcal{F}$ is a morphism of G-homogeneous $\mathcal{O}_X$-modules,

(ii) the actions of $\mathfrak{g}$ on $\mathcal{F}$ given by the action of $\mathcal{D}$ and the differential of the action of G coincide.

A.3. Functors

A.3.1. Twist. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ has a natural structure of right $\mathcal{D}$-module by right multiplication on the second factor. Let $\mathcal{D}^{\mathcal{L}}$ be the sheaf of differential endomorphisms of the $\mathcal{O}_X$-module $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ (for the $\mathcal{O}_X$-module structure given by left multiplication) which commute with the right $\mathcal{D}$-module structure. Then $\mathcal{D}^{\mathcal{L}}$ is a twisted sheaf of differential operators on X. We can define the *twist functor* from $\mathcal{M}^L(\mathcal{D})$ into $\mathcal{M}^L(\mathcal{D}^{\mathcal{L}})$ by

$$\mathcal{V} \rightsquigarrow (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V}$$

for $\mathcal{V} \in \mathcal{M}^L(\mathcal{D})$. As $\mathcal{O}_X$-module,

$$(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{V}.$$

The operation of twist is visibly an equivalence of categories.

In the case of a homogeneous space X, if $\mathcal{L}$ is the homogeneous invertible $\mathcal{O}_X$-module on X determined by the character of B_{x_0} with differential $\mu \in (\mathfrak{b}_{x_0}^*)^{B_{x_0}}$, we have

$$(\mathcal{D}_{X,\lambda})^{\mathcal{L}} = \mathcal{D}_{X,\lambda+\mu}.$$

A.3.2. Inverse image. Let $\mathcal{V} \in \mathcal{M}^L(\mathcal{D})$. Put

$$f^+(\mathcal{V}) = \mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{D}} f^{-1}\mathcal{V}.$$

Then $f^+(\mathcal{V}) \in \mathcal{M}^L(\mathcal{D}^f)$ is the *inverse image* of $\mathcal{V}$ (in the category of $\mathcal{D}$-modules), and f^+ is a right exact covariant functor from $\mathcal{M}^L(\mathcal{D})$ into $\mathcal{M}^L(\mathcal{D}^f)$. Considered as $\mathcal{O}_Y$-module,

$$f^+(\mathscr{V}) = \mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_X} f^{-1}\mathscr{V} = f^*(\mathscr{V}).$$

The left derived functors $L^p f^+ : \mathscr{M}^L(\mathscr{D}) \to \mathscr{M}^L(\mathscr{D}^f)$ of f^+ have analogous properties. Moreover, if $g: Z \to Y$ is another morphism of smooth algebraic varieties, we have the Grothendieck spectral sequence

$$L^p g^+ \circ L^q f^+ \Rightarrow L^{p+q}(f \circ g)^+.$$

A.3.3. Direct image. To define the direct image functor for $\mathscr{D}$-modules efficiently one has to introduce derived categories (although in this paper we deal only with affine morphisms where one can take also a more elementary approach). In addition, it is simpler to define it for right $\mathscr{D}^{\mathscr{L}}$-modules. Naively speaking, one would like to define the direct image as the functor $\mathscr{V} \rightsquigarrow f_*(\mathscr{V} \otimes_{\mathscr{D}^f} \mathscr{D}_{Y\to X})$ for $\mathscr{V} \in \mathscr{M}^R(\mathscr{D}^f)$. Unfortunately, the direct image functor f_* is left exact and the tensor product $-\otimes_{\mathscr{D}^f} \mathscr{D}_{Y\to X}$ is right exact, so this definition is unsatisfactory. We first give the correct general definition of the direct image using derived categories, and then discuss the special cases needed in the paper, where more naive constructions apply.

Let $D^b(\mathscr{M}^R(\mathscr{D}^f))$ be the derived category of bounded complexes of quasi-coherent right $\mathscr{D}^f$-modules. Then we define

$$Rf_+(\mathscr{V}^\cdot) = Rf_*\left(\mathscr{V}^\cdot \overset{L}{\underset{\mathscr{D}^f}{\otimes}} \mathscr{D}_{Y\to X}\right)$$

for any $\mathscr{V}^\cdot \in D^b(\mathscr{M}^R(\mathscr{D}^f))$ (here we denote by Rf_* and $\overset{L}{\underset{\mathscr{D}^f}{\otimes}}$ the derived functors of direct image and tensor product) [5]. Moreover, if $g: Z \to Y$ is another morphism of smooth algebraic varieties, we have

