# Hall effect metamaterials: guiding fields in the unit cell

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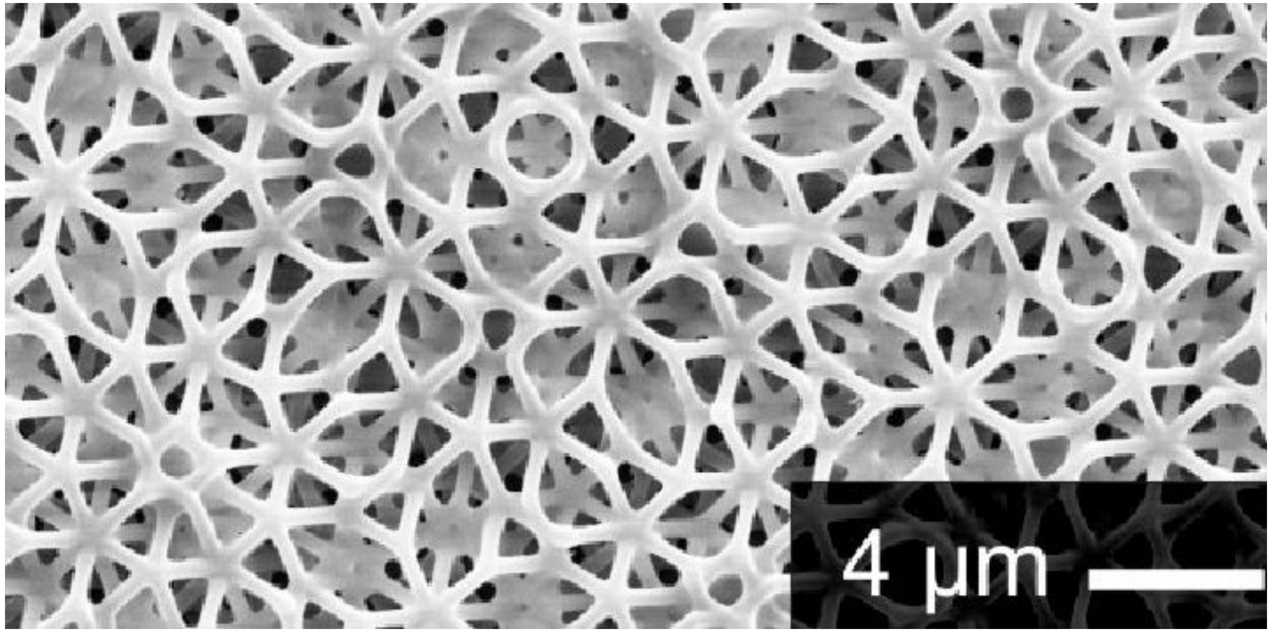
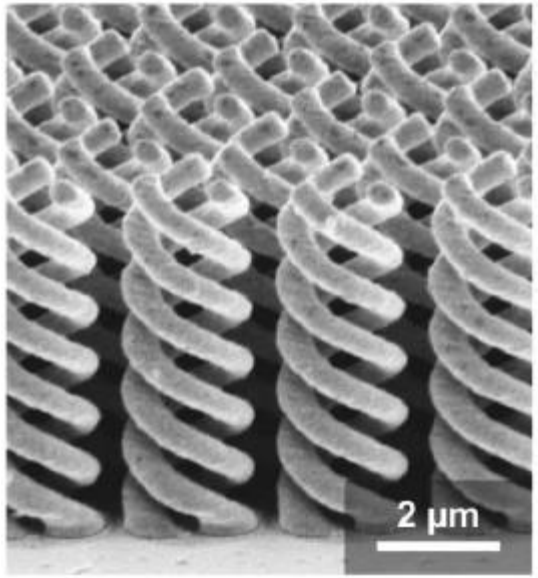
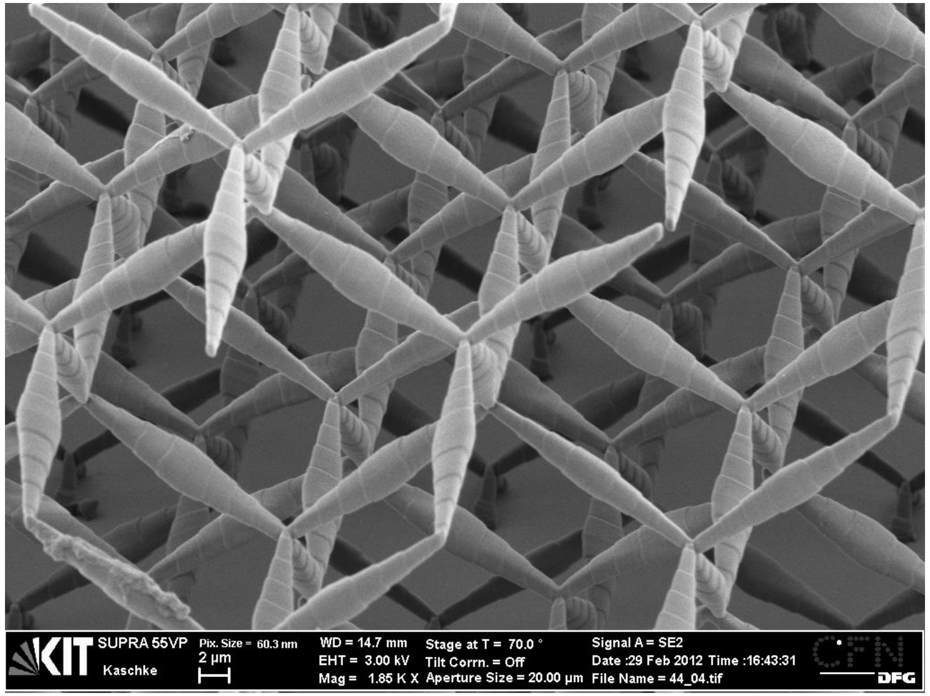
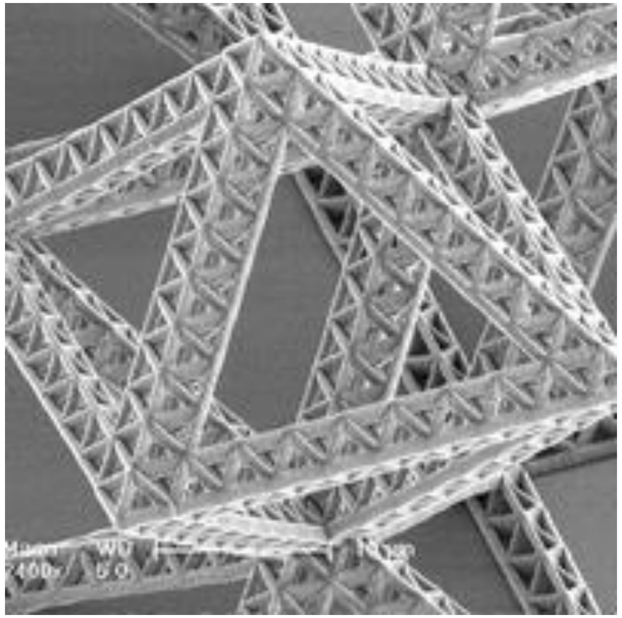
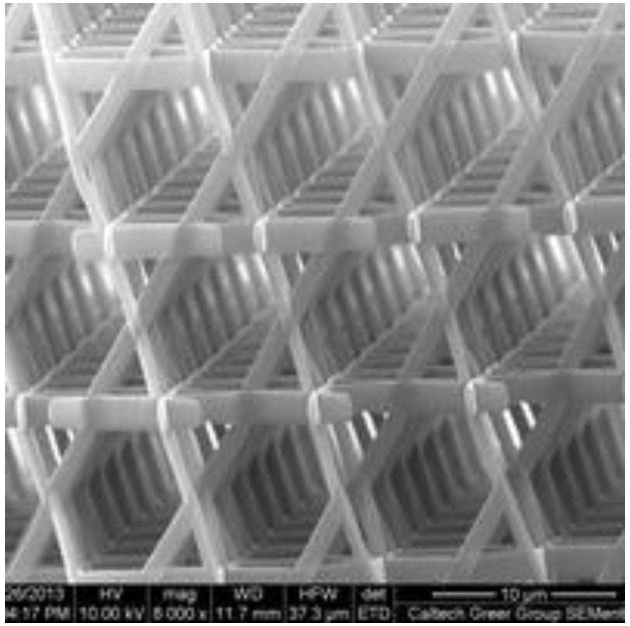


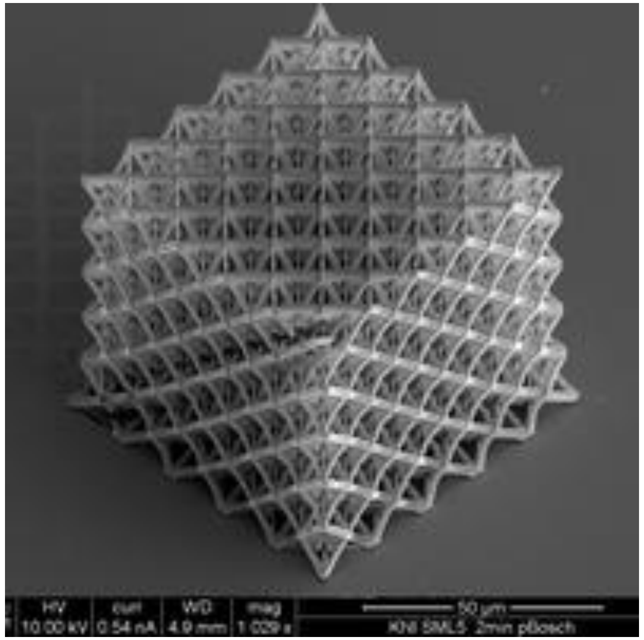




Group of
Martin Wegener






Group of Julia Greer

Walser 1999:

Macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of *two or more responses* to specific excitation.

Browning and Wolf 2001:

Metamaterials are a new class of ordered composites that exhibit exceptional properties not readily observed in nature.

Wegener (2018):

Metamaterials are rationally designed composites made of tailored building blocks or unit cells, which are composed of one or more constituent bulk materials. The metamaterial properties go beyond those of the ingredient materials – qualitatively or quantitatively.

With an addition:

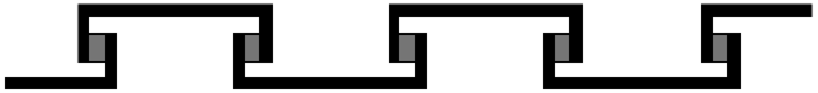
….The properties of the metamaterial can be mapped onto effective-medium parameters

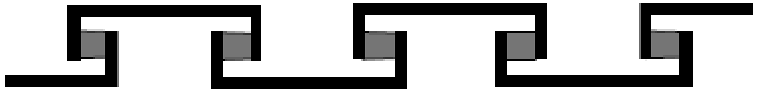
# Metamaterials are not new:

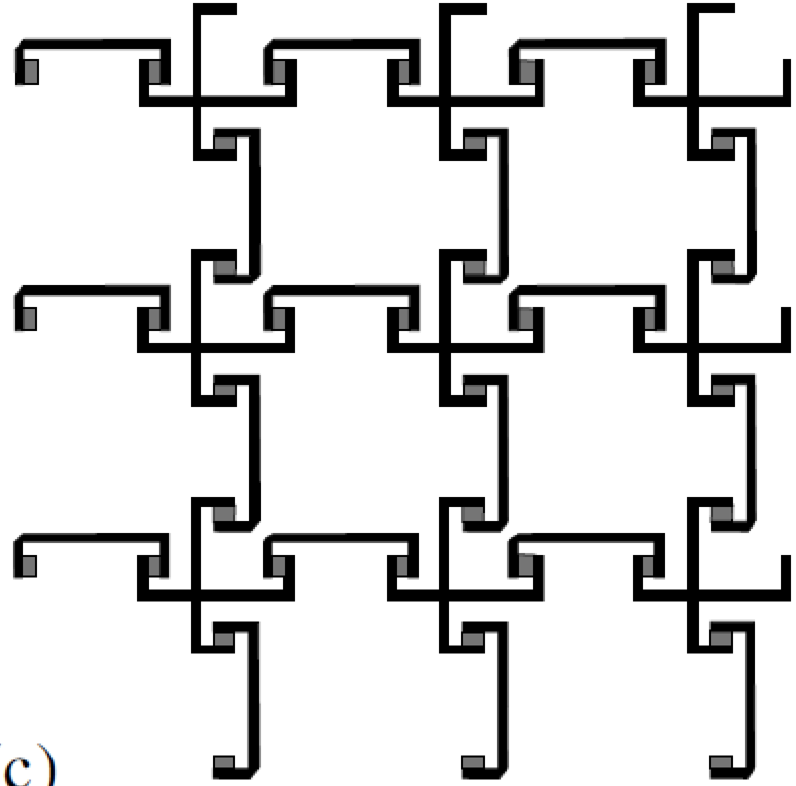
-Dispersions of metallic particles for optical effects
in stained glasses (Maxwell-Garnett, 1904 )
-Bubbly fluids for absorbing sound (masking submarine prop. noise)
-Split ring resonators for artificial magnetic
permeability (Schelkunoff and Friis, 1952)
-Wire metamaterials with artificial electric permittivity (Brown, 1953)
-Metamaterials with negative
and anisotropic mass densities (Auriault and Bonnet, 1985, 1994)
-Metamaterials with negative Poisson's ratio (Lakes 1987, Milton 1992)

What is new is the unprecedented ability to tailor-make structures
the explosion of interest, and the variety of emerging novel directions.

Another example: negative expansion from positive expansion

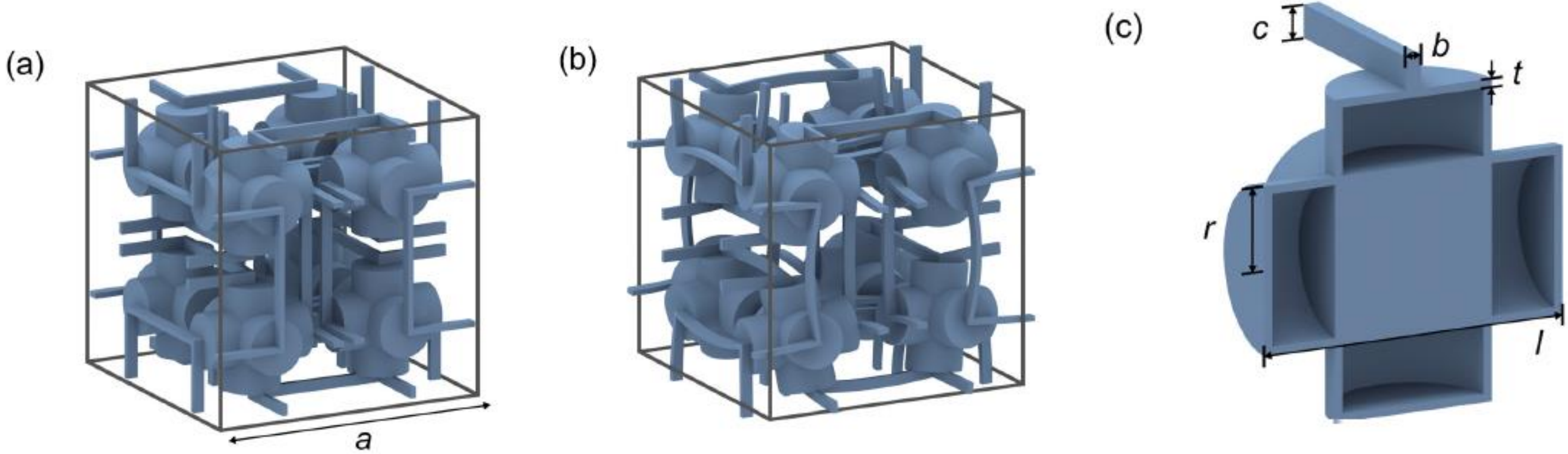

(a)


(b)


(c)

Original designs: Lakes (1996); Sigmund & Torquato (1996, 1997)

# One can get a similar effect for poroelasticity

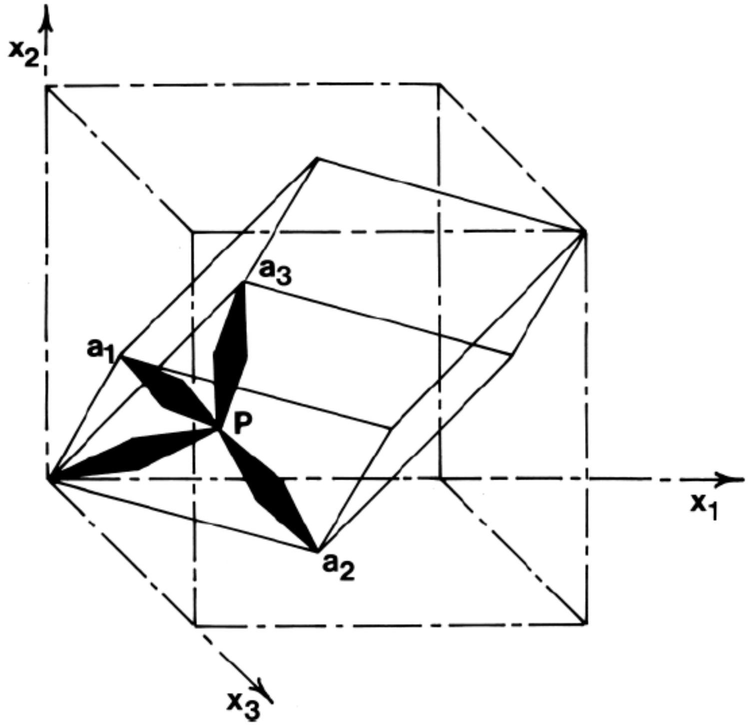



Qu, et.al 2017

# New classes of elastic materials (with Cherkaev, 1995)

Pentamodes, useful for guiding stress and the building block for getting any desired elasticity tensor.

A three dimensional pentamode material which can support any prescribed loading



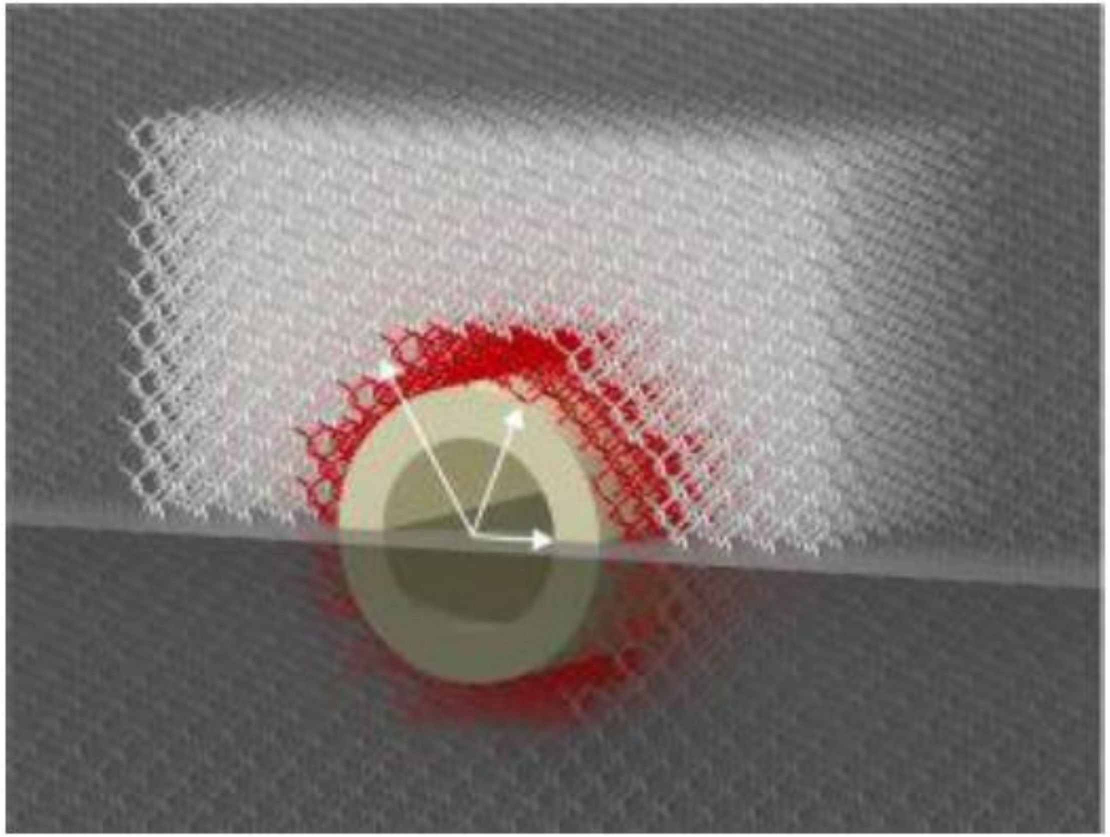

Many other important papers on pentamodes.

