Exact relations for Green's functions in linear PDE and boundary field equalities: a generalization of conservation laws

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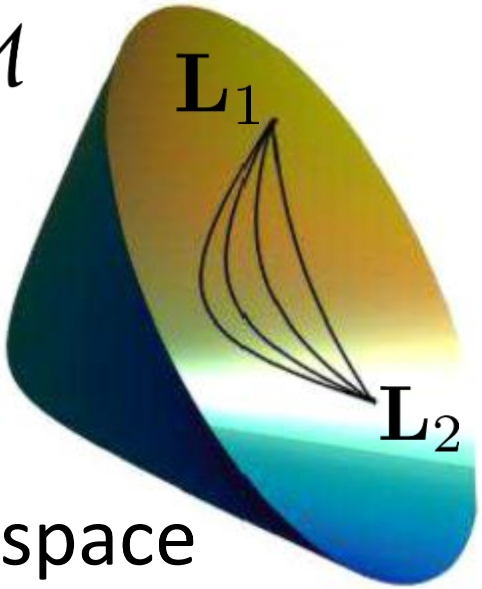
Classic example of an exact relation: Keller-Mendelson-Dykhne relation for 2-dimensional conductivity

$$\det \boldsymbol{\sigma}_* = c \qquad \text{when } \det \boldsymbol{\sigma}(\mathbf{x}) = c \ \text{ for all } \ \mathbf{x}.$$

$\mathcal{M}$



$\mathbf{L}$ -space

Picture courtesy of Y. Grabovsky

The manifold
$$\mathcal{M} = \{\boldsymbol{\sigma} : \det \boldsymbol{\sigma} = c\}$$
is stable under homogenization.

Goal of the theory of exact relations: identify manifolds of tensors, $\mathcal{M}$ that are Stable under homogenization

$$\text{Given periodic } \mathbf{L}(\mathbf{x}) \text{ with } \mathbf{L}(\mathbf{x}) \in \mathcal{M} \ \forall \mathbf{x} \text{ then } \mathbf{L}_* \in \mathcal{M}$$

Classic example of an exact link: Keller-Matheron-Mendelson reciprocal relation for 2-dimensional conductivity

Consider two conductivity problems with tensor fields $\boldsymbol{\sigma}(\mathbf{x})$ and $\widetilde{\boldsymbol{\sigma}}(\mathbf{x})$ related via

$$\widetilde{\boldsymbol{\sigma}}(\mathbf{x}) = [\mathbf{R}_\perp^T \boldsymbol{\sigma}(\mathbf{x}) \mathbf{R}_\perp]^{-1}, \quad \text{with} \quad \mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the effective conductivities are related in the same way: $\widetilde{\boldsymbol{\sigma}}_* = [\mathbf{R}_\perp^T \boldsymbol{\sigma}_* \mathbf{R}_\perp]^{-1}$.

Treat it as a trivial "coupled field problem" with no couplings!

$$\begin{pmatrix} \mathbf{j}(\mathbf{x}) \\ \widetilde{\mathbf{j}}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}(\mathbf{x}) & 0 \\ 0 & \widetilde{\boldsymbol{\sigma}}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} -\nabla V(\mathbf{x}) \\ -\nabla \widetilde{V}(\mathbf{x}) \end{pmatrix}$$

Then we can take $\mathcal{M}$ to consist of all matrices of the form $\mathbf{L} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & [\mathbf{R}_\perp^T \boldsymbol{\sigma} \mathbf{R}_\perp]^{-1} \end{pmatrix}$