$$Rf_+ \circ Rg_+ = R(f \circ g)_+.$$

Let $\mathscr{V}^\cdot$ be the complex in $D^b(\mathscr{M}^R(\mathscr{D}^f))$ which is zero in all degrees except 0, where it is equal to a quasi-coherent right $\mathscr{D}^f$-module $\mathscr{V}$. Then we put

$$R^i f_+(\mathscr{V}) = H^i(Rf_+(\mathscr{V}^\cdot)) \quad \text{for } i \in \mathbb{Z},$$

i.e. we get a family $R^i f_+$, $i \in \mathbb{Z}$, of functors from $\mathscr{M}^R(\mathscr{D}^f)$ into $\mathscr{M}^R(\mathscr{D})$. We call $R^i f_+$ the i^{th} *direct image*. This also leads to the spectral sequence

$$R^i f_+ \circ R^j g_+ \Rightarrow R^{i+j}(f \circ g)_+.$$

Now one can check that

(i) if f is an immersion, $R^i f_+ = 0$ for $i < 0$,

(ii) if f is affine, $R^i f_+ = 0$ for $i > 0$.

In the first case, if f is an immersion, $\mathscr{D}_{Y\to X}$ is a locally free $\mathscr{D}^f$-module. This implies that

$$R^i f_+(\mathscr{V}) = R^i f_*(\mathscr{V} \otimes_{\mathscr{D}^f} \mathscr{D}_{Y\to X}) \quad \text{for } \mathscr{V} \in \mathscr{M}^R(\mathscr{D}^f),$$

i.e. $R^0 f_+$ is our "naive" direct image functor, it is left exact, and $R^i f_+$ are its right derived functors. In the second case, if f is an affine morphism, $R^i f_+$

are the left derived functors of R^0f_+. Moreover, for an affine open subset U of X and $V=f^{-1}(U)$, it follows that

$$\Gamma(U, R^0f_+(\mathscr{V}))=\Gamma(V,\mathscr{V})\otimes_{\Gamma(V,\mathscr{D}^f)}\Gamma(V,\mathscr{D}_{Y\to X}),$$

which completely determines $R^0f_+(\mathscr{V})$.

Finally, if $f\colon Y\to X$ is an affine immersion, $R^if_+=0$ for $i\neq 0$ and R^0f_+ is exact.

We need more precise information in two special cases.

Closed immersions. If Y is a closed smooth subvariety of X and $f\colon Y\to X$ the natural immersion, R^0f_+ is an exact functor from $\mathscr{M}^R(\mathscr{D}^f)$ into $\mathscr{M}^R(\mathscr{D})$. It is an equivalence of the category $\mathscr{M}^R(\mathscr{D}^f)$ with the full subcategory of $\mathscr{M}^R(\mathscr{D})$ consisting of modules supported in Y. As we remarked before,

$$R^0f_+(\mathscr{V})=f_*(\mathscr{V}\otimes_{\mathscr{D}^f}\mathscr{D}_{Y\to X})$$

for any $\mathscr{V}\in\mathscr{M}^R(\mathscr{D}^f)$. If $\mathscr{I}_Y$ is the ideal of $\mathscr{O}_X$ consisting of germs vanishing on Y, we can define an increasing filtration of $\mathscr{D}_{Y\to X}$ by left $\mathscr{D}^f$- and right $f^{-1}\mathscr{O}_X$-modules, by setting

$$F_p\mathscr{D}_{Y\to X}=\{T\in\mathscr{D}_{Y\to X}\mid T\varphi=0 \text{ for } \varphi\in(\mathscr{I}_Y)^{p+1}\},$$

for $p\in\mathbb{Z}_+$. We call this filtration the filtration by *normal degree*. By the previous discussion, it induces also a natural $\mathscr{O}_X$-module filtration of $\mathscr{D}$-modules supported on Y. In particular, if $\mathscr{V}\in\mathscr{M}^R(\mathscr{D}^f)$,

$$Gr\,R^0f_+(\mathscr{V})=f_*(\mathscr{V}\otimes_{\mathscr{O}_Y}S(\mathscr{N}_{X|Y})),$$

where $\mathscr{N}_{X|Y}=f^*(\mathscr{T}_X)/\mathscr{T}_Y$ denotes the normal sheaf of Y and $S(\mathscr{N}_{X|Y})$ the corresponding sheaf of symmetric algebras.