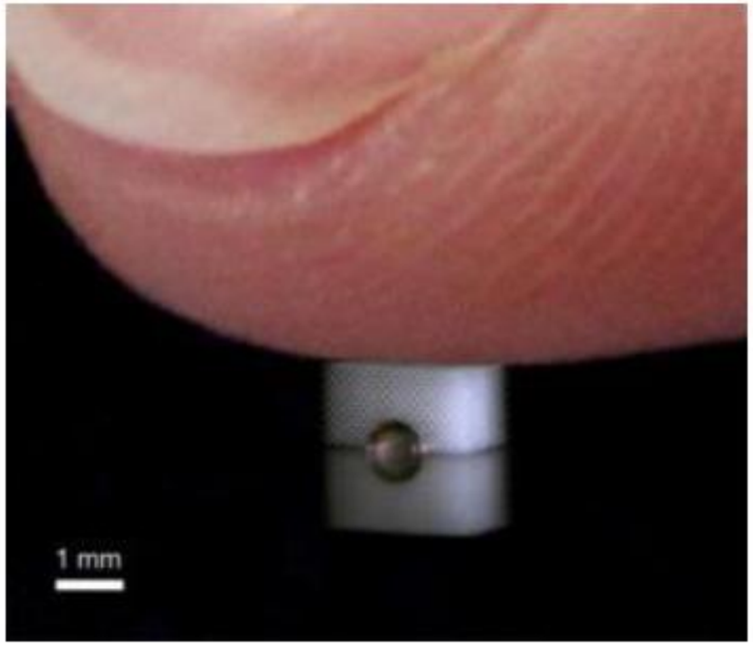
Like a fluid it only supports one loading, unlike a fluid that loading may be anisotropic. Desired support of a given anisotropic loading is achieved by moving P to another position in the unit cell.

KEY POINT is the coordination number of 4 at each vertex: the tension in one double cone connector, by balance of forces, determines uniquely the tension in the other 3 connecting double cones, and by induction the entire average stress field in the material.

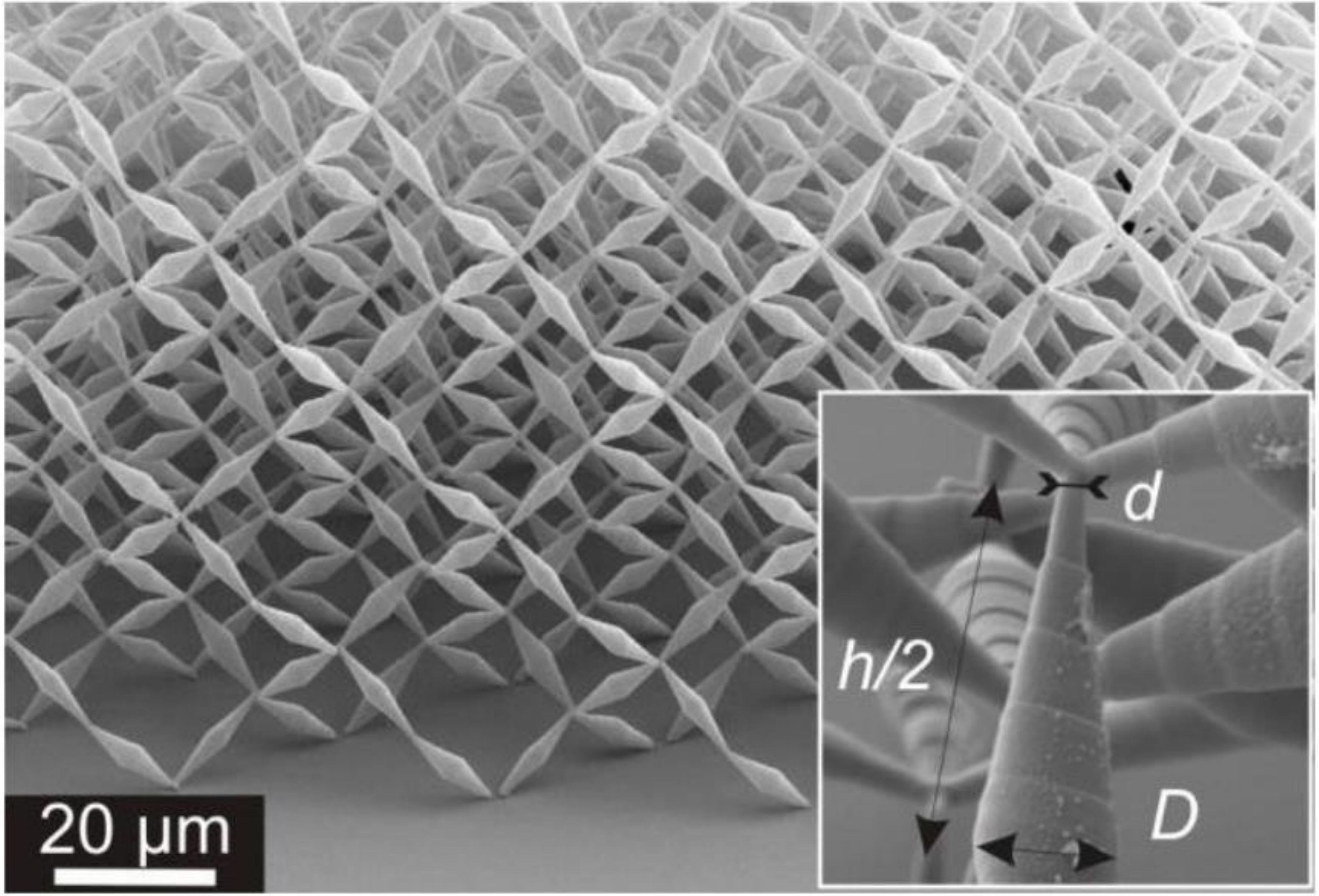
Application of Pentamodes:

Cloak making an object "unfeelable":
Buckmann et. al. (2014)



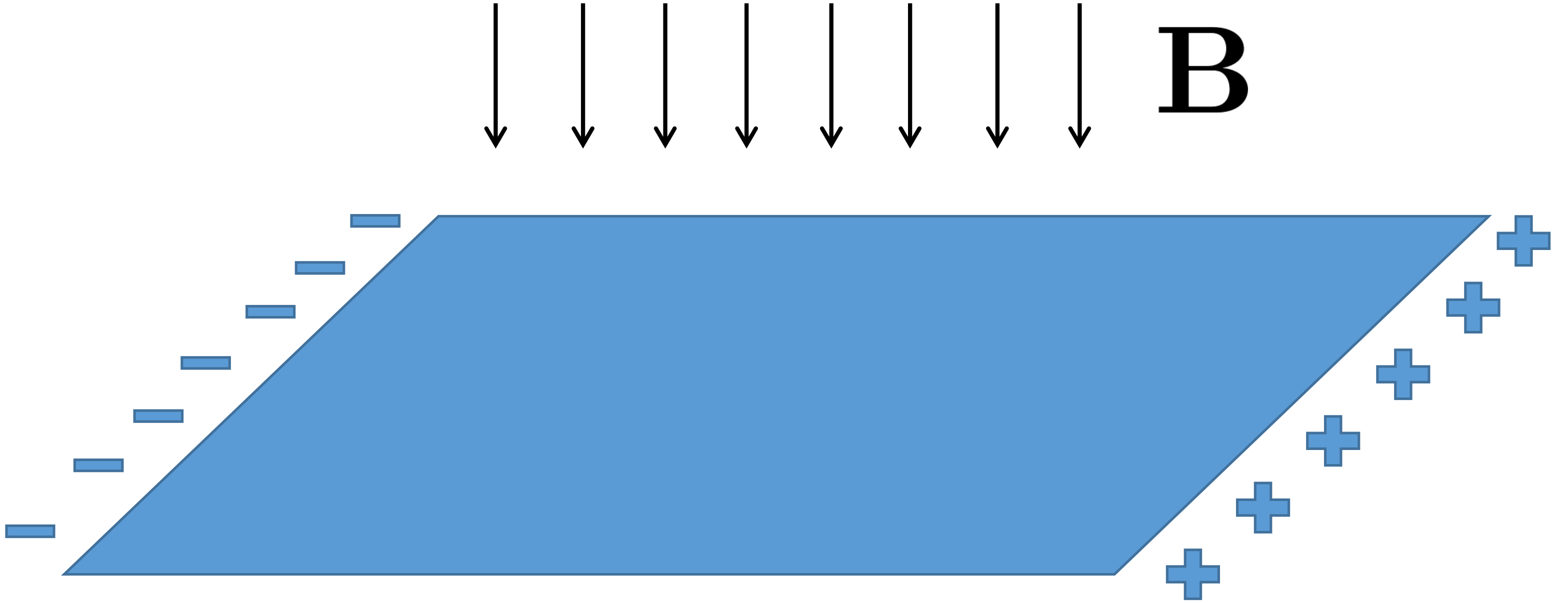


# Realization of Kadic et.al. 2012



It's constantly a surprise to find what properties a composite can exhibit.

One interesting example:




Hall Voltage

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

However there is a counterexample!

## Geometry suggested by artist Dylon Whyte

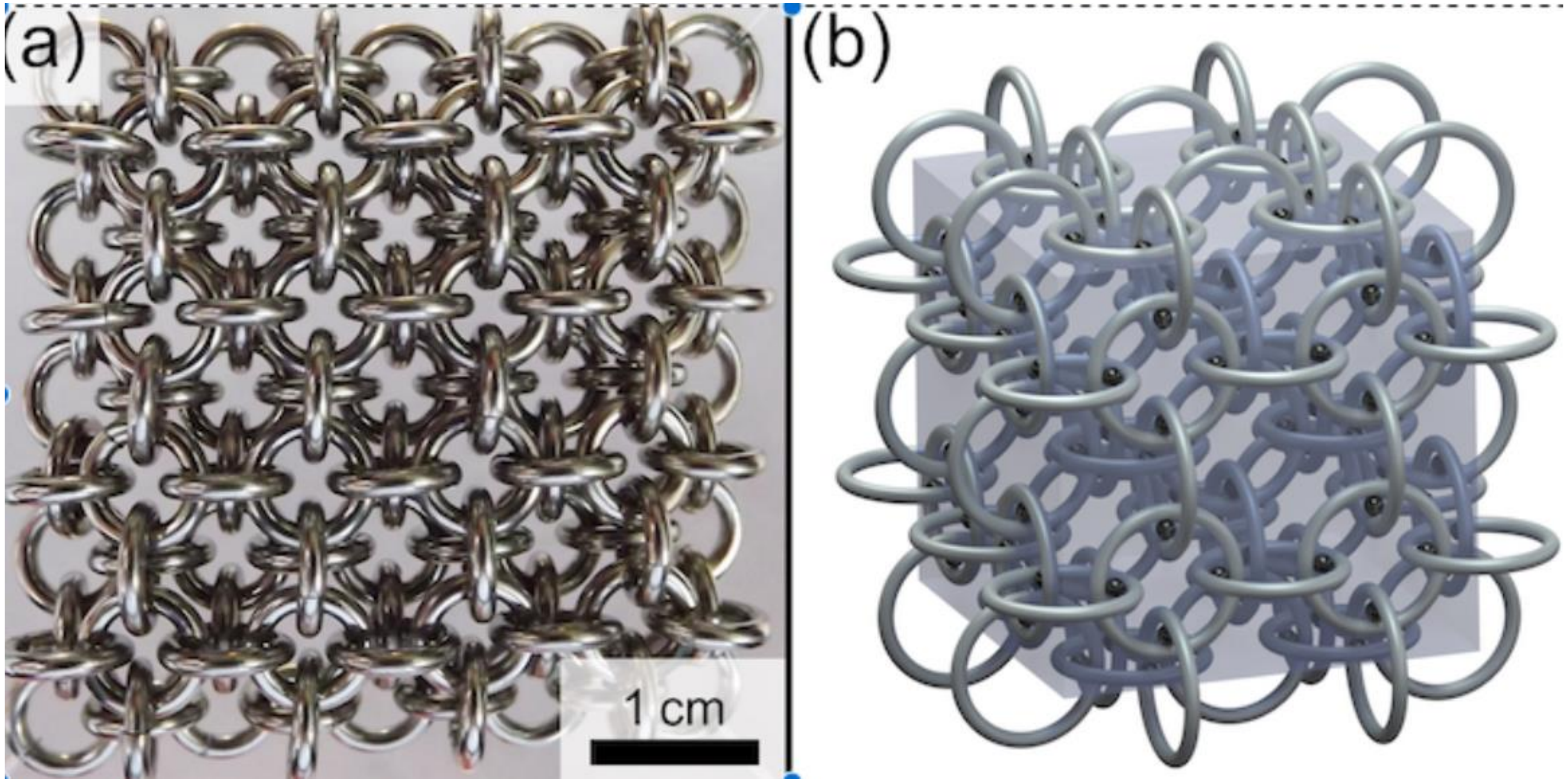
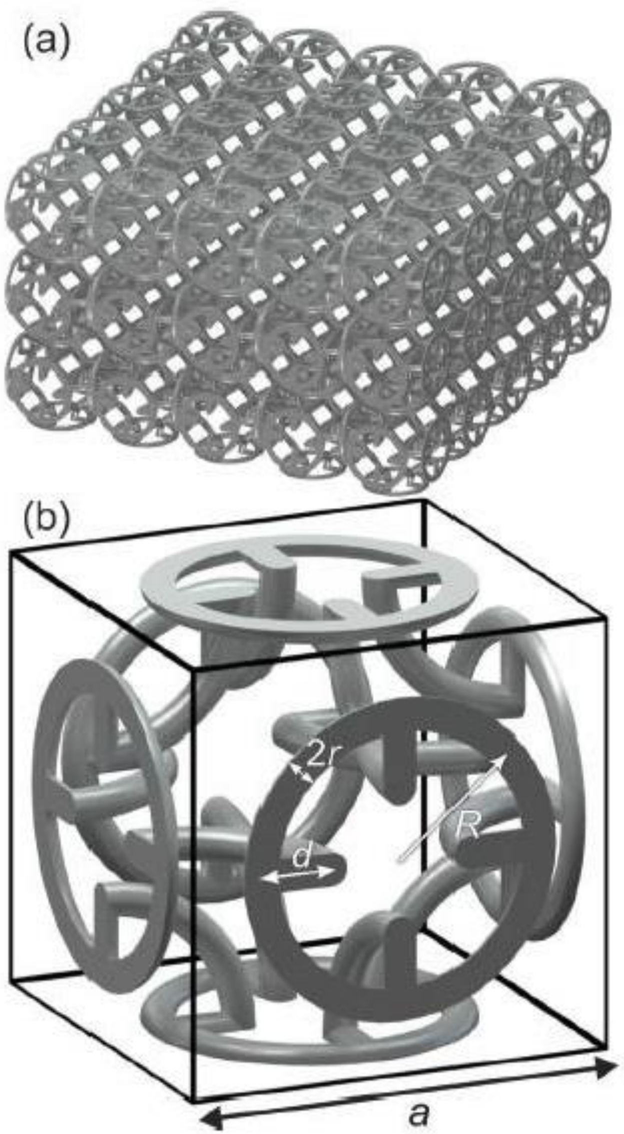
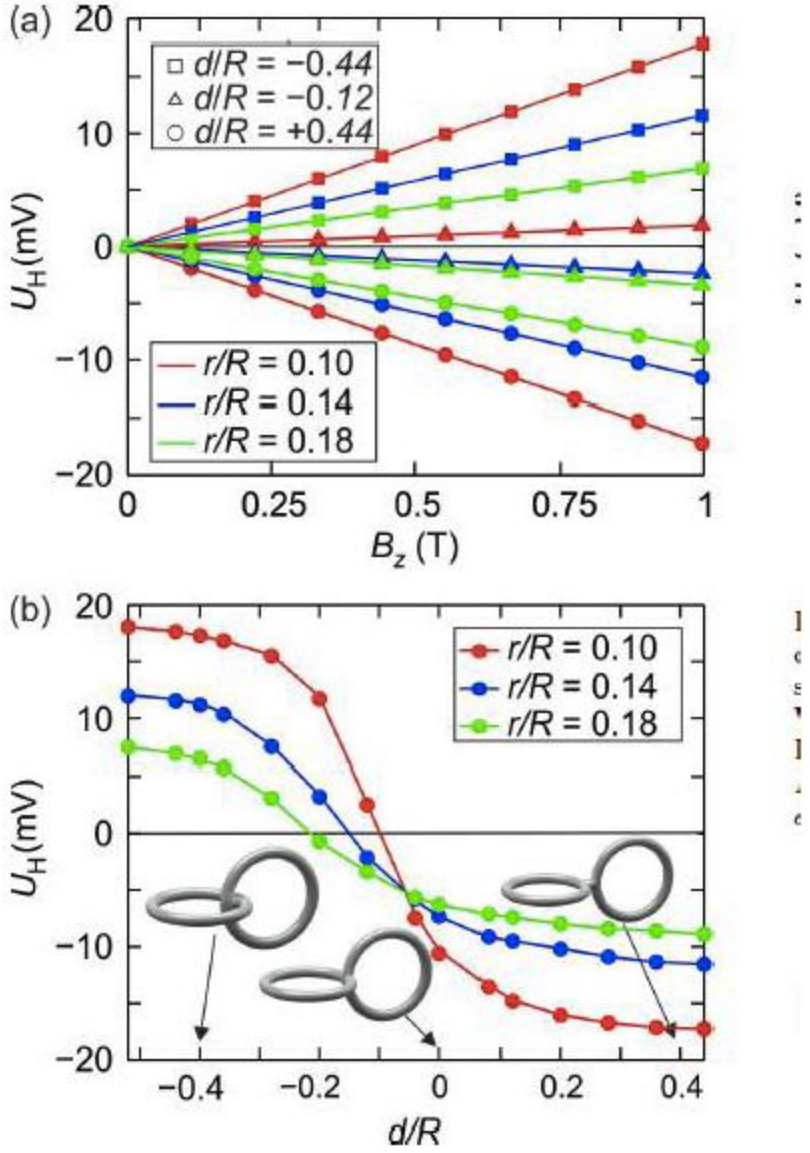