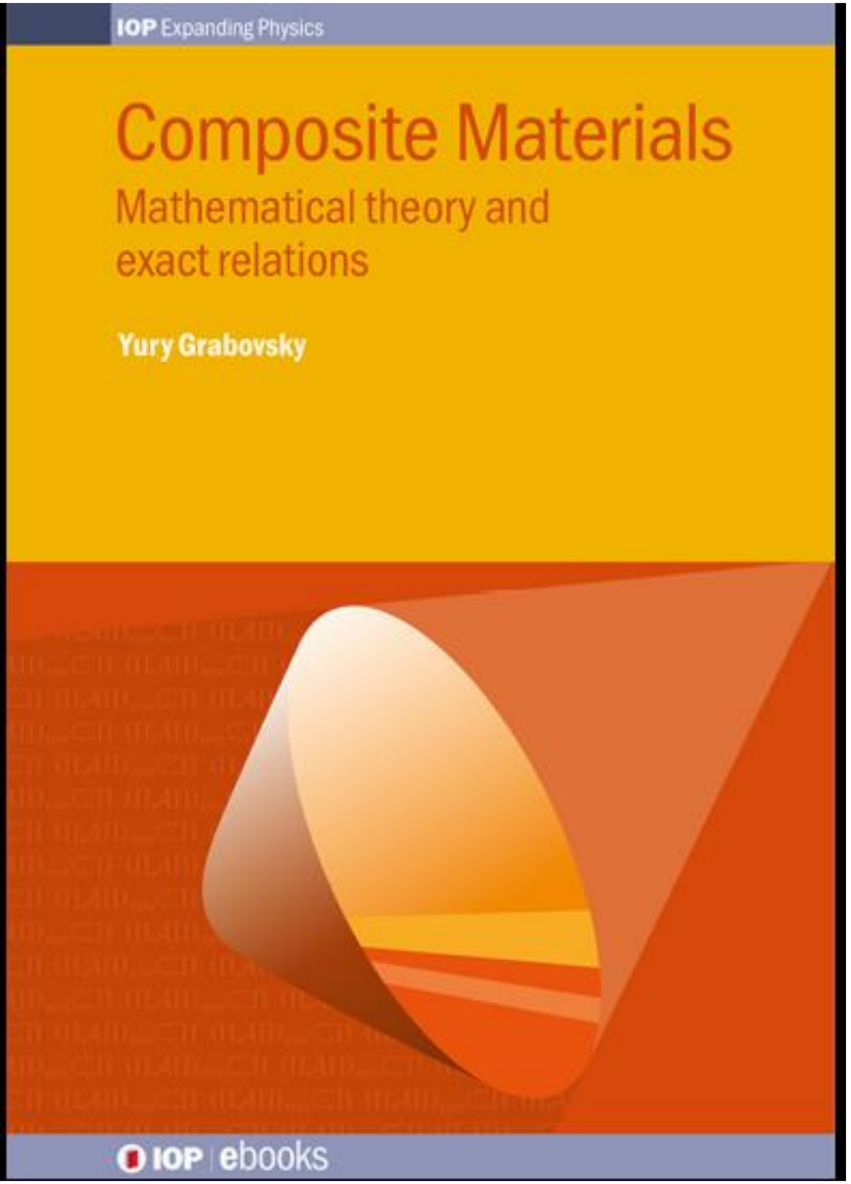
Many Scientists discovered exact relations one at a time:

Benveniste (Piezoelectricity)
Bergman (Hall-effect)
Berryman (Poroelasticity)
Chen (Coupled equations, Elasticity)
Cherkaev (Plate equations)
Cribb (Thermoelasticity)
Dvorak (Piezoelectricity)
Dykhne (Conductivity, Hall Effect)
Gassman (Poroelasticity)
Hashin (Elasticity)
He (Elasticity)
Helsing (Elasticity)
Hill (Elasticity)
Keller (Conductivity)

Levin (Thermoelasticity)
Lurie (Plate equations, Elasticity)
Matheron (Conductivity)
Milgrom (Coupled equations)
Milton (Complex conductivity, Hall effect, elasticity)
Movchan (elasticity)
Murat (Null-Lagrangians)
Shklovskii (Hall effect)
Shtrikman (Coupled equations)
Straley (Coupled Equations)
Strelniker (Hall effect)
Rosen (Thermoelasticity)
Schulgasser (Piezoelectricity)
Tartar (Null-Lagrangians)