Surjective submersions. Let $f\colon Y\to X$ be a surjective submersion. Denote by $\mathscr{T}_{Y|X}$ the sheaf of local vector fields tangent to the fibres of f. Then $\mathscr{T}_{Y|X}$ is a sheaf of Lie subalgebras of the sheaf $\mathscr{T}_Y$. From the construction of $\mathscr{D}^f$ it is evident that $\mathscr{T}_{Y|X}\subset\mathscr{D}^f$. Therefore, for any $\mathscr{V}\in\mathscr{M}^R(\mathscr{D}^f)$ we can form the complex $C_{Y|X}(\mathscr{V})$,

$$C^k_{Y|X}(\mathscr{V})=\mathscr{V}\otimes_{\mathscr{O}_Y}\wedge^{-k}\mathscr{T}_{Y|X},\qquad k\in\mathbb{Z},$$

with differential

$$\begin{aligned} d(u\otimes v_1\wedge v_2\wedge\ldots\wedge v_k) &= \sum_{i=1}^{k}(-1)^{i+1}uv_i\otimes v_1\wedge v_2\wedge\ldots\wedge\hat{v}_i\wedge\ldots\wedge v_k\\ &\quad+\sum_{1\leq i<j\leq k}(-1)^{i+j}u\otimes[v_i,v_j]\wedge v_1\wedge\ldots\wedge\hat{v}_i\wedge\ldots\wedge\hat{v}_j\wedge\ldots\wedge v_k,\end{aligned}$$

for $u\in\mathscr{V}$, $v_1,v_2,\ldots,v_k\in\mathscr{T}_{Y|X}$. The complex $C_{Y|X}(\mathscr{V})$ is, up to a shift in degree, the relative de Rham complex for the right $\mathscr{D}^f$-module $\mathscr{V}$. One can show that

$C_{Y|X}(\mathcal{D}^f)$ is a left resolution of $\mathcal{D}_{Y\to X}$ by locally free left $\mathcal{D}^f$- and right $f^{-1}\mathcal{O}_X$-modules. Therefore, as complex of $\mathcal{O}_X$-modules,

$$Rf_+(\mathcal{V}^\cdot)=Rf_*(\mathcal{V}^\cdot\otimes_{\mathcal{D}^f} C_{Y|X}(\mathcal{D}^f))=Rf_*(\mathcal{V}^\cdot\otimes_{\mathcal{O}_Y}\wedge\mathcal{T}_{Y|X})$$

for any $\mathcal{V}^\cdot\in D^b(\mathcal{M}^R(\mathcal{D}^f))$. Moreover, if we assume in addition that f is an affine morphism, we have

$$R^if_+(\mathcal{V})=H^i(f_*(\mathcal{V}\otimes_{\mathcal{O}_Y}\wedge\mathcal{T}_{Y|X}))=H^i(f_*(C_{Y|X}(\mathcal{V}))),$$

for any $\mathcal{V}\in\mathcal{M}^R(\mathcal{D}^f)$. This describes the direct images for f as $\mathcal{O}_X$-modules. One can describe the action of $\mathcal{D}$ locally in a noncanonical way.

Fortunately, in the homogeneous situation, the $\mathcal{D}$-module actions are induced by the action of the enveloping algebra $\mathcal{U}(\mathfrak{g})$, which makes it possible to describe the $\mathcal{D}$-module structure on $H^i(f_*(C_{Y|X}(\mathcal{V})))$ in a canonical manner. Assume that $X=G/B$ and $Y=G/H$ are homogeneous spaces for an algebraic group G such that $H\subset B$. Let $f\colon Y\to X$ be the natural map. Assume in addition that B/H is an affine variety, so that f is an affine morphism ([3], Lemma 1.1). Let $\mathcal{D}_{X,\lambda}$ be a homogeneous twisted sheaf of differential operators on X. Then the linear form $\lambda\in(\mathfrak{b}^*)^B$ determines, by restriction, $\mu=\lambda\,|\,\mathfrak{h}\in(\mathfrak{h}^*)^H$, and therefore a homogeneous twisted sheaf of differential operators $\mathcal{D}_{Y,\mu}=(\mathcal{D}_{X,\lambda})^f$ on Y. In this situation, $\wedge\mathcal{T}_{Y|X}$ is a G-homogeneous $\mathcal{O}_Y$-module on Y, and by differentiation it has a natural structure of left $\mathcal{U}(\mathfrak{g})$-module. It follows that we can define a right action of $\mathcal{U}(\mathfrak{g})$ on $C_{Y|X}(\mathcal{V})$ via

$$(u\otimes v)\,\xi=u\xi\otimes v-u\otimes\xi v$$

for $u\in\mathcal{V}$ and $v\in\wedge\mathcal{T}_{Y|X}$. The action induces a right action of $\mathcal{U}(\mathfrak{g})$ on the cohomology groups of $f_*(C_{Y|X}(\mathcal{V}))$ that generates the right action of $\mathcal{D}_{X,\lambda}$.