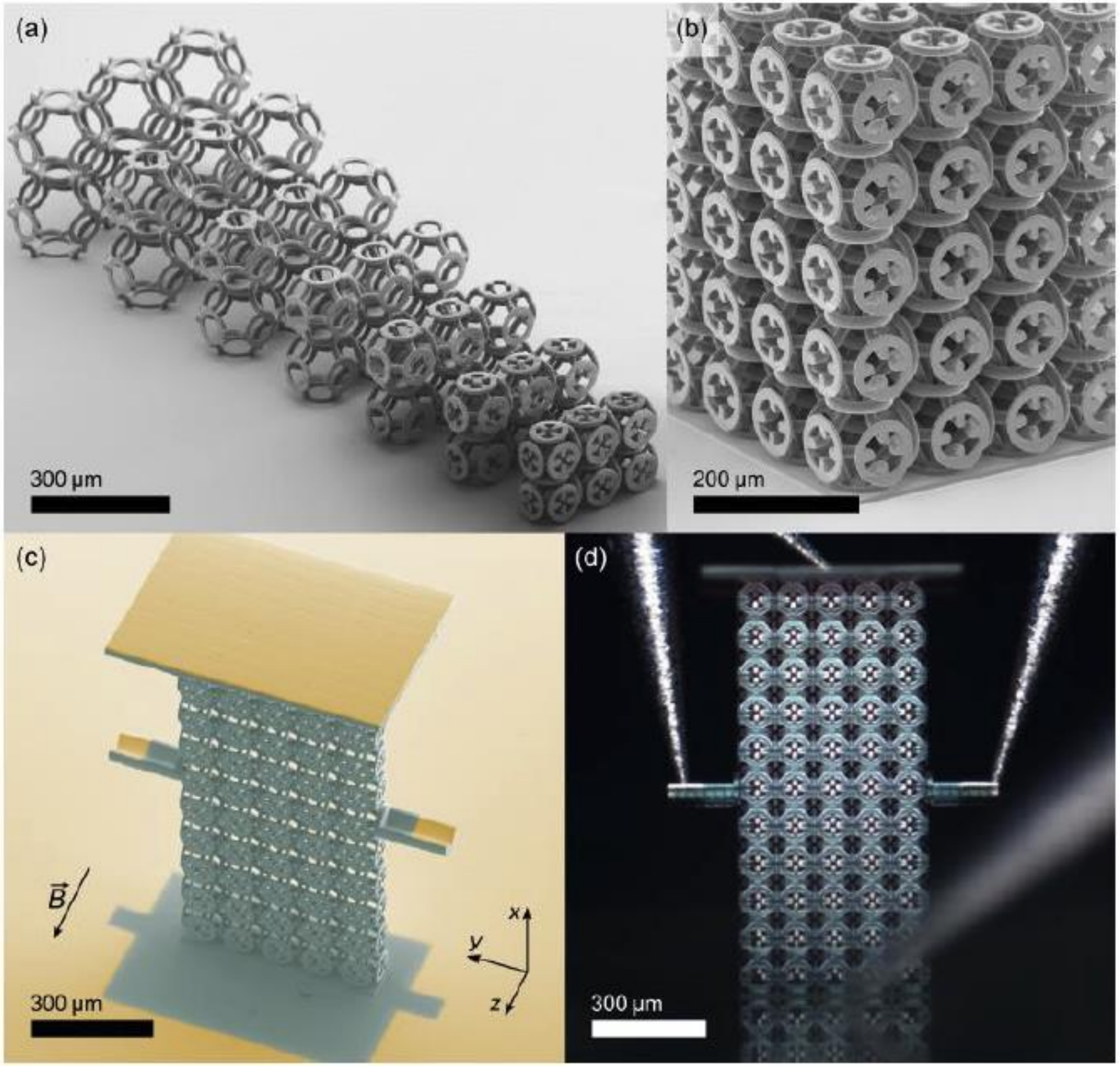

Image courtesy of Christian Kern

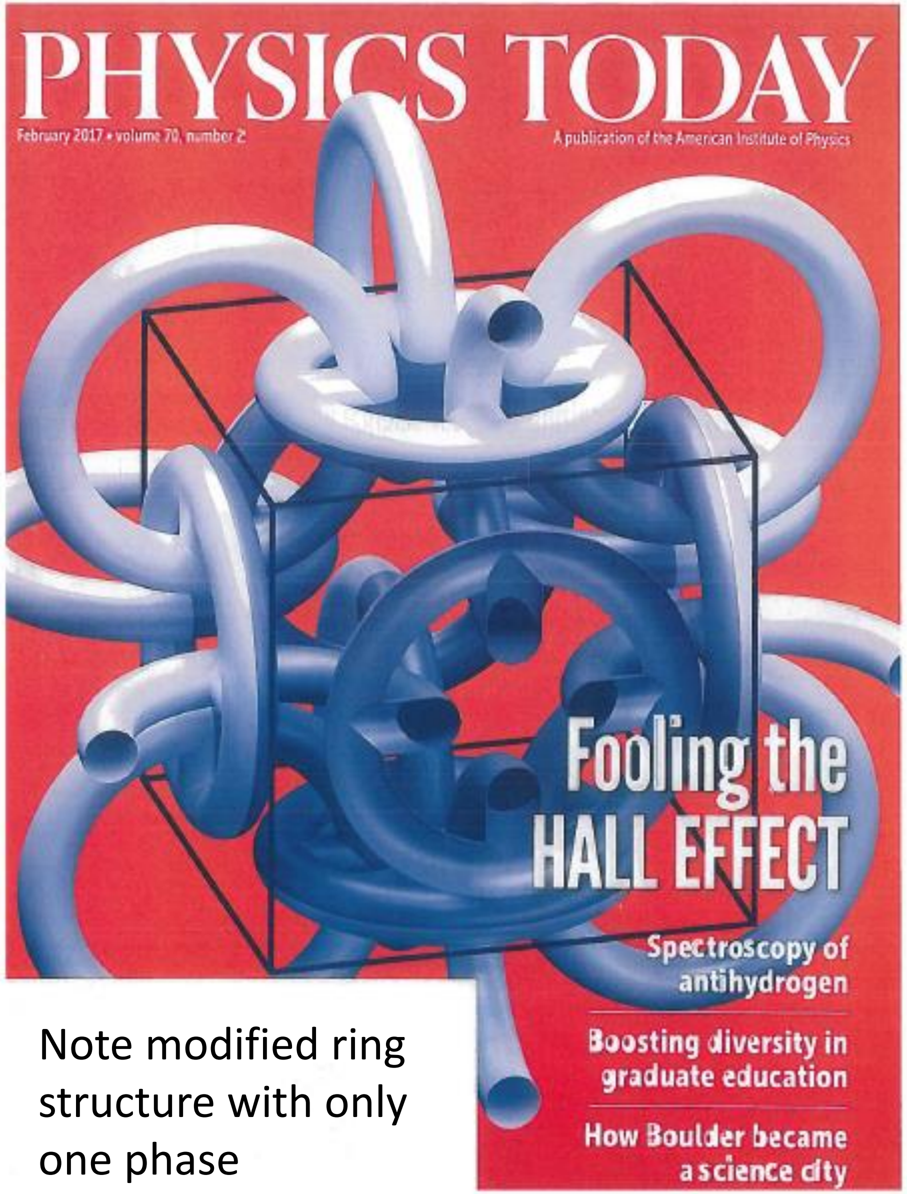
A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)

# Simplification of Kadic et.al. (2015)






Note modified ring structure with only one phase

# Experimental Realization of Kern, Kadic, Wegener

# Two natural problems:

(1) Concentrating a field into a region.

(2) Shielding a region from fields.

Sharp corners concentrate fields

Large Fields also very important for Raman Spectroscopy:

Effect goes as the $4^{th}$ power of the field intensity.

Well known that rough surfaces enhance Raman Spectroscopy,
by orders of magnitude (SERS)

Shielding: Think of Faraday cage to shield Electromagnetic Field,
Shielding from Magnetic Fields, Thermal Currents
Shielding from Vibrations, Sonar

How to measure this?
Threshold exponents on $L^\gamma$ integrability:

$$\gamma^- \equiv \inf_\gamma : \int_B |\mathbf{E}(\mathbf{x})|^\gamma \, d\mathbf{x} < \infty$$

$$\gamma^+ \equiv \sup_\gamma : \int_B |\mathbf{E}(\mathbf{x})|^\gamma \, d\mathbf{x} < \infty$$

$B$ is any Ball containing $\Omega$.

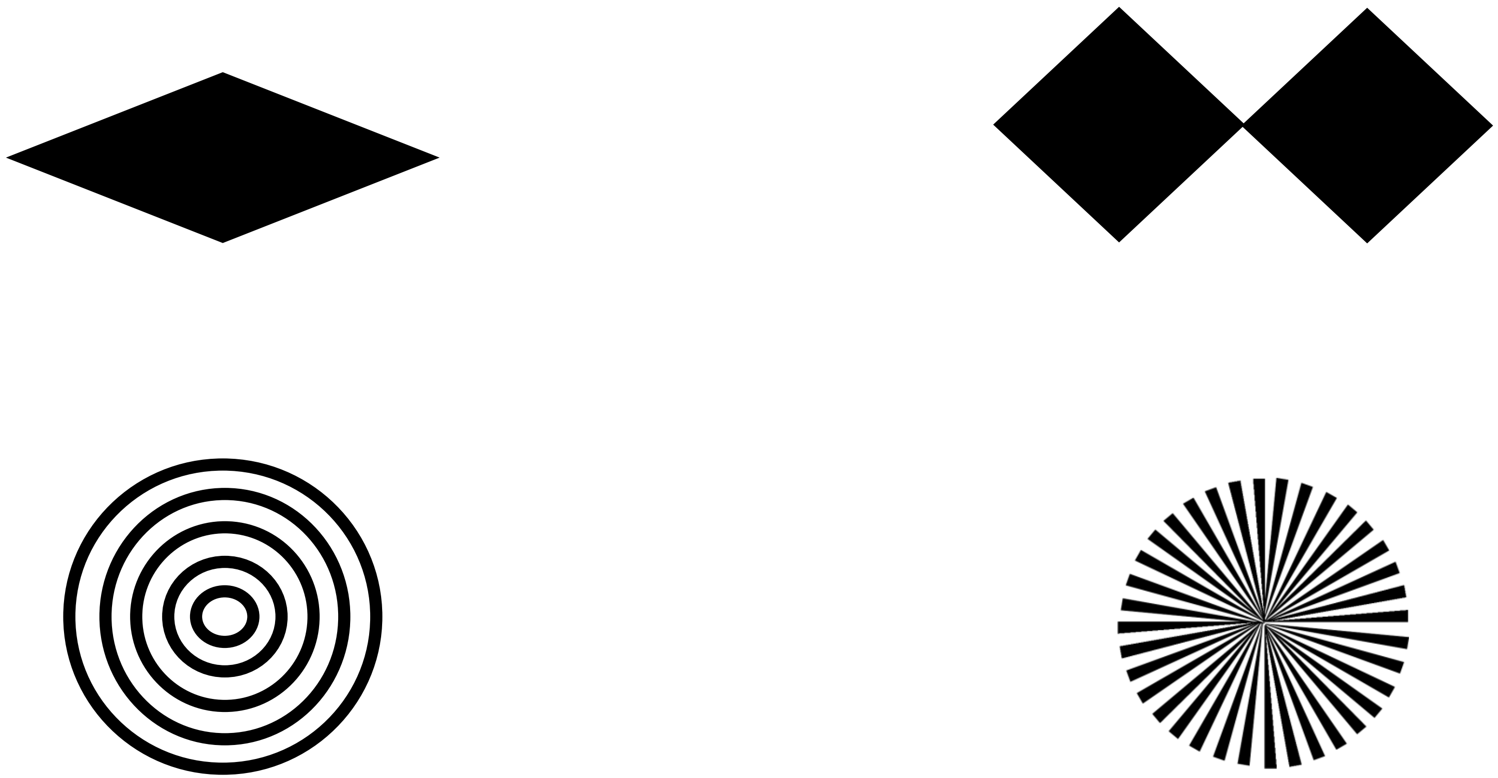
Equivalently, given a (possibly disconnected) subregion $Q \subset \Omega$ of small subvolume $|Q|$ one can maximize or minimize

$$\int_Q |\mathbf{E}(\mathbf{x})|^2 \, d\mathbf{x}$$

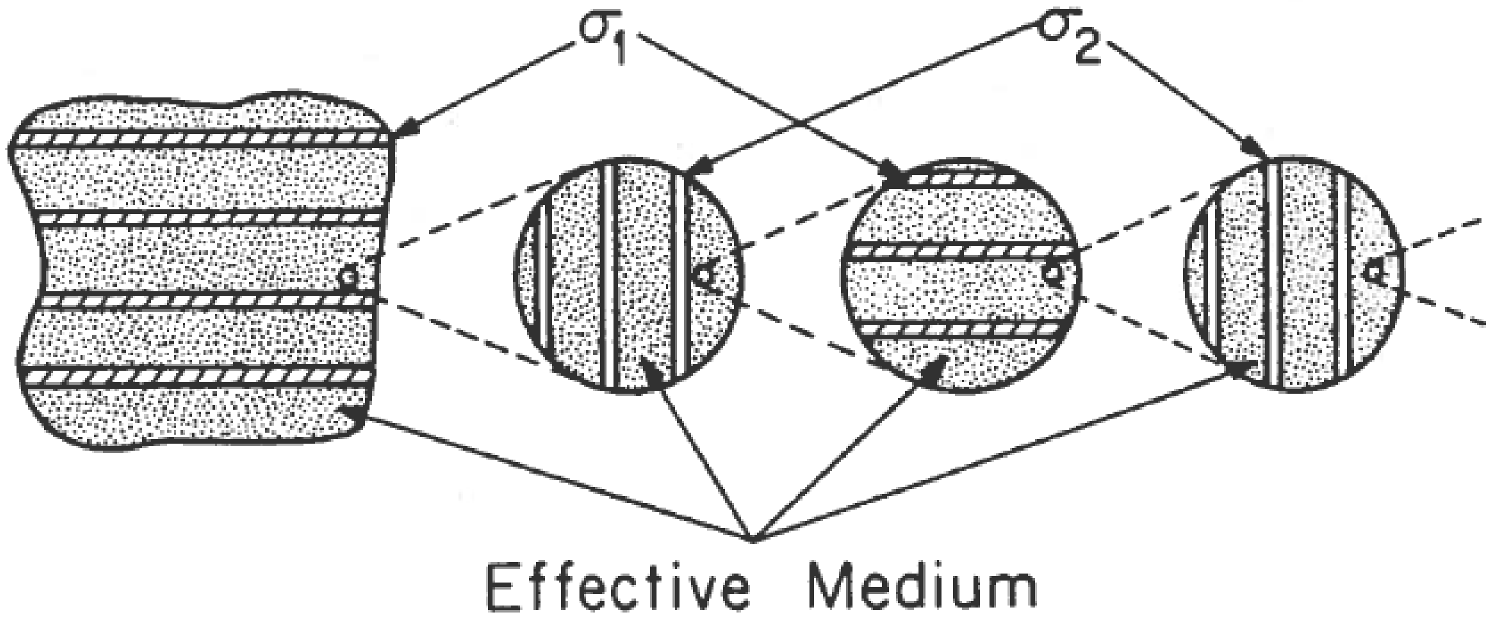
and ask how this depends on $|Q|$ asymptotically as $|Q| \to 0$

Two isotropic conductors, conductivities $\sigma_1$, $\sigma_2$.
Uniform field at infinity

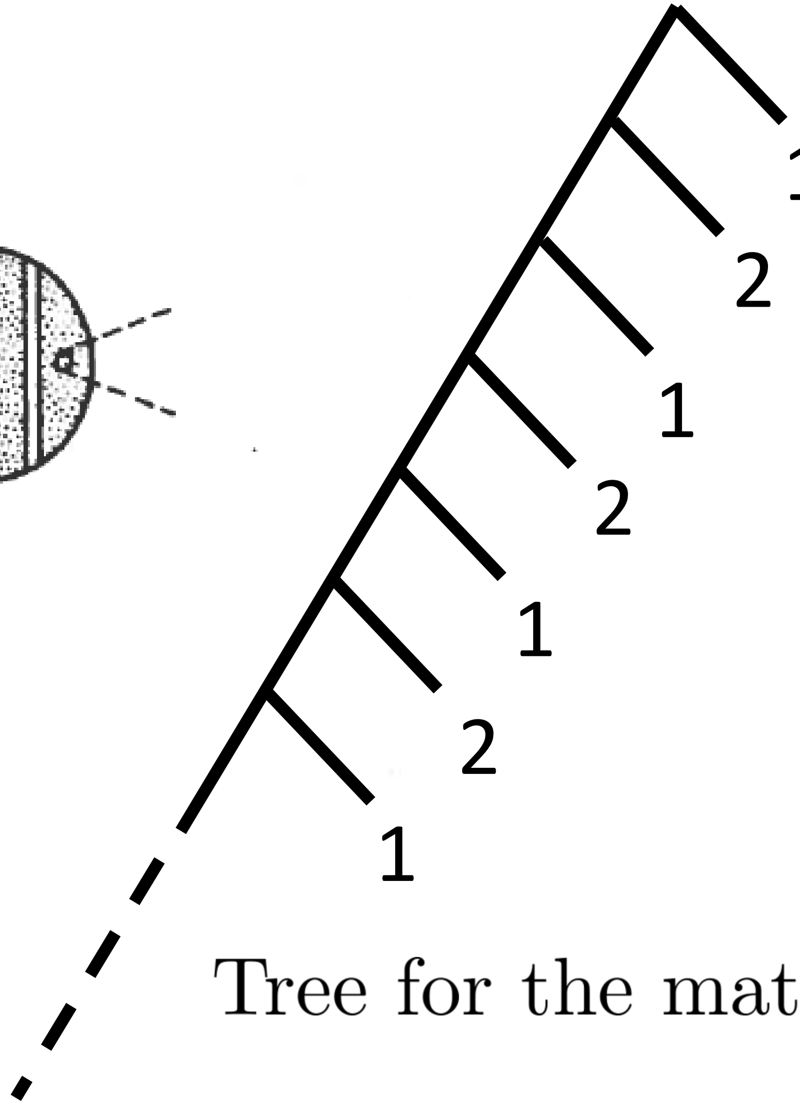
Some Candidates:



# Best:




GWM (1986)




Tree for the material

# Beauty Contest (GWM, 1986):

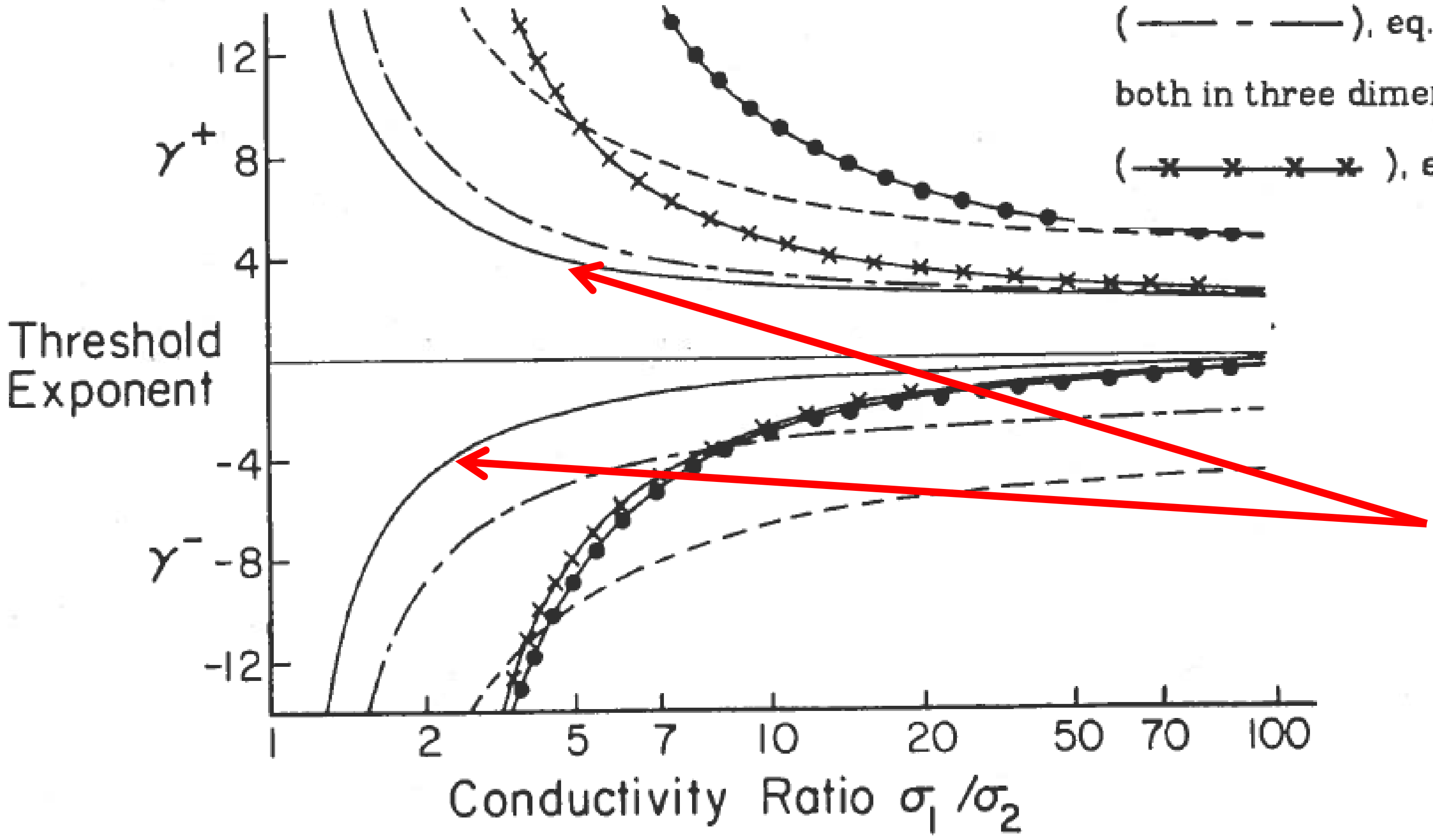



Fig.9. Comparison of threshold exponents for the laminate of Fig.8. (————), eqs. (4.13) and (4.18); an array of diamond shaped grains (– – – – –), eq. (4.8); a checkerboard of the two components (—— - ——), eq. (4.9); and Schulgasser's symmetric material, both in three dimensions (—•—•—•—•—) and in two dimensions (—×—×—×—×—), eqs. (4.11) and (4.12).