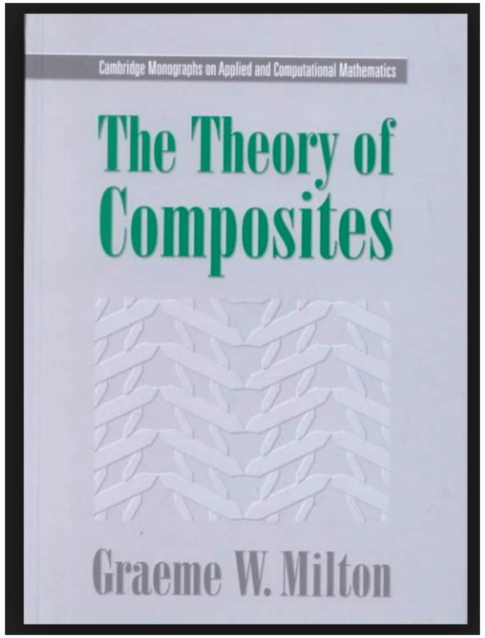
Yury Grabovsky and coworkers discovered hundreds,
(many intersections of more fundamental ones)

Theory of exact relations for composites reviewed in the books:



Grabovsky 2016

Milton 2002



Relevant Chapters:

3. Duality transformations in two-dimensional media
4. Translations and equivalent media
5. Some microstructure-independent exact relations
6. Exact relations for coupled equations
9. Laminate materials
12. Reformulating the problem of finding effective tensors
14. Series expansions for the fields and effective tensors
17. The general theory of exact relations and links between effective tensors

First major breakthrough: Grabovsky (1998)

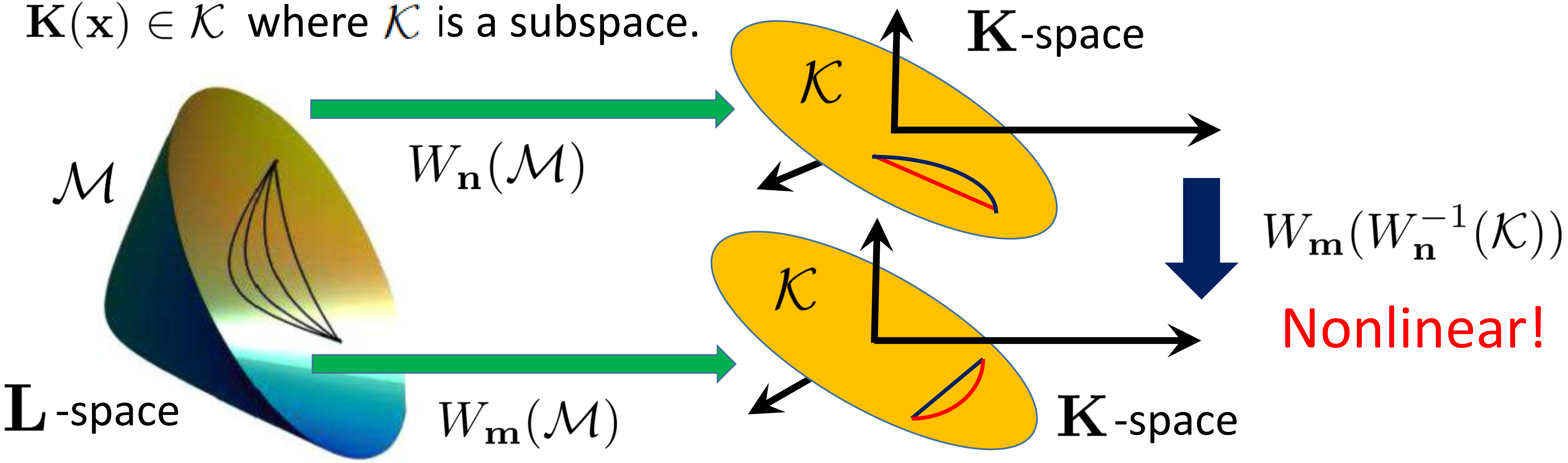
As an exact relation holds for all geometries it must certainly hold for laminate geometries

The transformation (Milton, 1990; Zhikov 1991)

$$W_{\boldsymbol{n}}(\boldsymbol{L}) = [\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}(\boldsymbol{n})]^{-1}(\boldsymbol{L} - \boldsymbol{L}_0) = \boldsymbol{K}, \quad \mathbf{L}_0 \in \mathcal{M}$$

converts lamination in direction $\boldsymbol{n}$ to a linear average: $\mathbf{L}_* = W_{\mathbf{n}}^{-1}(\langle W_{\mathbf{n}}(\mathbf{L})\rangle)$

Therefore in $\boldsymbol{K}$ -space an exact relation must be a linear relation, $\mathbf{K}_* \in \mathcal{K}$, when $\mathbf{K}(\mathbf{x}) \in \mathcal{K}$ where $\mathcal{K}$ is a subspace.