Appendix B: Extension to a larger class of groups

In this appendix we show how our arguments can be extended to more general situations which play some role in applications of the duality theorem. We have avoided this degree of generality in the main body of the paper to keep the notation simple and to make the basic ideas as transparent as possible. Here we restrict ourselves to brief indications of the ideas involved in these generalizations.

To begin with we may assume that the Lie algebra $\mathfrak{g}$ is reductive, rather than semisimple. Also, in the definition of Harish-Chandra pair, the connectedness assumption on the group K can be dropped and replaced with the assumption that the image $\varphi(K)$ is contained in $\operatorname{Int}(\mathfrak{g})$. We no longer insist that the differential of φ be injective; instead we require that it factors through an injective morphism of $\mathfrak{k}$ into $\mathfrak{g}$. In this case, K acts trivially on $\mathcal{Z}(\mathfrak{g})$, and the categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$ and $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$ are well-defined if we add to the compatibility condition (CM) the condition that the action of $\mathcal{U}_\theta$ is K-equivariant. Moreover, these categories have the same properties with respect to the cohomology and localization functors. The only essential difference comes from the fact that K-orbits in X are not necessarily connected. To deal with this difficulty we now analyze more carefully the structure of standard Harish-Chandra sheaves $\mathcal{I}(Q, \tau)$ and their unique irreducible subobjects $\mathcal{L}(Q, \tau)$.

For any subgroup $K' \subset K$ of finite index, there is a natural induction functor from $\mathcal{M}_{fg}(\mathcal{U}_\theta, K')$ into $\mathcal{M}_{fg}(\mathcal{U}_\theta, K)$, as follows. Let $R(K)$ be the ring of regular functions on K. For any $\mathcal{V} \in \mathcal{M}_{fg}(\mathcal{U}_\theta, K')$ we can view $R(K) \otimes_{\mathbb{C}} V$ as the space of regular V-valued functions on K. Let $\operatorname{Ind}_{K'}^{K}(V)$ be the subspace of all such functions F which satisfy

$$F(kh) = \pi(h^{-1}) F(k) \quad \text{for } h \in K' \text{ and } k \in K.$$

We can then define the action of $\mathfrak{g}$ and K on $\operatorname{Ind}_{K'}^{K}(V)$ by

$$(v(k)F)(h) = F(k^{-1}h) \qquad \text{for } h, k \in K,$$
$$(v(X)F)(k) = \pi(\operatorname{Ad}(k^{-1})X)F(k) \qquad \text{for } X \in \mathfrak{g},\ k \in K;$$

one checks that $\operatorname{Ind}_{K'}^{K}(V) \in \mathcal{M}_{fg}(\mathcal{U}_\theta, K)$.

Fix a K-orbit Q and a point $x \in Q$. Denote by S_x the stabilizer of x in K. If K^0 is the identity component of K, K^0 and S_x generate a subgroup $K^1 \subset K$ of finite index which stabilizes the component Q^0 of Q with $x \in Q^0$. Then $(\mathfrak{g}, K^1)$ is again a Harish-Chandra pair. Let i be the natural immersion of Q into X, and τ an irreducible $((\mathcal{D}_\lambda)^i, K)$-connection on Q. The restriction $\tau | Q^0$ is an irreducible $((\mathcal{D}_\lambda)^i | Q^0, K^1)$-connection on Q^0. The standard module $\mathcal{I}(Q^0, \tau | Q^0)$ is therefore a well-defined object of $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K^1)$, and its cohomology modules are in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K^1)$. We claim that

$$H^p(X, \mathcal{I}(Q, \tau)) = \operatorname{Ind}_{K^1}^{K}(H^p(X, \mathcal{I}(Q^0, \tau | Q^0)))$$

for any $p \in \mathbb{Z}_+$. This follows directly from the construction of the direct image. Indeed, as $\mathcal{U}_\theta$-module, $H^p(X, \mathcal{I}(Q, \tau))$ is the direct sum of $H^p(X, \mathcal{I}(Q', \tau | Q'))$ taken over all connected components Q' of the orbit Q. Moreover, the elements of K/K^1 parametrize connected components of Q. Let $k \in K$ be a representative of the right K^1-coset corresponding to the connected component Q' of Q, and $\tau_k: X \to X$ translation by k. Then the action of k identifies $\mathcal{I}(Q^0, \tau | Q^0)$ and $\tau_k^*(\mathcal{I}(Q', \tau | Q'))$, which in turn identifies $H^p(X, \mathcal{I}(Q^0, \tau | Q^0))$ and $H^p(X, \mathcal{I}(Q', \tau | Q'))$. Our statement follows from the classical block matrix construction of induced representations.