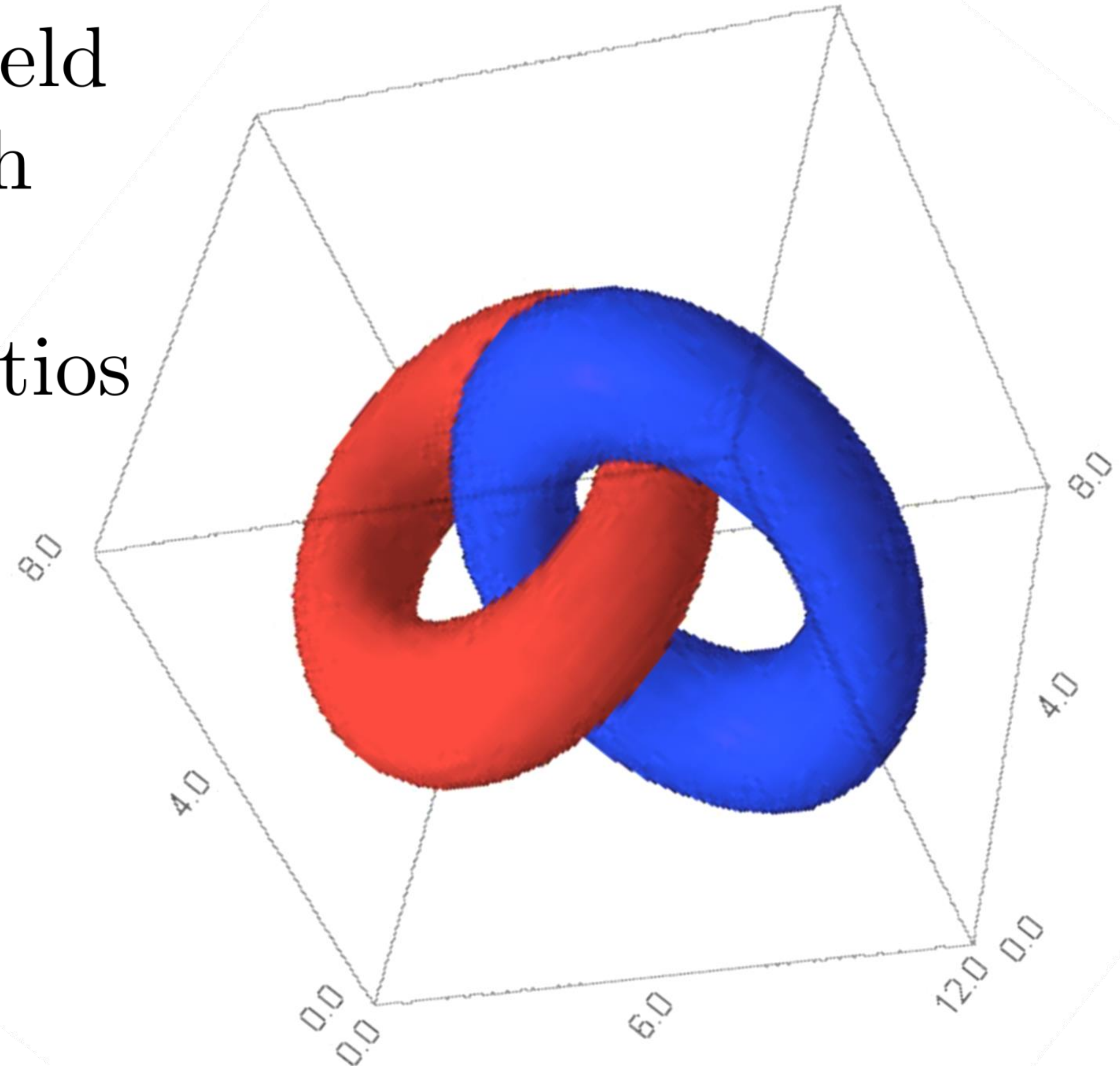
~~Conjectured~~ bounds
Now Established

Proof of this micostructure independent Lower Bound on $\gamma^+$: Morrey (1938); Boyarski (1957)

Proof of this microstructure Upper Bound on $\gamma^-$: Leonetti and Nesi (1997)

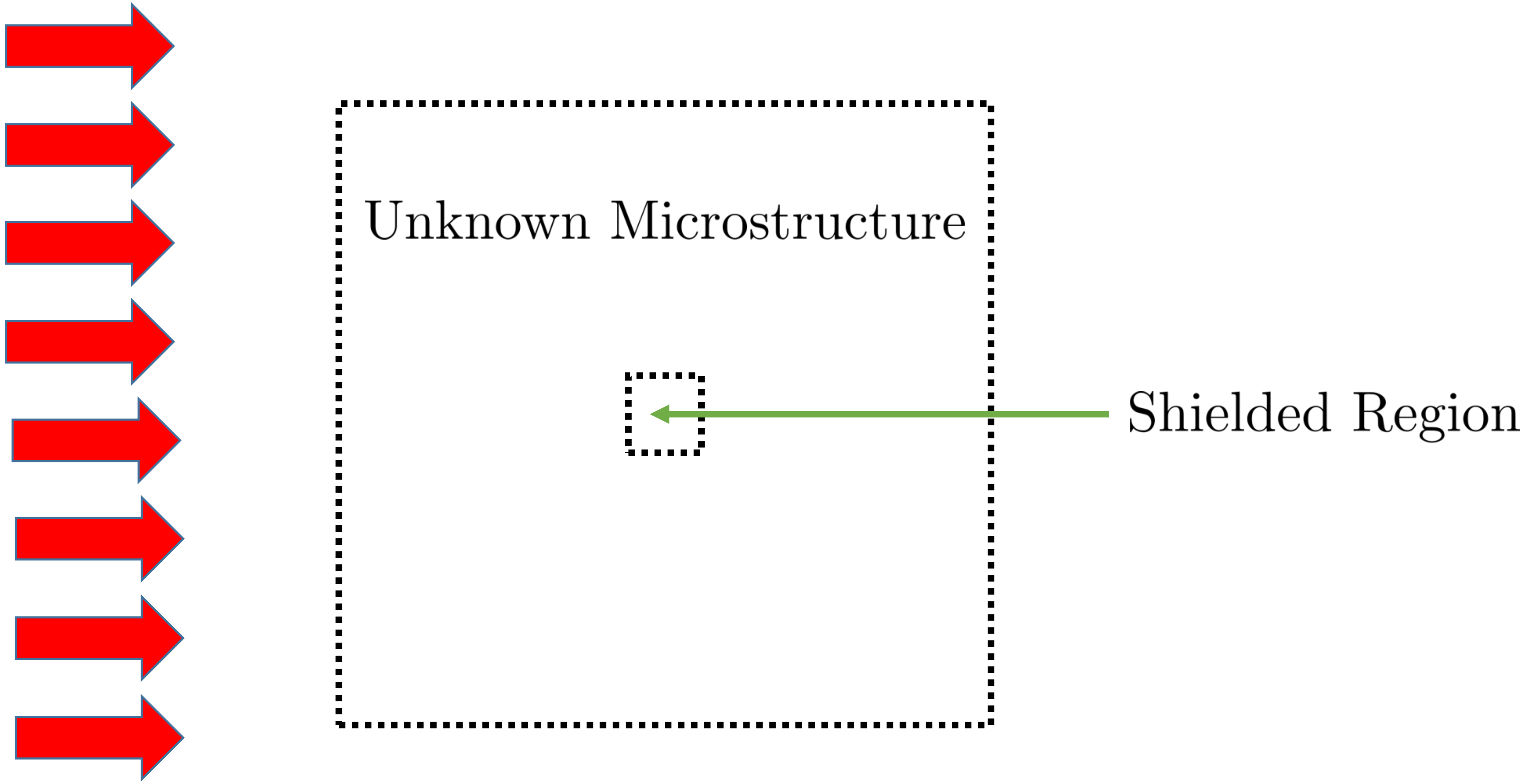
See Also Faraco (2003)

What about 3d?
For a uniform applied field the local field can vanish between the torii, even at finite conductivity ratios

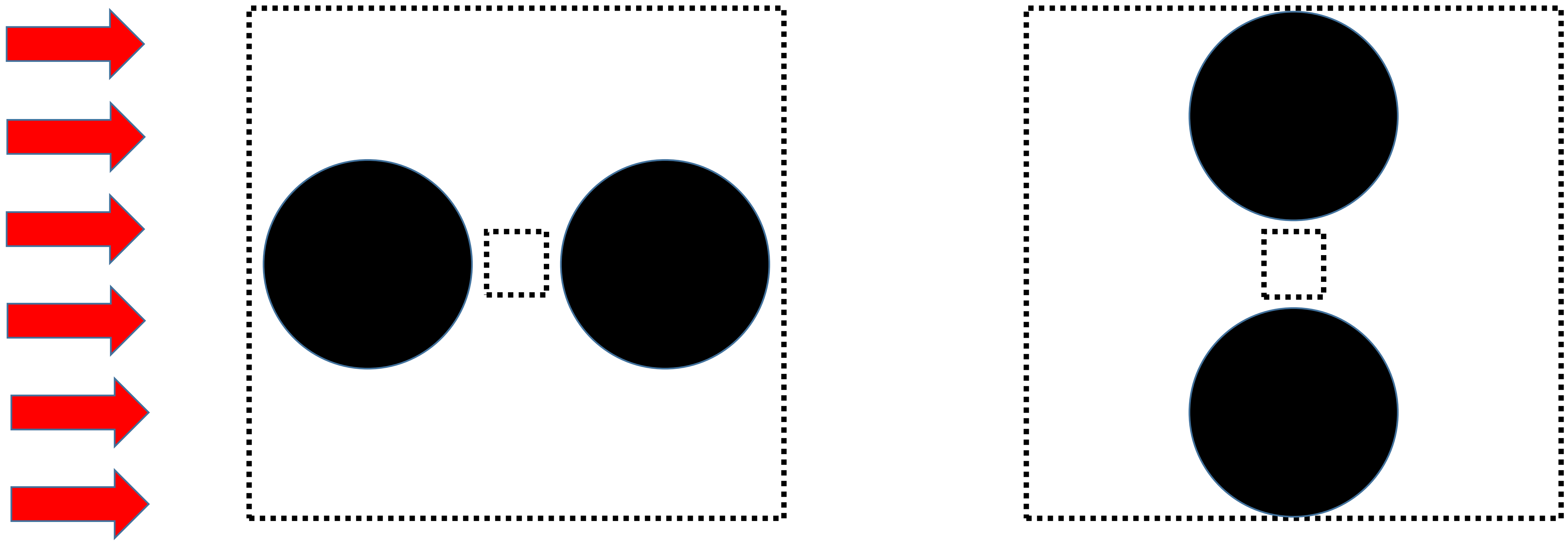


Back to the shielding problem:

It seems more reasonable to require that there is no microstructure in the shielded region and that the microstructure is localized in a box.



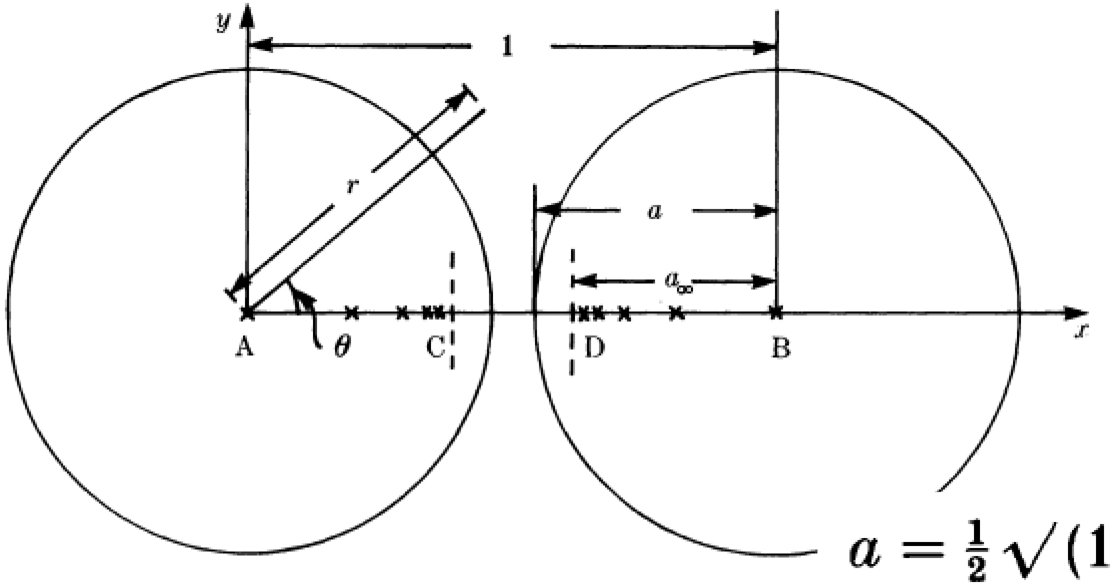
# Using Disks:



Concentration

Shielding

# Field between two highly conducting disks close to touching




$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}.$$

$$\rho_-(a^2/x) = -\eta\rho_+(x)$$

$$\eta = (\sigma-1)/(\sigma+1).$$

McPhedran, Poladian, GWM (1988)

$$\rho_+(1-x) = -\rho_-(x),$$

$$\rho_-[a^2/(1-x)] = \eta\rho_-(x)$$

$$B_1 = \frac{-(c/2)\,(1-1/c)}{2s\ln(c)+1-2s[\gamma+\psi(1+s)]}.$$

$$\rho_-(x) = A[(a_\infty - x)/(1-a_\infty - x)]^s$$

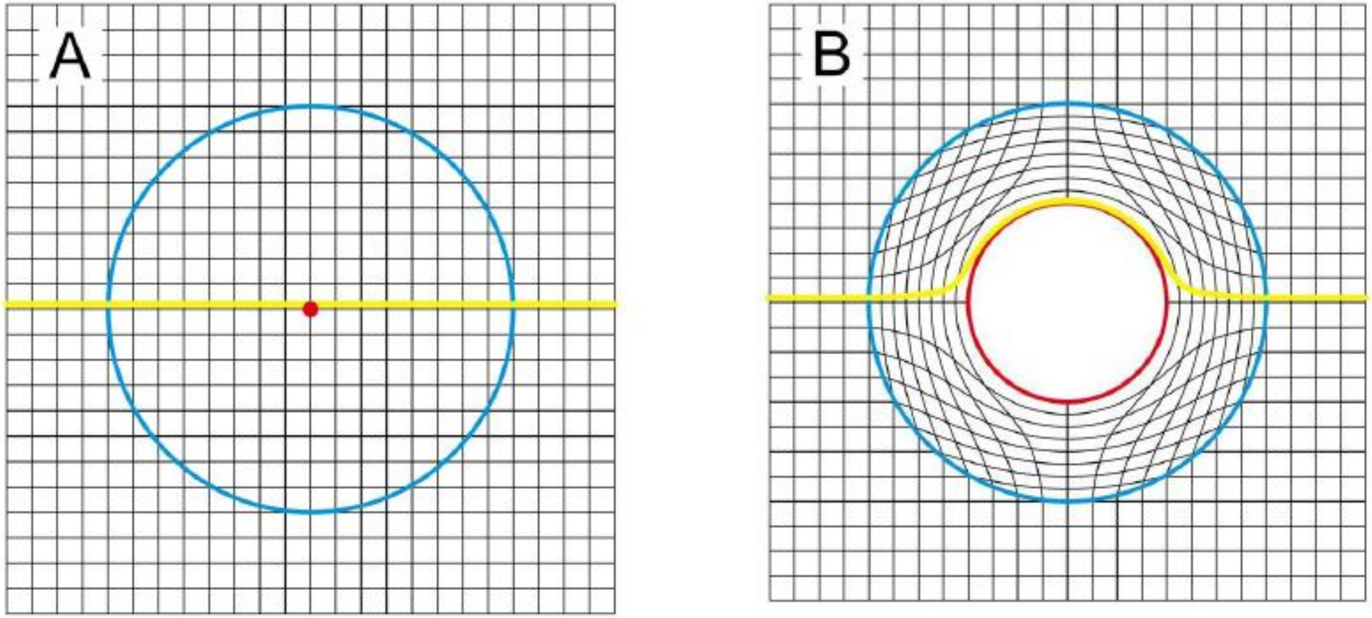
$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}. \quad a_\infty = \tfrac{1}{2}(1-1/c).$$

$\psi$: Psi or Digamma function

$$s = \ln(\eta)/\ln[a_\infty/(1-a_\infty)]$$

Rigorous Analysis: Lim and Yu (2015)

Could use the transformation based approach of Greenleaf, Lassas, and Uhlmann

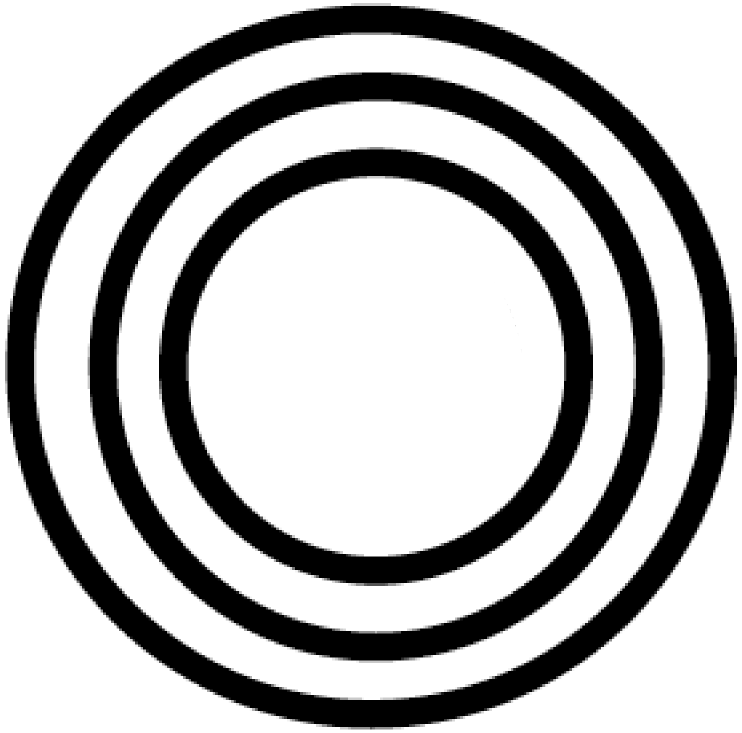


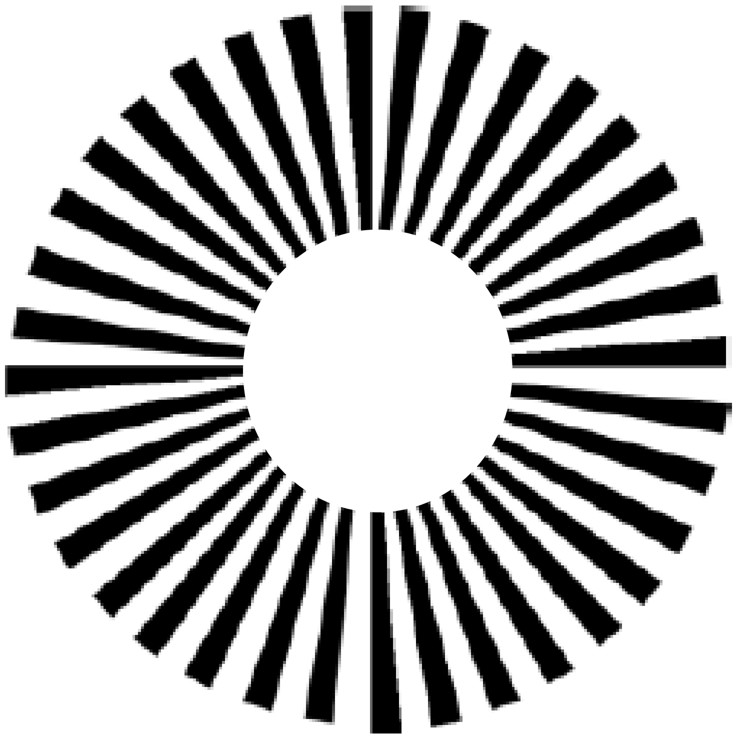

Stretching space (From Ulf Leonhardt)

Advantages: Works for any external field and creates no disturbance

Disadvantages: Requires extreme conductivities, and if one truncates the solution there is no reason to expect it is optimal.