Expansion of the non-linear transformation. Set $\mathbf{A}(\boldsymbol{m}) = \boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})$.

$$\begin{aligned} W_{\boldsymbol{m}}(W_{\boldsymbol{n}}^{-1}(\epsilon \mathbf{K})) &= \epsilon \mathbf{K}\{\mathbf{I} - [\boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})]\epsilon \mathbf{K}\}^{-1} \\ &= \epsilon \mathbf{K} + \epsilon^2 \mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K} + \epsilon^3 \mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K} \\ &\quad + \epsilon^4 \mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K} + \cdots, \end{aligned}$$

So $\mathcal{K}$ independent of $\boldsymbol{n}$ and

$\mathbf{K}\mathbf{A}(\boldsymbol{m})\mathbf{K} \in \mathcal{K}$ for all $\boldsymbol{m}$ and for all $\mathbf{K} \in \mathcal{K}$. (Necessary Condition)

Then all terms in the series lie in $\mathcal{K}$

The search for candidate exact relations becomes a search for subspaces $\mathcal{K}$ satisfying this algebraic constraint.

Example: Two-dimensional conductivity

Take $\mathbf{L}_0 = \sigma_0 \mathbf{I}$. Then

$$\mathbf{A}(\mathbf{m}) = \frac{\mathbf{n}\mathbf{n}^T}{(\sigma_0 \mathbf{n} \cdot \mathbf{n})} - \frac{\mathbf{m}\mathbf{m}^T}{(\sigma_0 \mathbf{m} \cdot \mathbf{m})}$$

is trace-free and symmetric. We can take $\mathcal{K}$ as the space of 2×2 symmetric trace-free matrices.

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} = \begin{pmatrix} ab & ac \\ -ac & ab \end{pmatrix}$$

But with 3 matrices:

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} \begin{pmatrix} d & e \\ e & -d \end{pmatrix} = \begin{pmatrix} abd + ace & abe - acd \\ abe - acd & -abd - ace \end{pmatrix}$$

Then $\mathcal{M} = W_{\mathbf{n}}^{-1}(\mathcal{K})$ consists of 2×2 symmetric matrices with determinant σ_0^2.

Second major breakthrough: (Grabovsky, Milton, Sage 2000)

The transformation $W_n(L)$ and series expansions of Milton and Golden (1990) [that formed the basis of the rapidly converging FFT approach of Eyre and Milton (1999)] provided the essential clues for a condition that guarantees a candidate exact relation holds for all geometries not just **laminate** ones.

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$$\mathbf{K}_* = W_{\mathbf{M}}(\mathbf{L}_*) = [\mathbf{I} + (\mathbf{L}_* - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}_* - \mathbf{L}_0)$$

Series expansion: let $AP = M(P - \langle P \rangle) - \Gamma P$ define A (acts locally in Fourier space)

$$K_* = \langle [I - KA]^{-1} K \rangle = \sum_{j=0}^{\infty} \langle (KA)^j K \rangle \qquad \begin{aligned} A(k) &= M - \Gamma(k) \quad \text{for } k \neq 0, \\ &= 0 \qquad\qquad\quad \text{for } k = 0. \end{aligned}$$