The proof of the duality theorem holds without any changes for $\mathcal{I}(Q^0, \tau | Q^0)$ considered as an object of $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K^1)$. Since K^1 is reductive, passage to the contragredient commutes with induction, hence the duality theorem for $\mathcal{I}(Q, \tau)$ follows from the description of standard Zuckerman modules for nonconnected K ([22], 6.2).

This discussion applies in particular to the representation theory of real reductive groups of the Harish-Chandra class, as discussed at the end of Sect. 4.

Another situation which can be treated by the duality theorem is the case of a real reductive group G_0 with infinite center. In this case the action of the natural complexification of the maximal compactly imbedded subgroup K_0 on the flag variety X is not algebraic, hence one has to modify the approach significantly. The way to do this is an algebraic analogue of a trick introduced in ([25], 3.3). For simplicity we assume that G_0 is a connected semisimple Lie group and $\mathfrak{g}$ its complexified Lie algebra. Let Z_0 be the center of G_0. Then Z_0 is a central subgroup of K_0 and K_0/Z_0 is a compact group.

Denote by $\mathcal{M}_{fg}(\mathfrak{g}, K_0)$ the category of Harish-Chandra modules, i.e. the category of modules V with the following properties

(i) V is finitely generated as $\mathcal{U}(\mathfrak{g})$-module,

(ii) as a K_0-module it is a sum of irreducible finite-dimensional continuous submodules,

(iii) the actions are compatible, i.e. the differential of the action of K_0 agrees with the action of the Lie algebra $\mathfrak{k}$.

If V is an irreducible object in $\mathcal{M}_{fg}(\mathfrak{g}, K_0)$, the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ and the center Z_0 of G_0 act on V by scalars. The action of Z_0 is given by a character $\zeta: Z_0 \to \mathbb{C}^*$ which we call the central character of V. Therefore, to describe all irreducible Harish-Chandra modules for $(\mathfrak{g}, K_0)$, it is sufficient to describe irreducible objects in the categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, K_0, \zeta)$ of all Harish-Chandra modules with infinitesimal character χ_λ and central character ζ. We claim that the study of $\mathcal{M}_{fg}(\mathcal{U}_\theta, K_0, \zeta)$ can be reduced to the study of the categories analogous to the ones we encountered before.

Let C be the center of K_0 and C_0 its connected component. Let $Z_1 = Z_0 \cap C_0$. One sees that K_0/Z_1 is a compact group. By going to a finite cover of K_0, if necessary, we can always assume that K_0 is the product of its commutant and C_0. With this additional assumption, there exists a one-dimensional representation ω of K_0 such that $\omega | Z_1 = \zeta | Z_1$. In the following we assume that K_0 satisfies the above assumption.

Let (π, V) be an object in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K_0, \zeta)$. Then $\pi \otimes \omega^{-1}$ is a representation of K_0 which is trivial on Z_1. We can therefore view it as a representation of the compact group K_0/Z_1. This group has a natural complexification K. It follows that V is a finitely generated $\mathcal{U}_\theta$-module which is also an algebraic K-module via the K-action ν, and these actions satisfy the compatibility condition:

(IC) for any $\xi \in \mathfrak{k}$,
$$\pi(\xi) = \nu(\xi) + \omega(\xi);$$
here we identify ω with its differential.

This leads us to the following definition of the category of *twisted* Harish-Chandra modules. Let ω be a K-invariant linear form on the Lie algebra $\mathfrak{k}$. The category $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$ consists of objects (π, ν, V) such that

(i) (π, V) is a finitely generated $\mathcal{U}_\theta$-module,

(ii) (ν, V) is an algebraic K-module,

(iii) the actions satisfy the compatibility condition (IC).