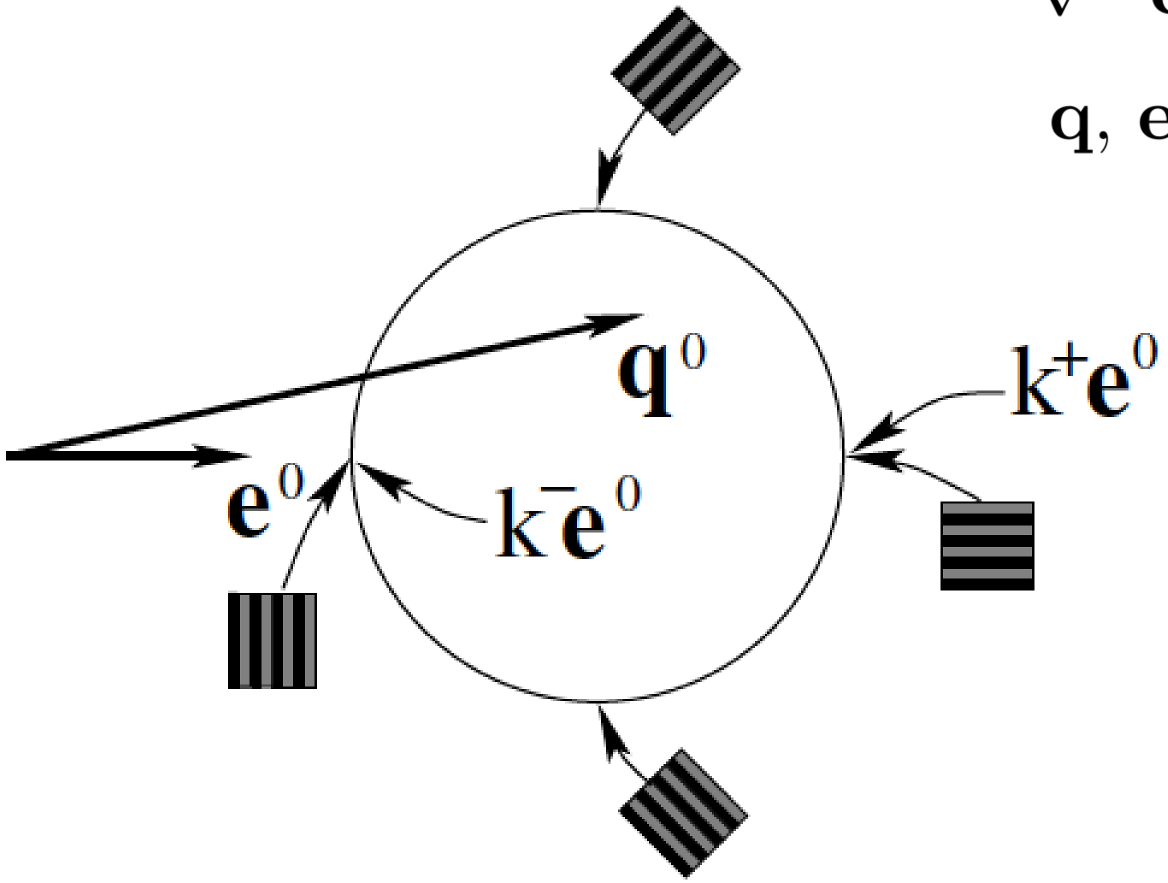
# Or Maybe?





Seems like we are just guessing. Is there a more systematic approach, at least in the case where we use just 2 conducting materials, and we are seeking shielding or concentration for just one applied field?

Possible (average heat current, $\mathbf{q}^0$, average temperature gradient, $\mathbf{e}^0$) pairs in a two phase conducting composite (Raitum, 1978).




$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q}(\mathbf{x}) = k(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \mathbf{e} = -\nabla T$$

$\mathbf{q}$, $\mathbf{e}$ periodic, $\langle \mathbf{q} \rangle = \mathbf{q}^0, \quad \langle \mathbf{e} \rangle = \mathbf{e}^0$,

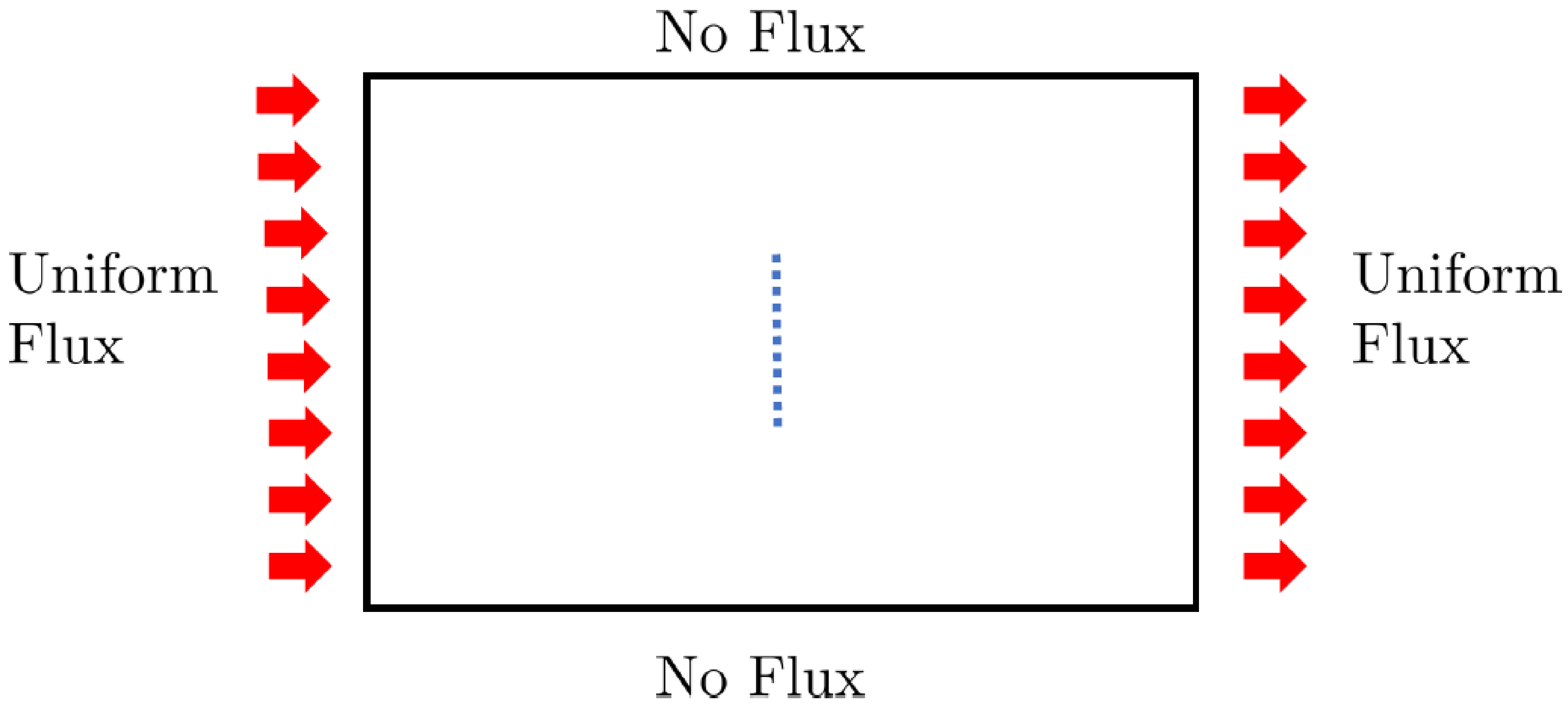
Follows from the Wiener bounds:

$$\begin{aligned} k^- \mathbf{I} &\leq \mathbf{k}^* \leq k^+ \mathbf{I} \\ k^+ &= f k_1 + (1-f) k_2 \\ k^- &= (f/k_1 + (1-f)/k_2)^{-1} \end{aligned}$$

Solution of the "weak G-closure" problem for conductivity

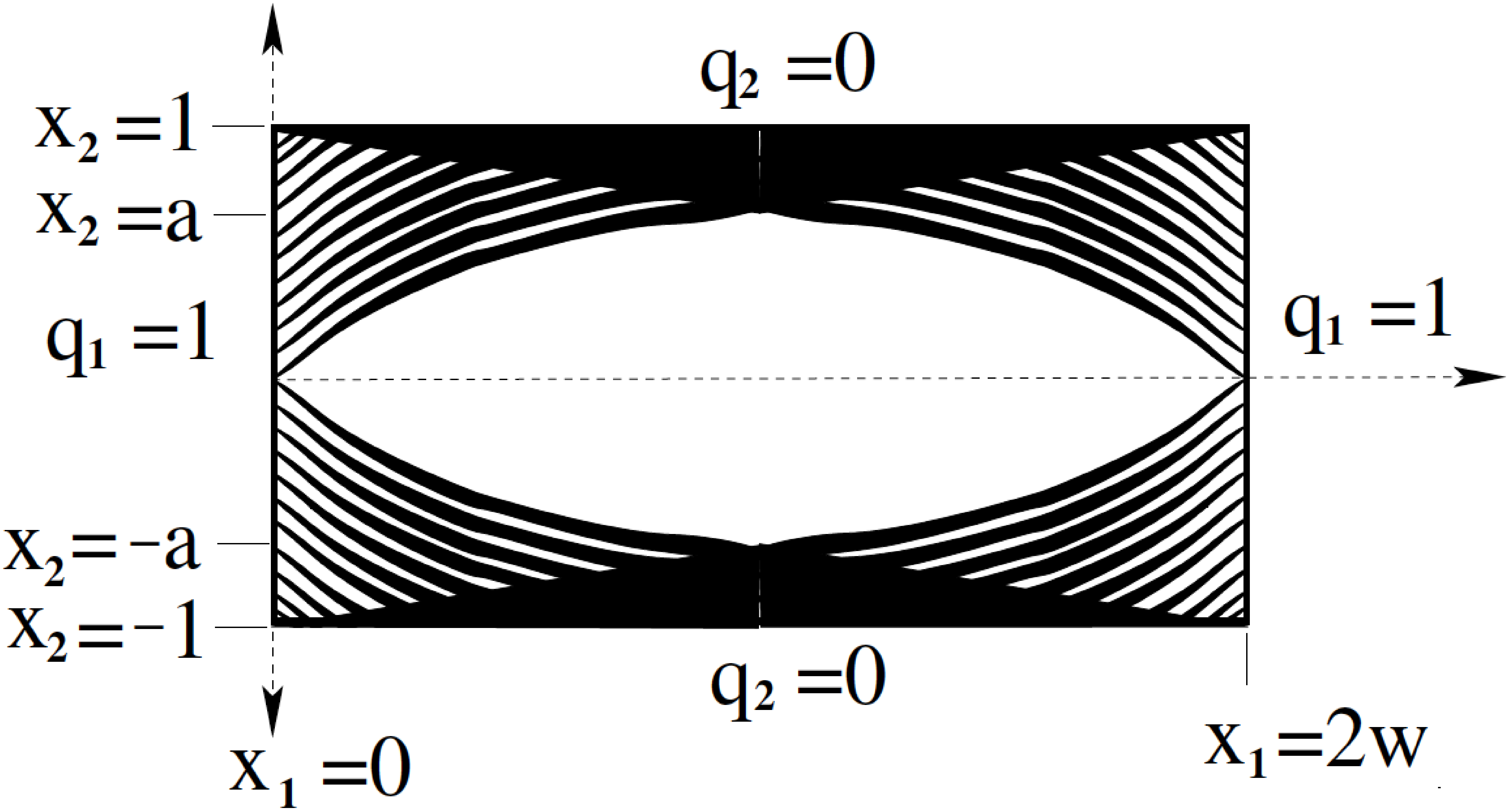
# The heat lens problem: Gibiansky, Lurie and Cherkaev (1988)

Aim: Shield or concentrate flux in the blue dashed interval

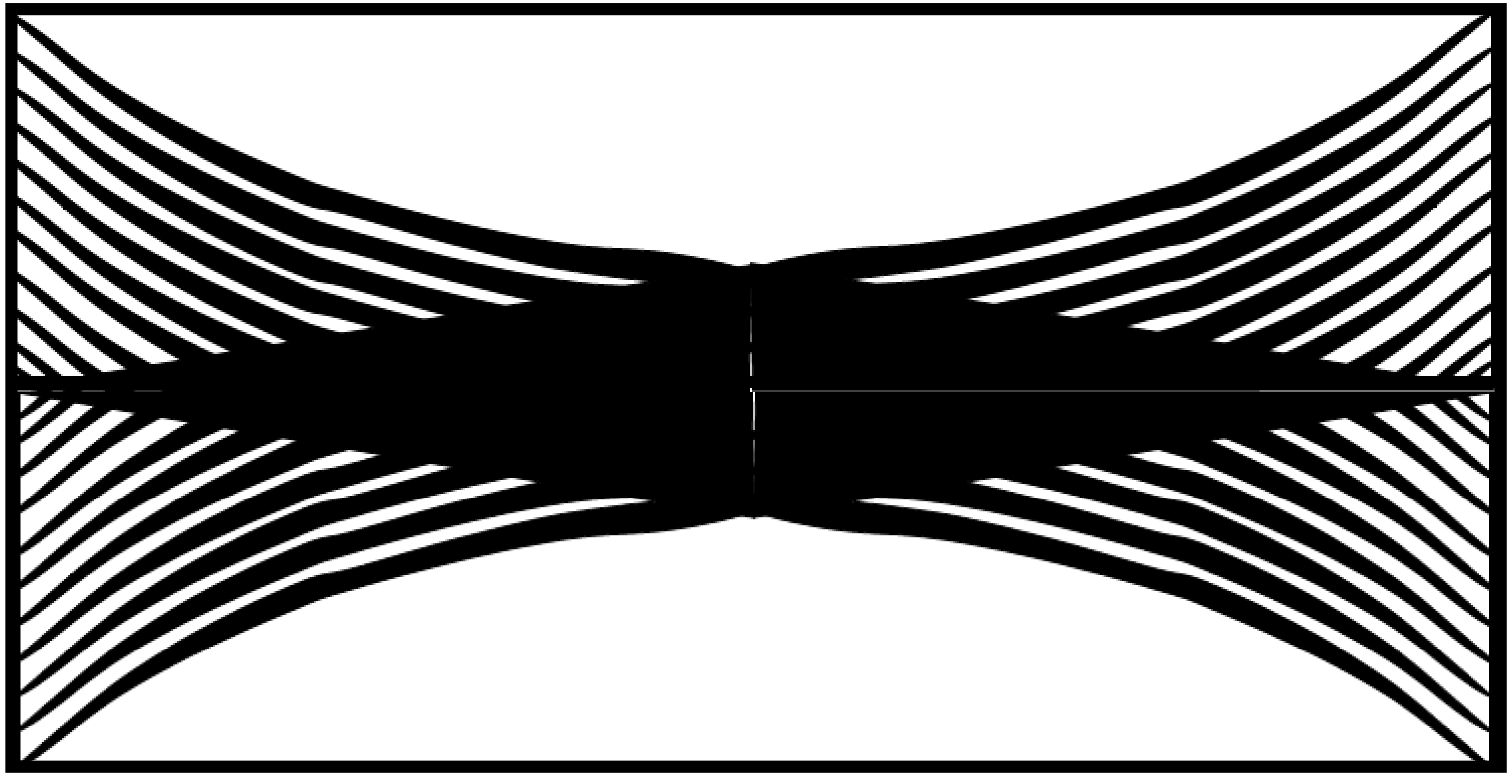



How does one optimally distribute a poor and good conductor to do this?

# Field Shield: (Black, good conductor)



# Field Concentrator:



# The Hall matrix.

In anisotropic materials one has

$$\boldsymbol{e} = \rho_0 \boldsymbol{j} + (\boldsymbol{A}_{\mathrm{H}} \boldsymbol{b}) \times \boldsymbol{j} \qquad \boldsymbol{j} = \boldsymbol{\sigma}_0 \boldsymbol{e} + (\boldsymbol{S} \boldsymbol{b}) \times \boldsymbol{e}$$

What are the constraints on the Hall matrix?

Can one use metamaterials to get unusual Hall matrices?