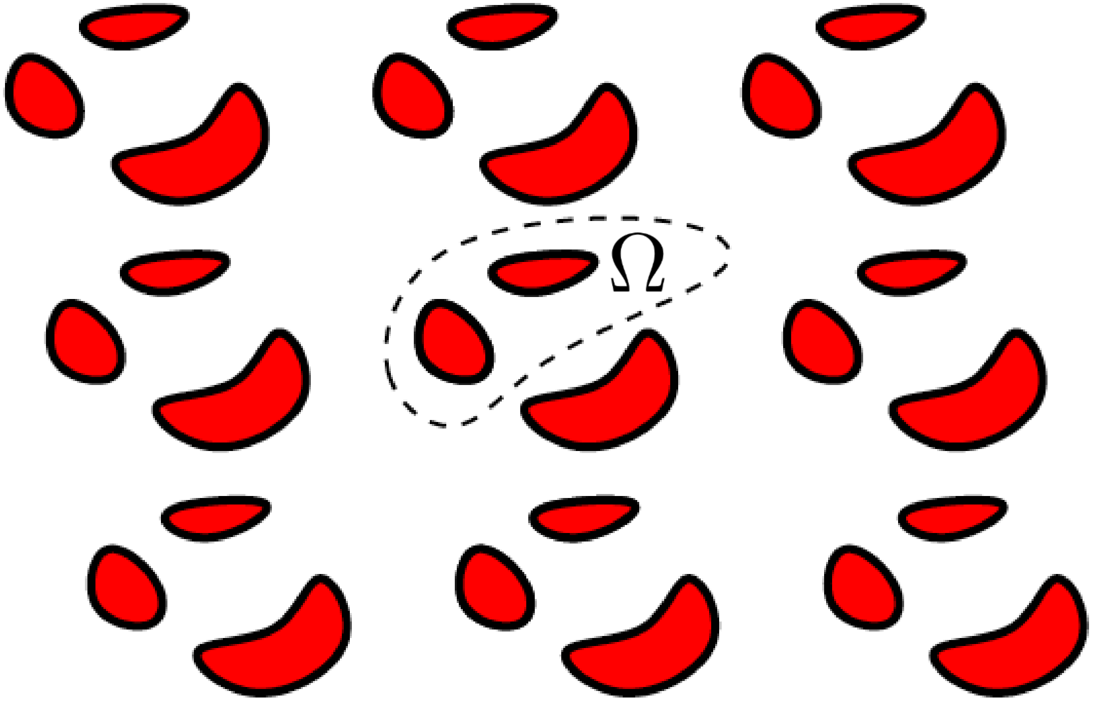
$K_1[M - \Gamma(n)]K_2 \in \overline{\mathcal{K}}$ and for all $K_1, K_2 \in \overline{\mathcal{K}}$, (Sufficient Condition)

Appropriately defined “polarization fields” within the material also are constrained to take values in $\overline{\mathcal{K}}$

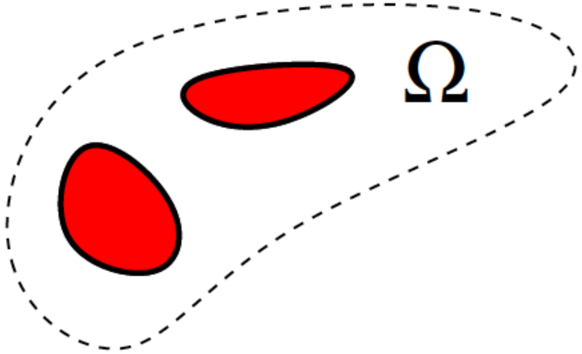
$\mathcal{K}$ can be taken to consist of all symmetric matrices in $\overline{\mathcal{K}}$

If the series does not converge, use analytic continuation

Third Major Breakthrough (Milton and Onofrei, arXiv:1712.03597, 2018)



Suppose we have a periodic composite for which an exact relation holds,
And hence the "polarization field" takes values in $\overline{\mathcal{K}}$ at each $\mathbf{x}$ in Ω



The region Ω marked by the "dashed lines" does not know it is in a periodic medium, but the boundary conditions on the potentials or fluxes on this dashed boundary must be such to force the "polarization field" inside Ω to take values in $\overline{\mathcal{K}}$ and this gives us additional Information about the boundary fields.