There is a natural equivalence of categories between $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$ and $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega')$ if $\omega - \omega'$ is a differential of a one-dimensional algebraic representation of K. Moreover, if ω is a one-dimensional representation of K_0 such that $\omega | Z_1 = \zeta | Z_1$, we have a natural equivalence of categories between $\mathcal{M}_{fg}(\mathcal{U}_\theta, K_0, \zeta)$ and a full subcategory of $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$. More precisely, $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$ is equivalent to the direct sum of categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, K_0, \zeta')$ for central characters ζ' such that $\zeta' | Z_1 = \omega | Z_1$.

On the other hand, the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$ can be studied by localization. Define $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \omega)$ to be the category of coherent $\mathcal{D}_\lambda$-modules on the flag variety X with an algebraic action of K such that the action of $\mathcal{D}_\lambda$ is K-equivariant and satisfies the compatibility condition analogous to (IC). Then the localization functor maps objects of $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$ into $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K, \omega)$. Moreover, the cohomology modules of objects of $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K, \omega)$ belong to $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \omega)$.

The classification of irreducible objects in $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K, \omega)$ in terms of standard twisted Harish-Chandra sheaves proceeds as in the nontwisted case with minor modifications. The same is true for the construction of standard Zuckerman modules and the proof of the duality theorem.

References

1. Beilinson, A., Bernstein, J.: Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris, Ser. I **292**, 15–18 (1981)
2. Beilinson, A., Bernstein, J.: A generalization of Casselman's submodule theorem. In: Representation theory of reductive groups, Progress in Mathematics, vol. 40, pp. 35–52. Boston: Birkhäuser 1983
3. Bialynicki-Birula, A.: On homogeneous affine spaces of linear algebraic groups. Am. J. Math. **85**, 577–582 (1963)
4. Borel, A.: Linear algebraic groups. New York: Benjamin 1969
5. Borel, A., et al.: Algebraic $\mathcal{D}$-modules. Boston-New York: Academic Press 1986
6. Brylinski, J.L., Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems. Invent. Math. **64**, 387–410 (1981)
7. Bourbaki, N.: Algébre, Ch. I–X. Paris: Masson 1980–1981
8. Bourbaki, N.: Groupes et algébres de Lie, Ch. I–VIII. Paris: Masson 1981
9. Cartan, H., Eilenberg, S.: Homological algebra. Princeton: Princeton University Press 1956
10. Dixmier, J.: Enveloping algebras. Amsterdam: North-Holland 1977
11. Enright, T.J., Wallach, N.R.: Notes on homological algebra and representations of Lie algebras. Duke Math. J. **47**, 1–15 (1980)
12. Grothendieck, A.: Sur quelques points d'algébre homologique. Tôhoku Math. J. **9**, 119–221 (1957)
13. Hochschild, G.: Introduction to affine algebraic groups. San Francisco: Holden-Day 1971
14. Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math. **53**, 165–184 (1979)
15. Langlands, R.P.: On the classification of irreducible representations of real algebraic groups, mimeographed notes. Institute for Advanced Study, 1973
16. Lusztig, G., Vogan, D.: Singularities of closures of K-orbits on flag manifolds. Invent. Math. **71**, 365–379 (1983)
17. Matsuki, T.: The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. J. Math. Soc. Japan **31**, 332–357 (1979)
18. Miličić, D.: Localization and representation theory of reductive Lie groups mimeographed notes; to appear
19. Miličić, D., Taylor, J.L.: Localization and coherent continuation. (Preprint)
20. Mumford, D.: Geometric invariant theory. Ergebnisse der Mathematik, Band 34. Berlin-Heidelberg-New York: Springer Berlin 1965
21. Schmid, W.: Recent developments in representation theory. In: Proceedings of Arbeitstagung, Bonn 1984, Lect. Notes Math., vol. 1111, pp 135–153. Berlin-Heidelberg-New York-Tokyo: Springer 1985
22. Vogan, D.: Representations of real reductive Lie groups. Progress in Mathematics, vol. 15. Boston: Birkhäuser 1981
23. Vogan, D.: Irreducible characters of semisimple Lie groups III: proof of the Kazhdan-Lusztig conjectures in the integral case. Invent. Math. **71**, 381–417 (1983)
24. Wolf, J.A.: Finiteness of orbit structure for real flag manifolds. Geom. Dedicata **3**, 377–384 (1974)
25. Wolf, J.A.: Unitary representations on partially holomorphic cohomology spaces. Mem. Amer. Math. Soc. **138** (1974)
26. Zuckerman, G.J.: Geometric methods in representation theory. In: Representation theory of reductive groups, Progress in Mathematics, vol. 40, pp. 283–290. Boston: Birkhäuser 1983

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