# Homogenization formula for the effective Hall matrix

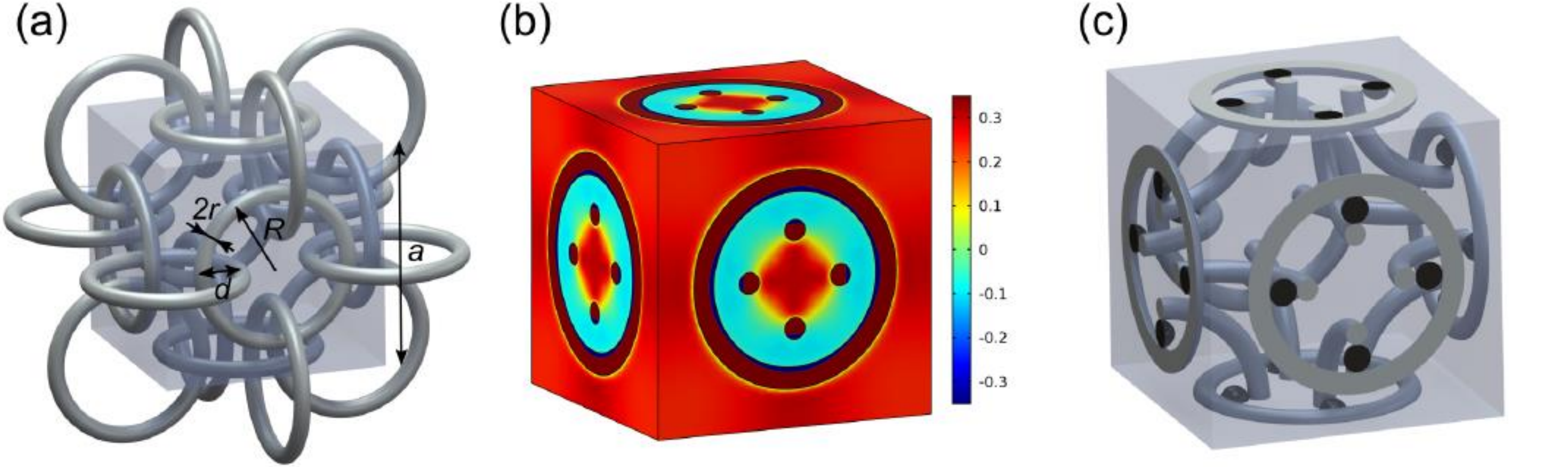
$$\boldsymbol{e} = -\left(\nabla\boldsymbol{\Phi}\right)\langle\boldsymbol{e}\rangle, \quad \left(\nabla\boldsymbol{\Phi}\right)_{ij} = \frac{\partial\phi_j}{\partial x_i} \qquad \nabla\cdot\left(\boldsymbol{\sigma}\nabla\boldsymbol{\Phi}\right) = 0 \qquad \langle\nabla\boldsymbol{\Phi}\rangle = \boldsymbol{I}$$

$$\langle \mathrm{Cof}\left(\boldsymbol{\sigma}_0\nabla\boldsymbol{\Phi}\right)^{\mathsf{T}}\boldsymbol{A}_{\mathrm{H}}\rangle = \mathrm{Cof}\left(\boldsymbol{\sigma}_0^*\right)\boldsymbol{A}_{\mathrm{H}}^*.$$

Generalizes a formula of Bergman for isotropic materials:

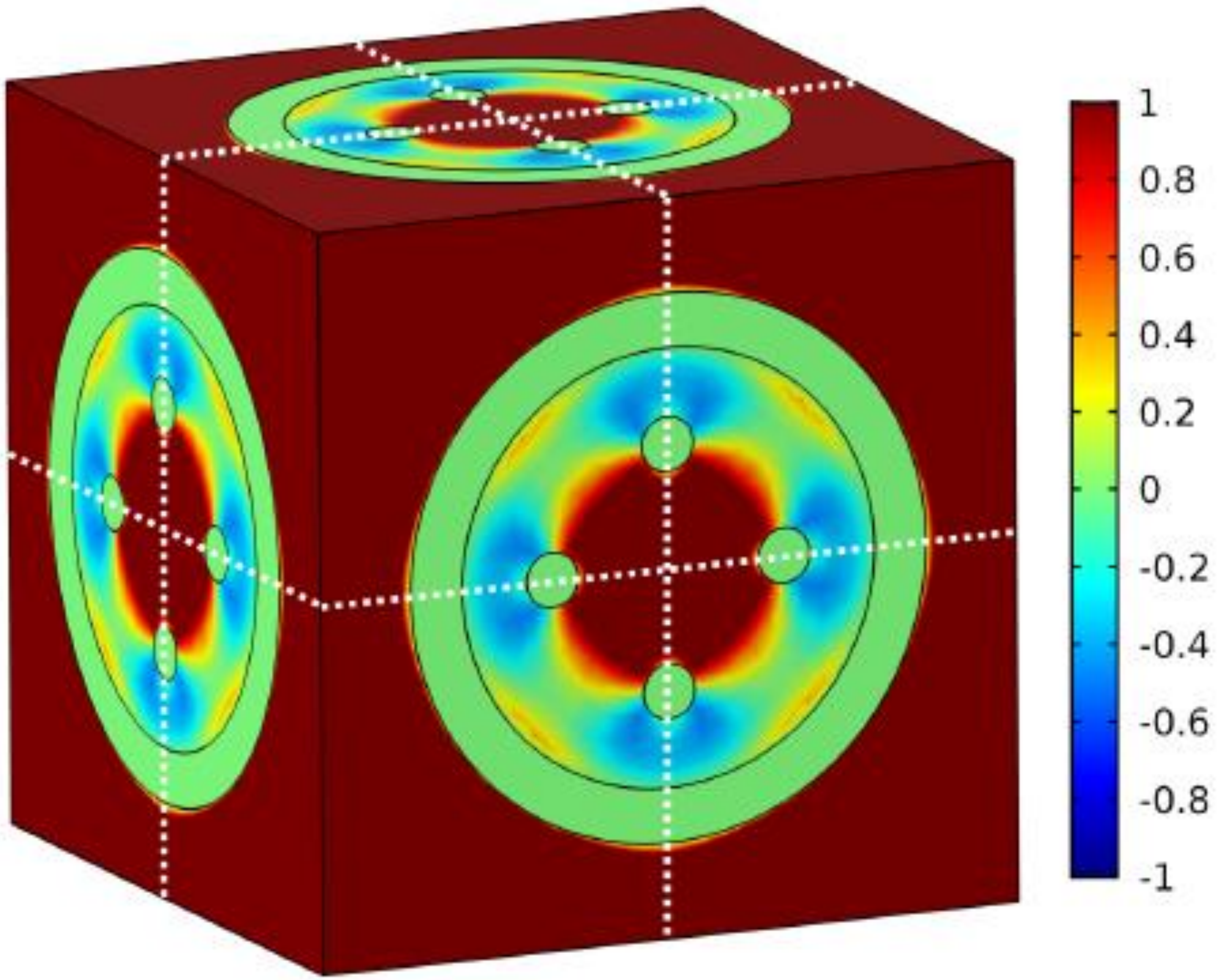
$$\langle\left(J_{11}J_{22} - J_{21}J_{12}\right)A_{\mathrm{H}}\rangle = \left(\sigma_0^*\right)^2 A_{\mathrm{H}}^*$$

# Geometry studied by Briane and Milton (2009)

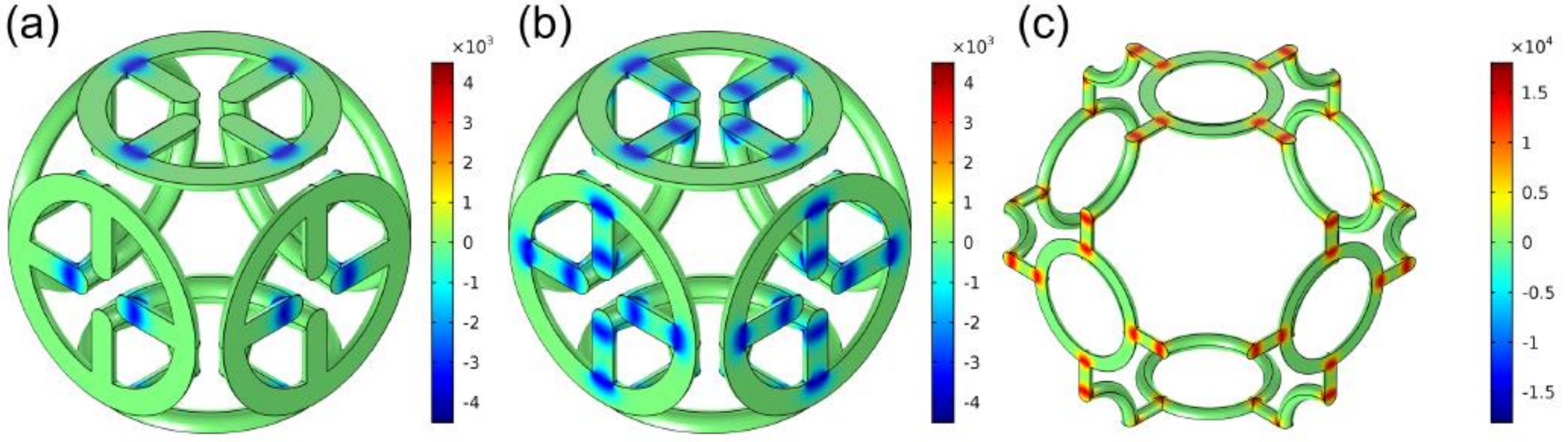



Suggestion to use this geometry came from chain-mail artist Dylon Whyte

# Plot of the determinant of the matrix valued electric field



# Plot of the cofactor matrices

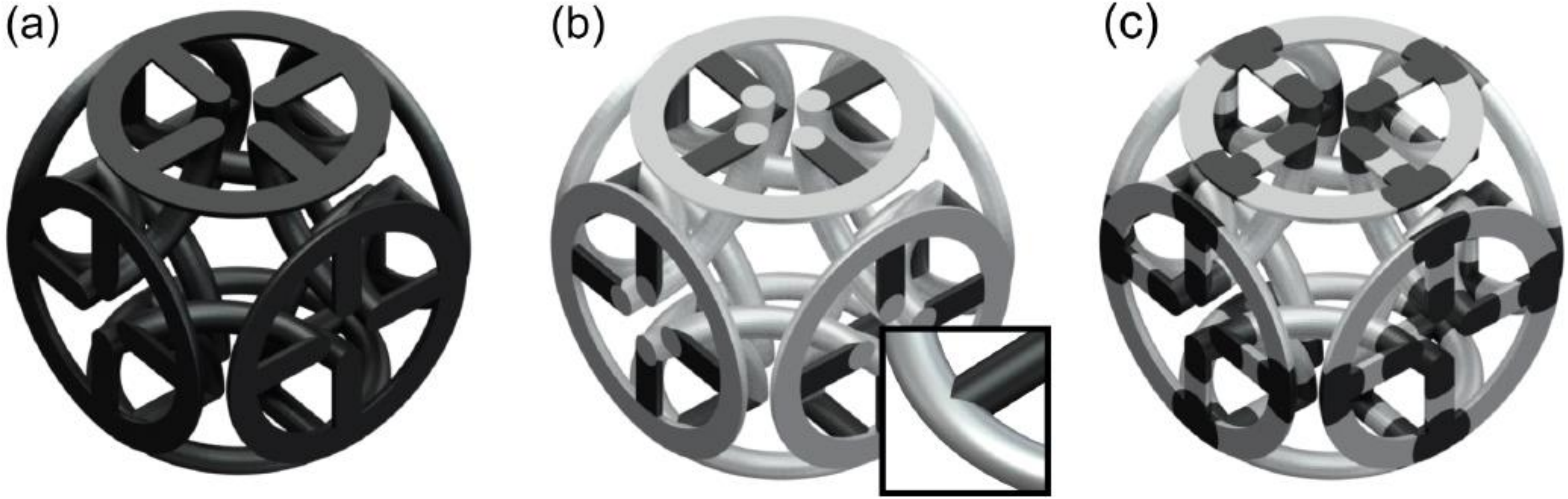



One cofactor

Cofactor trace

Cofactor trace

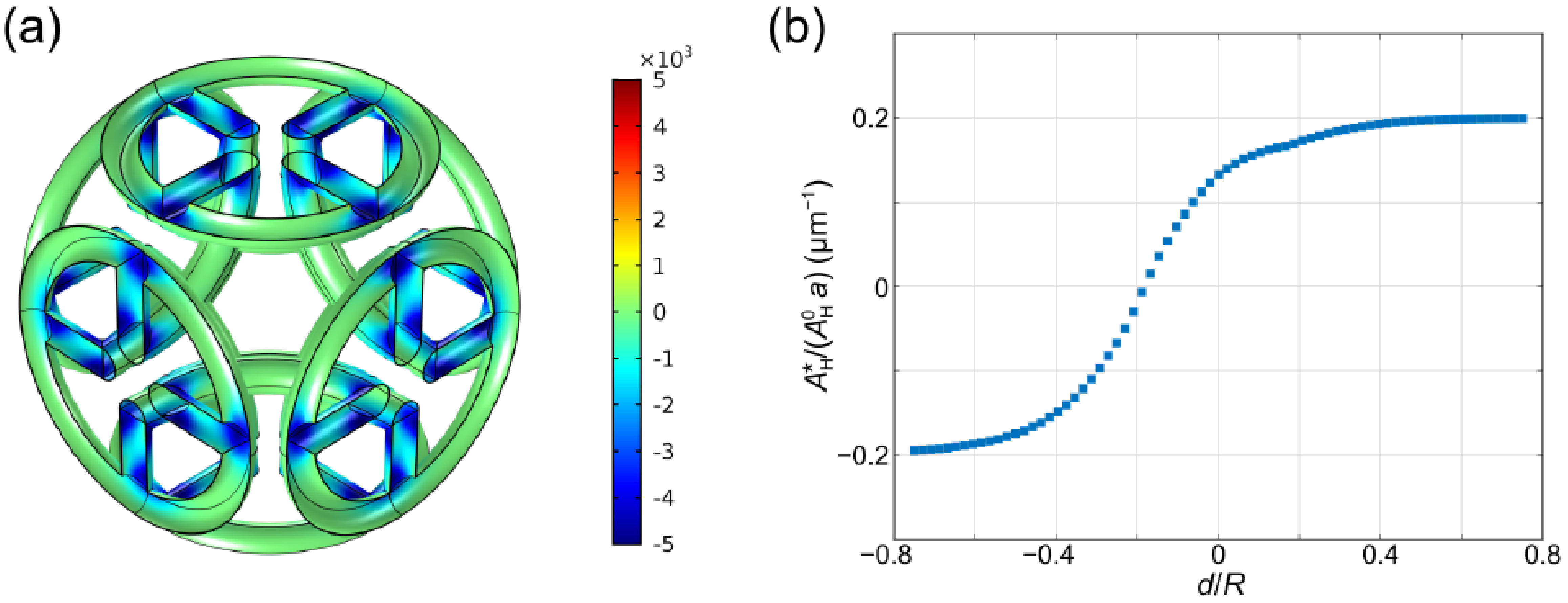
# Hall coefficient for 3 different geometries



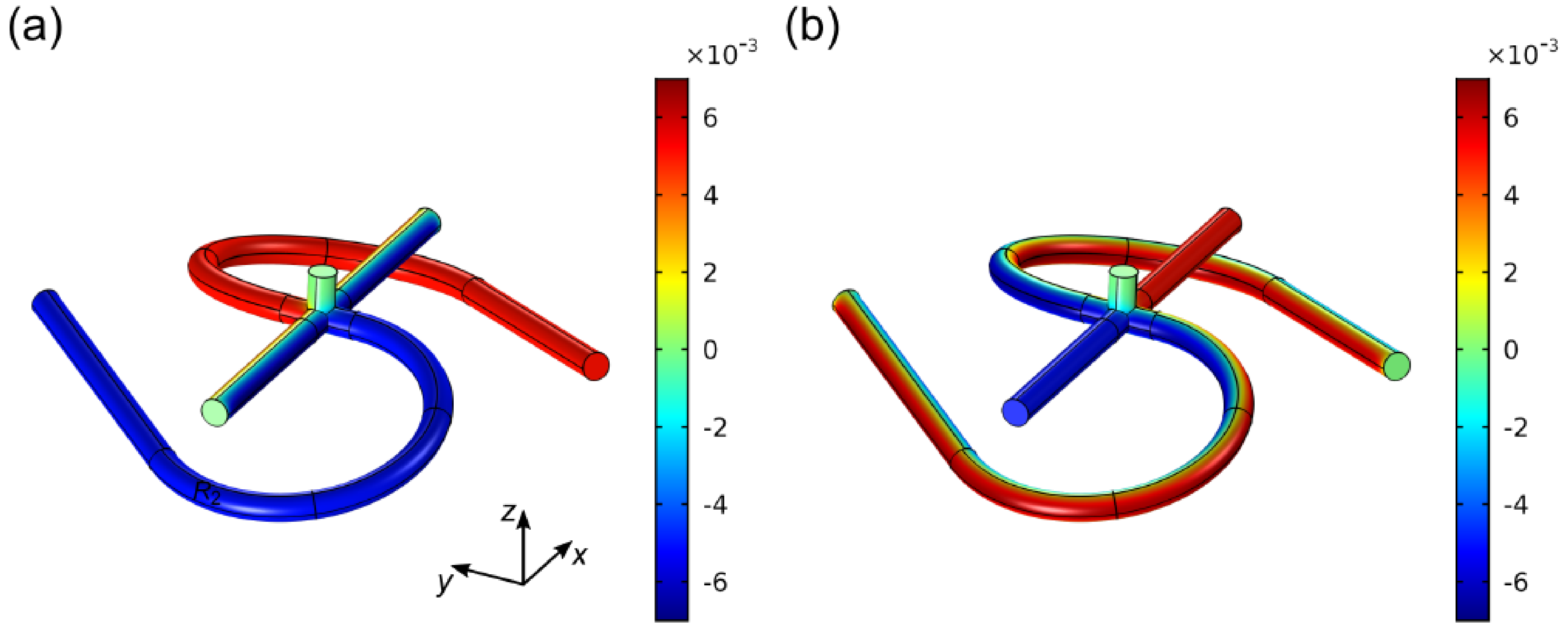

$A_{\mathrm{H}}^{*} = -5.73\, A_{\mathrm{H}}^{0}$ $A_{\mathrm{H}}^{*} = -1.86\, A_{\mathrm{H}}^{0}$ $A_{\mathrm{H}}^{*} = -5.39\, A_{\mathrm{H}}^{0}$

Dark Grey= Semiconductor, Light Grey=perfect conductor

# Plot of the trace of the cofactor matrix

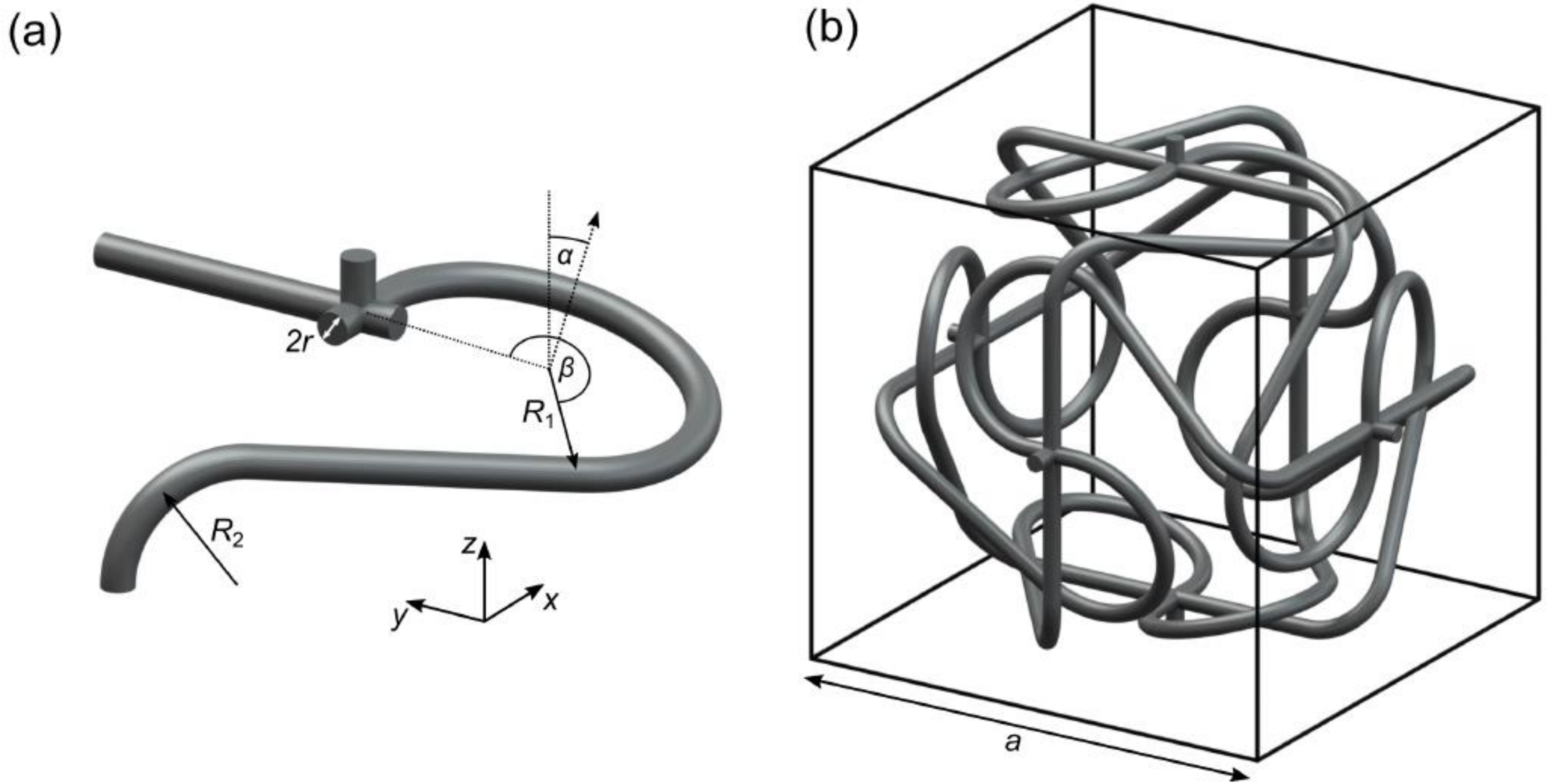


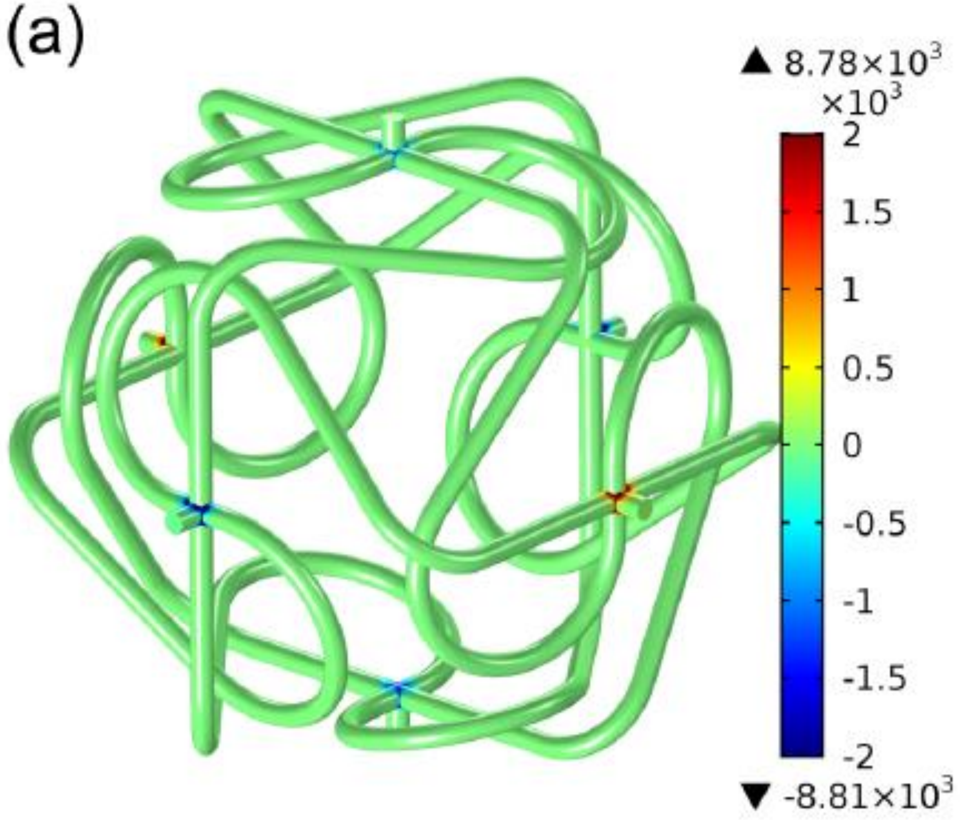
A simple idea for reversing the Hall voltage….
just swap the connecting leads



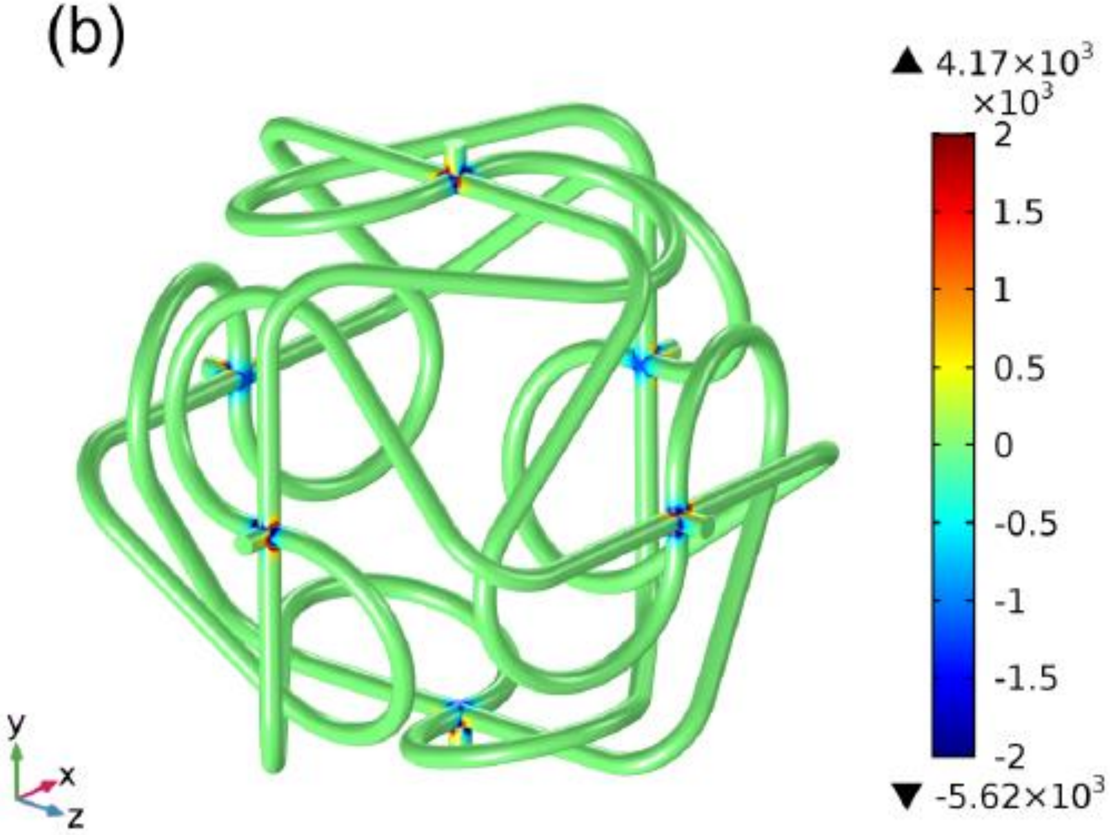

But can one incorporate this idea in a metamaterial
to reverse the Hall coefficient?

# Yes: the incredible geometry of Christian Kern:






Plot of a cofactor




Plot of the trace of the cofactor matrix

# The parallel Hall effect:

twisting the induced electric field in each unit cell

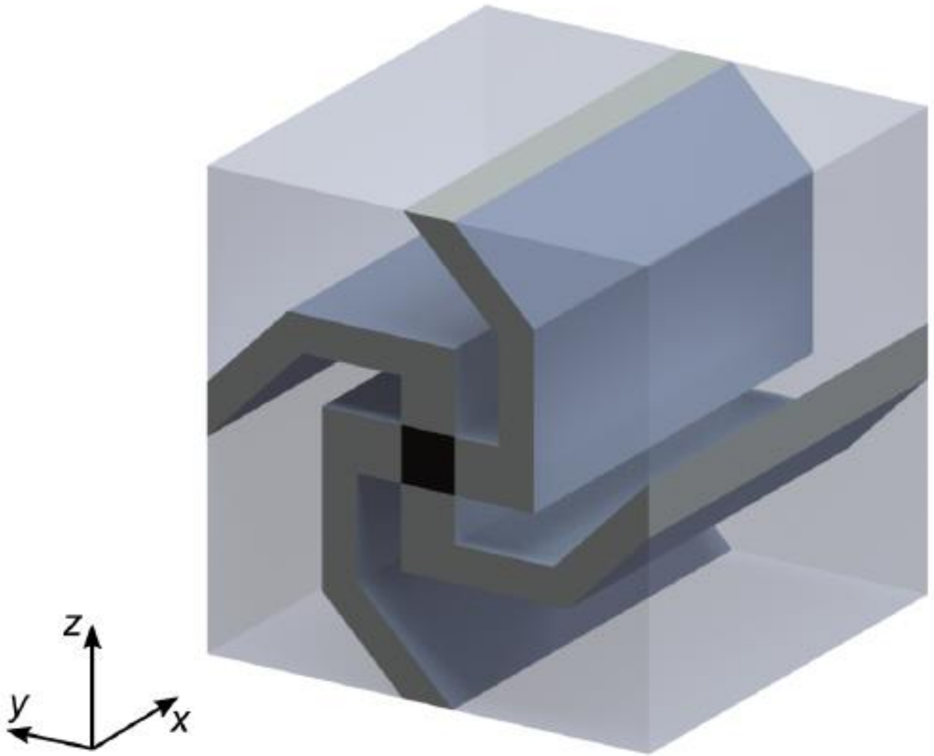


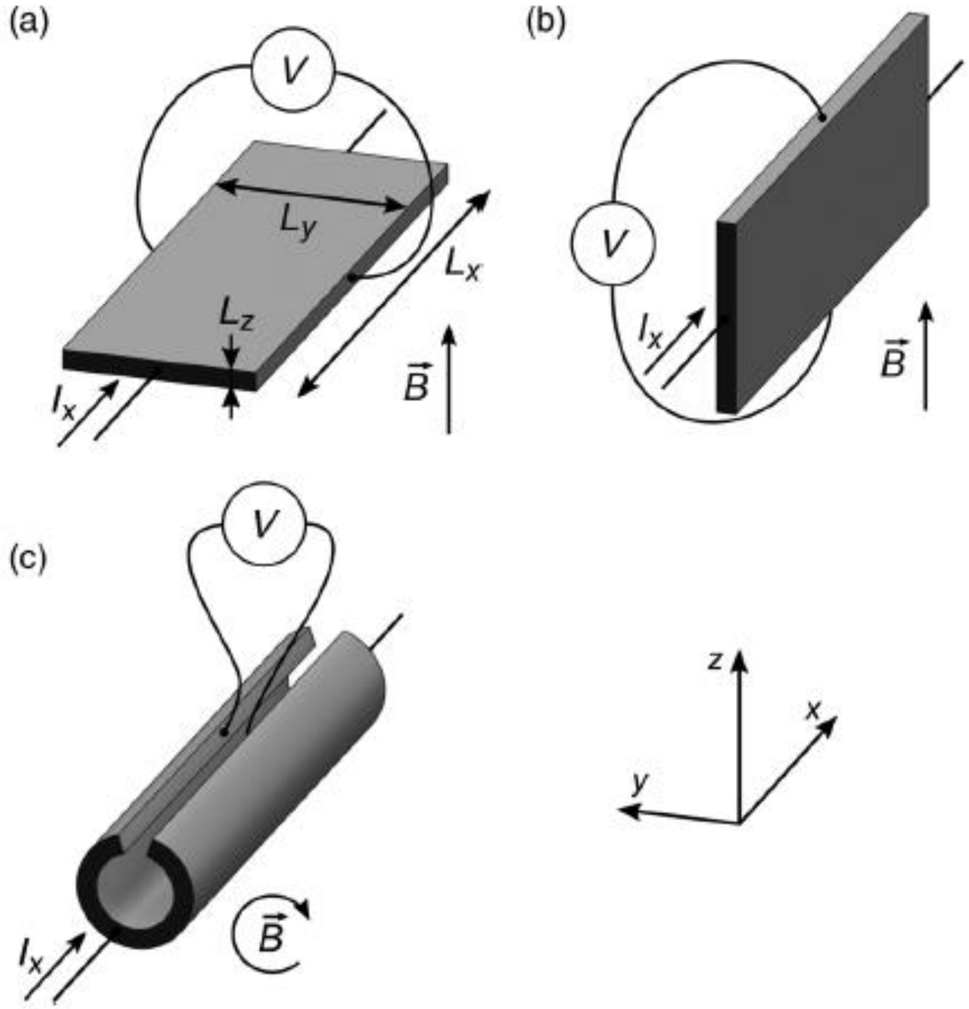

Image courtesy Christian Kern

$$\boldsymbol{A}_{\mathrm{H}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -A_{23} & 0 \end{pmatrix}$$

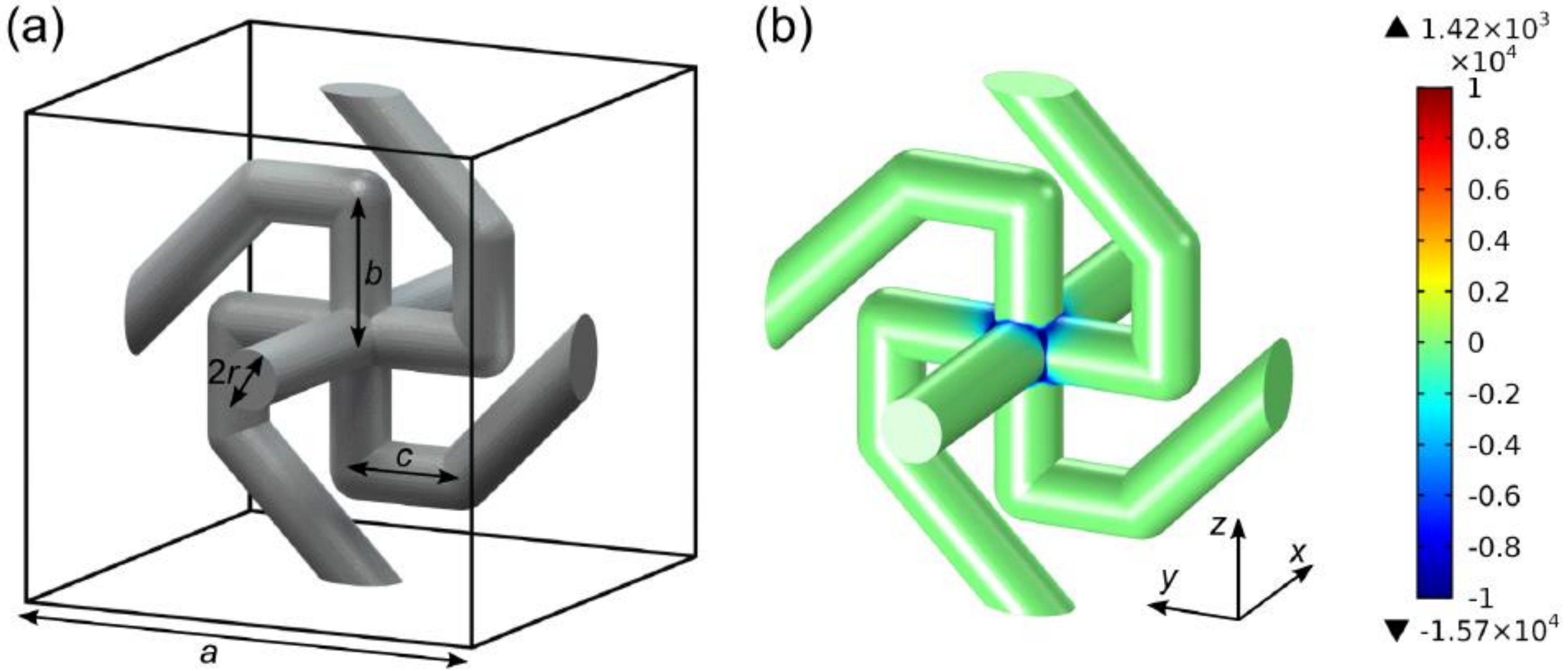
$$\boldsymbol{e}_{\mathrm{H}} = -A_{23} j_x \left( b_y \hat{\boldsymbol{y}} + b_z \hat{\boldsymbol{z}} \right)$$

The Hall matrix becomes asymptotically an antisymmetric matrix.
(Milton and Briane, 2010)

# Measuring the curl of the magnetic field using the parallel Hall effect: Kern et.al (2017)



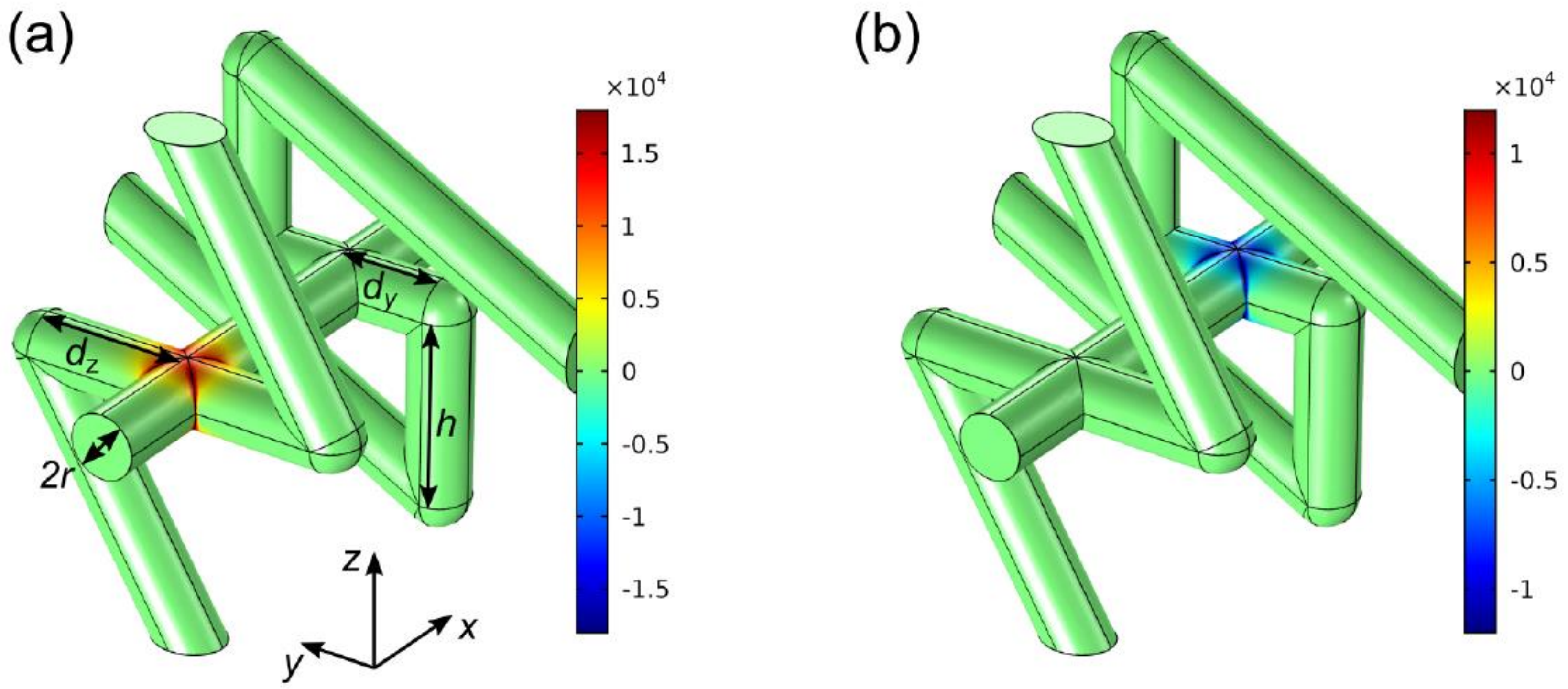
# Simplified Design: (Kern, Kadic, Wegener 2015)




$$\boldsymbol{A}_{\mathrm{H}}^{*} = \begin{pmatrix} 6.85 & 0 & 0 \\ 0 & 0.04 & 6.84 \\ 0 & -6.84 & 0.04 \end{pmatrix} A_{\mathrm{H}}^{0}$$

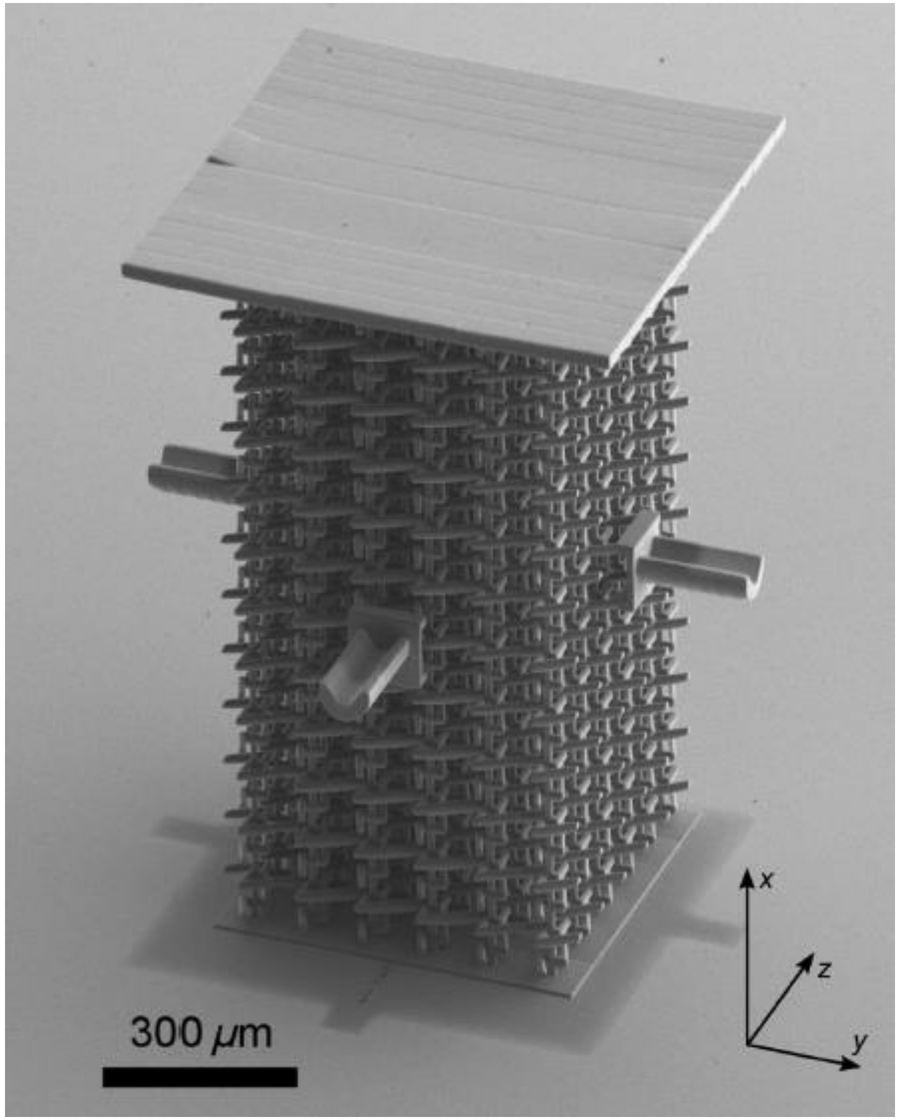
Plot of the cofactor

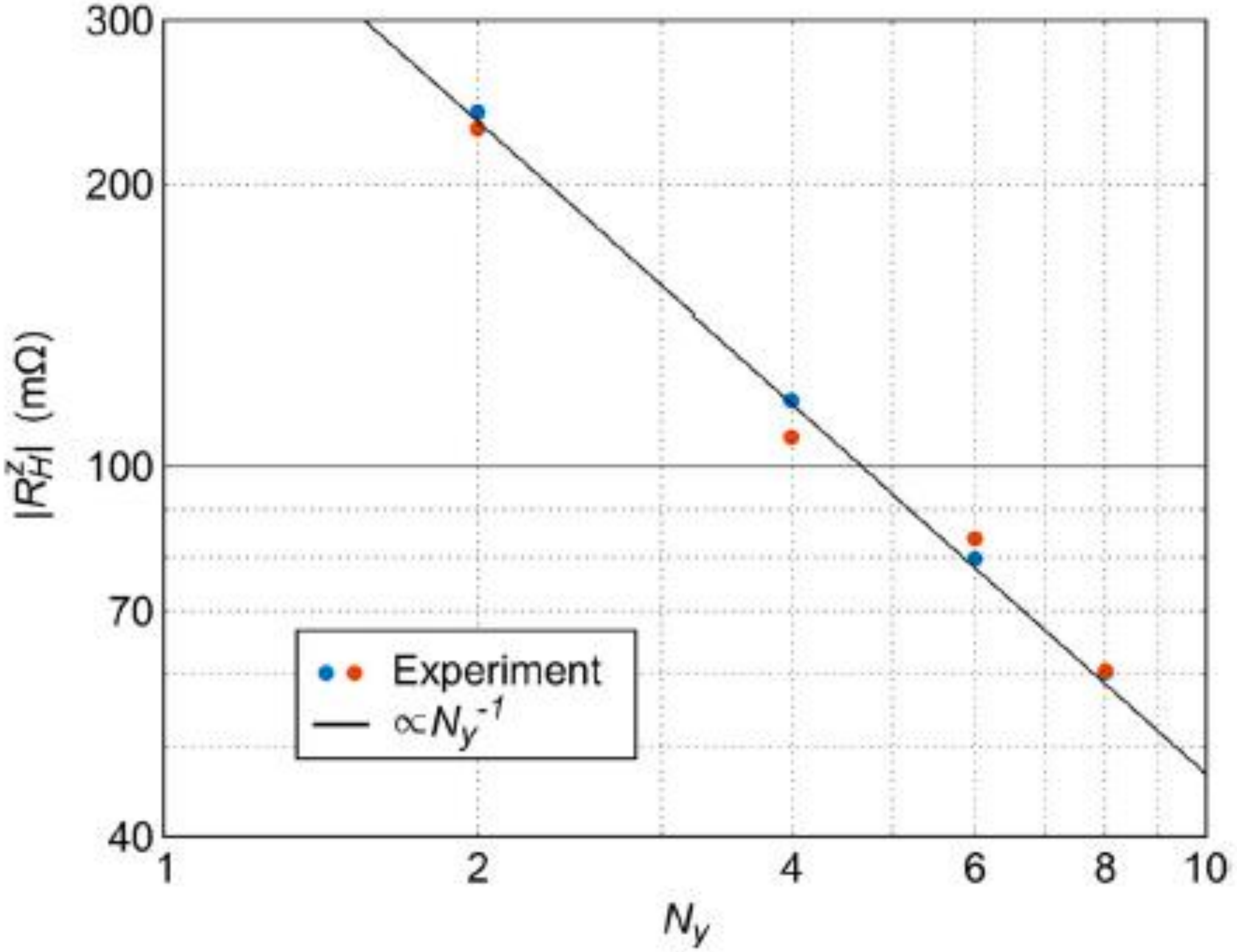
# A modified structure with an almost antisymmetric Hall matrix




$$\boldsymbol{A}_{\mathrm{H}}^{*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.05 & 8.81 \\ 0 & -8.81 & 0.05 \end{pmatrix} A_{\mathrm{H}}^{0}$$

# Experiments: Kern, Schuster, Kadic, and Wegener (2017)



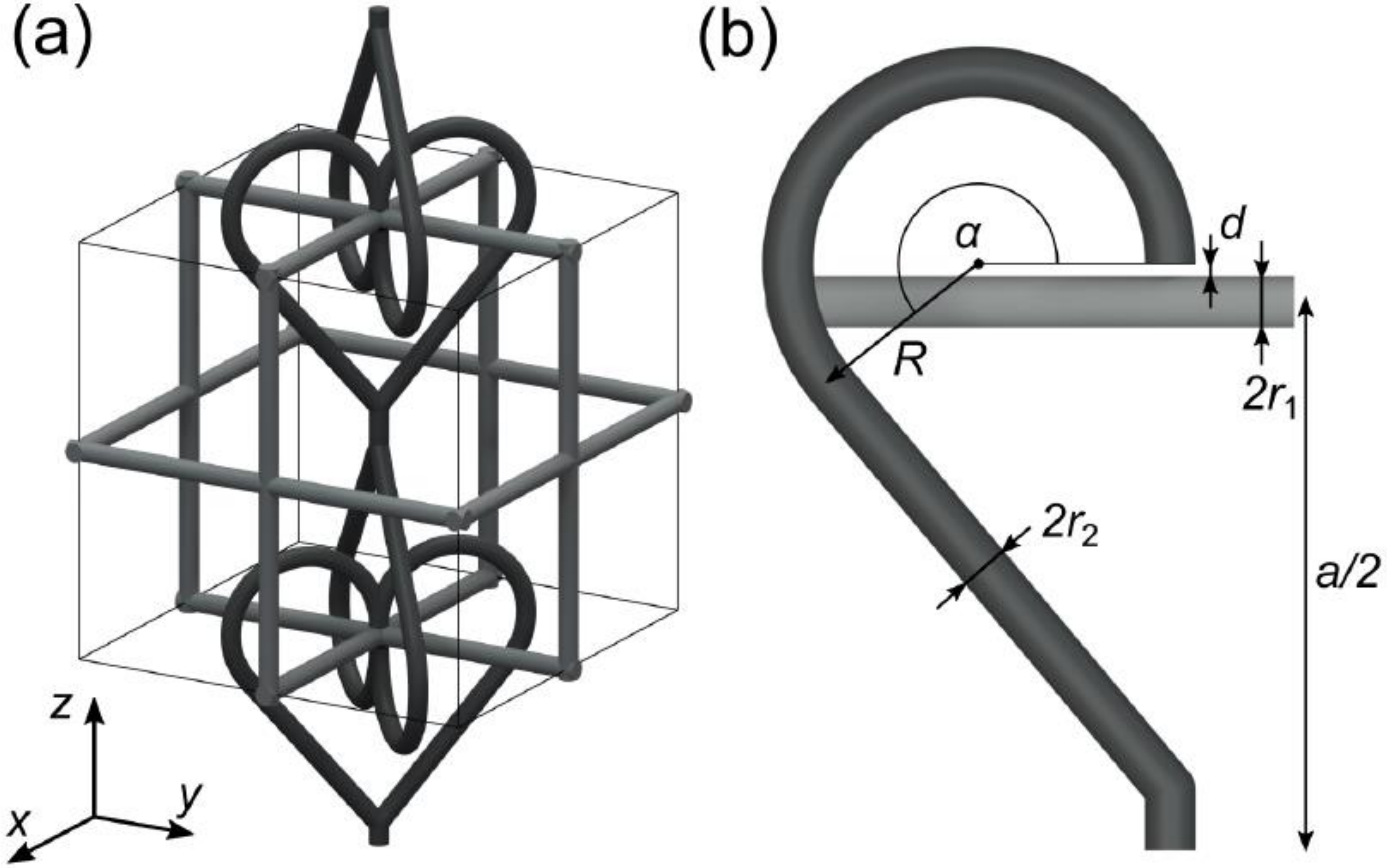
So far we have been manipulating the conductivity
to channel current in the structure to achieve desired effects.

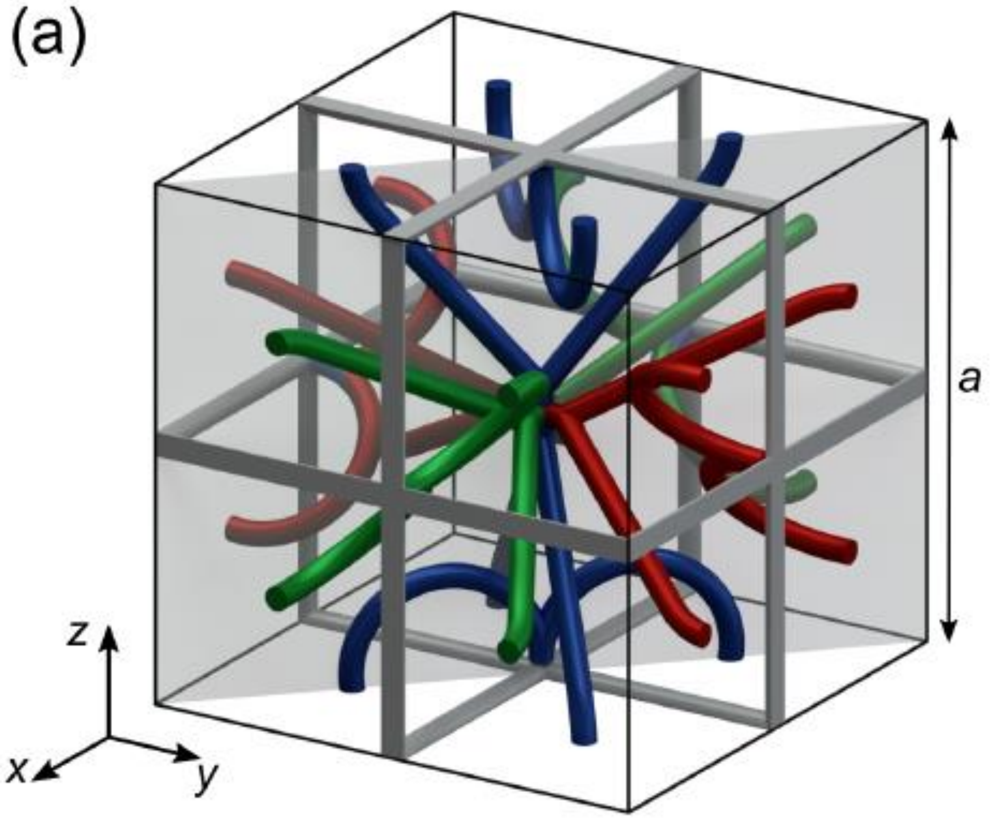
What about if we also manipulate the magnetic permeability
to channel the magnetic field to achieve desired effects?

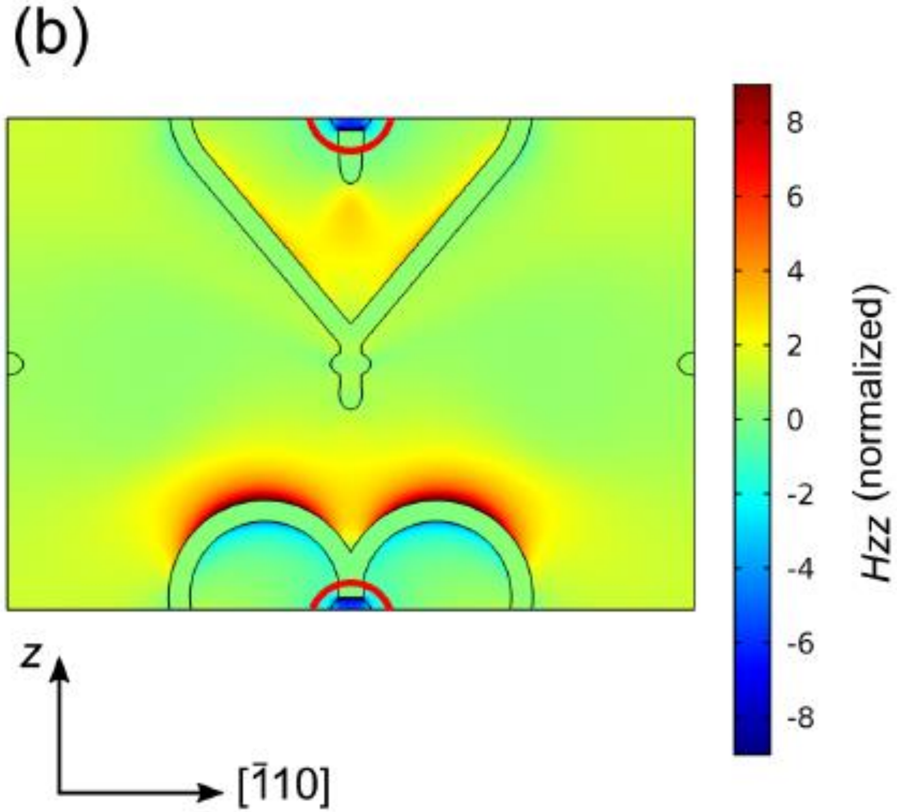
## Formula for the effective Hall matrix with magnetic permeability variations

$$\mathrm{Cof}\left(\boldsymbol{\sigma}_0^*\right)\boldsymbol{A}_{\mathrm{H}}^*\boldsymbol{\mu}^* = \left\langle \mathrm{Cof}\left(\boldsymbol{\sigma}_0\nabla\boldsymbol{\Phi}\right)^{\mathsf{T}}\boldsymbol{A}_{\mathrm{H}}\boldsymbol{\mu}\left(\nabla\boldsymbol{\Phi}_{\mathrm{m}}\right)\right\rangle$$

$$\boldsymbol{b} = -\mu_0\boldsymbol{\mu}\left(\nabla\boldsymbol{\Phi}_{\mathrm{m}}\right)\langle\boldsymbol{h}\rangle \qquad \nabla\cdot\left(\boldsymbol{\mu}\nabla\boldsymbol{\Phi}_{\mathrm{m}}\right) = 0 \qquad -\left(\nabla\boldsymbol{\Phi}_{\mathrm{m}}\right)\langle\boldsymbol{h}\rangle = \boldsymbol{h}.$$


(a)
z
x
y
(b)
α
d
R
2r1
2r2
a/2


(a)
a
z
x
y


(b)
8
6
4
2
0
-2
-4
-6
-8
Hzz (normalized)
z
[$\bar{1}$10]

# Some References

See: https://sinews.siam.org/Details-Page/surprises-regarding-the-hall-effect-an-extraordinary-story-involving-an-artist-mathematicians-and-physicists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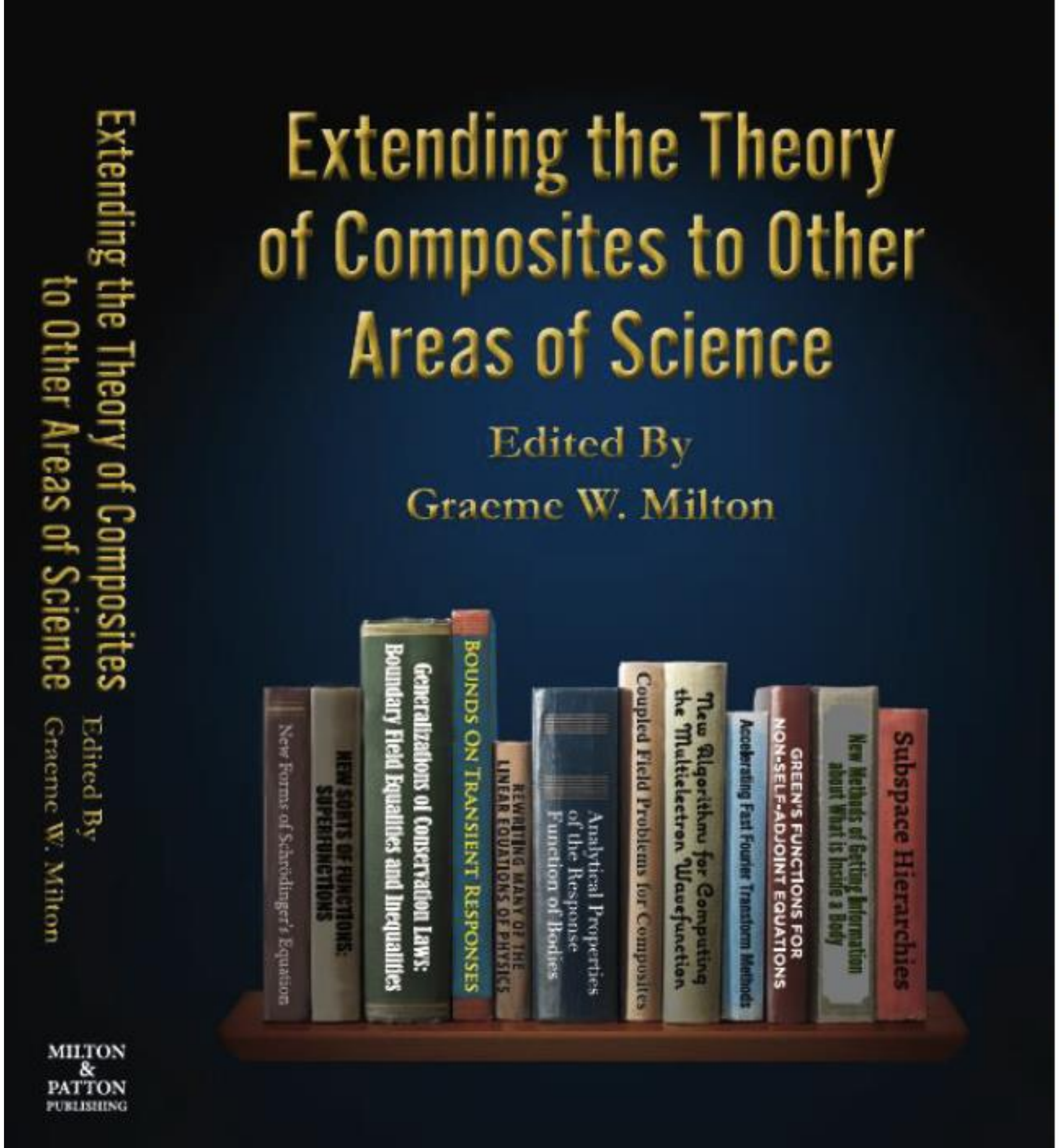
Also, in particular, arXiv:1806.04914 [cond-mat.mes-hall]

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