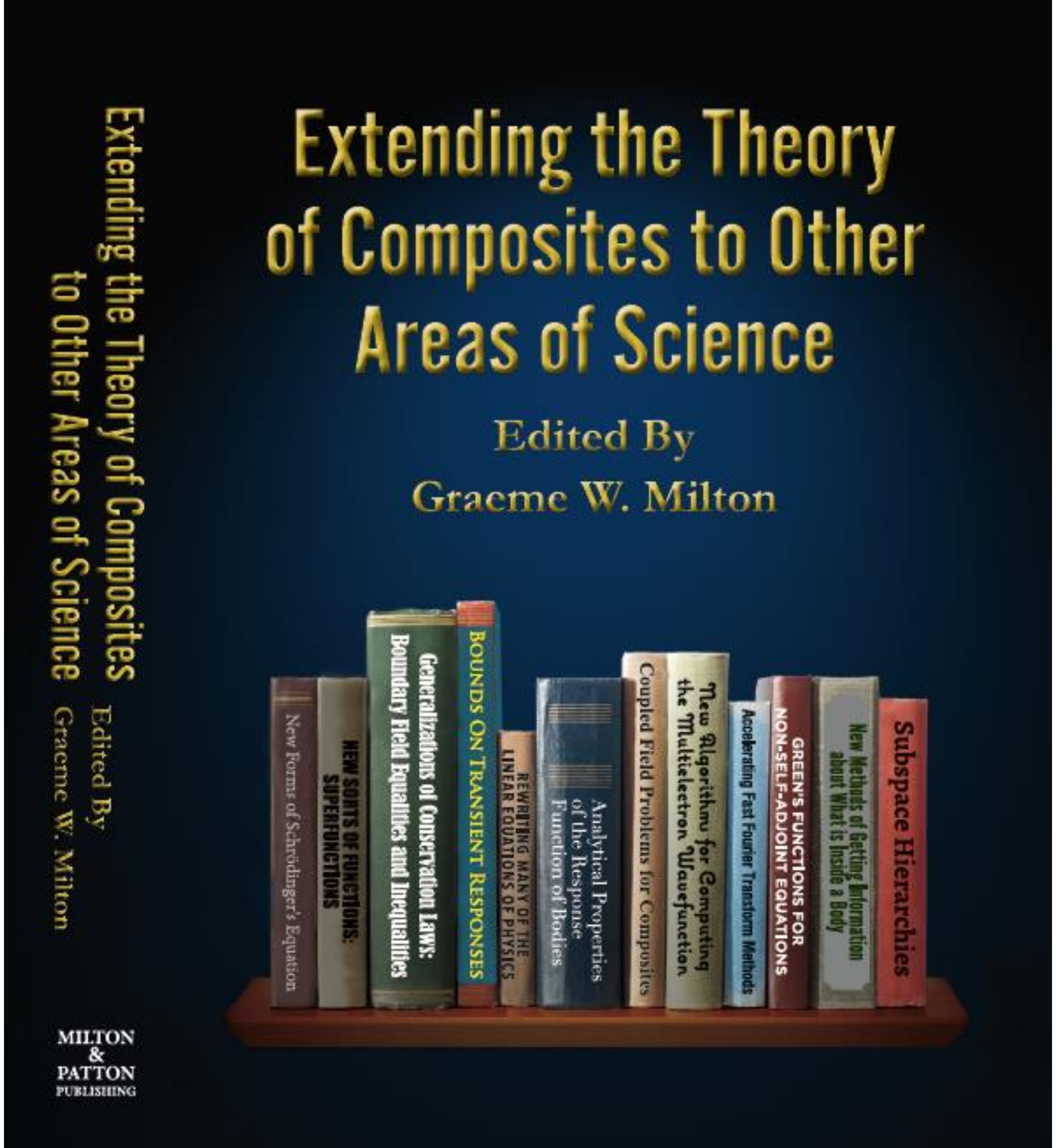
Aim: identify these boundary conditions, and find the associated exact identities (boundary field equalities) satisfied by the "Dirichlet-to-Neumann map".

A new perspective on **conservation laws**:
Boundary field equalities and inequalities

$$\text{If } \nabla \cdot \mathbf{Q} = 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} = 0$$

$$\text{If } \nabla \cdot \mathbf{Q} \geq 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} \geq 0$$

Requires information about what is happening inside Ω namely that $\nabla \cdot \mathbf{Q} = 0$ or $\nabla \cdot \mathbf{Q} \geq 0$ in Ω .



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Are there other boundary field equalities or inequalities that use partial information about what is inside the body?

Simple example, our theory much more powerful

$$\begin{pmatrix}\mathbf{j}(\mathbf{x})\\ \widetilde{\mathbf{j}}(\mathbf{x})\end{pmatrix} = \begin{pmatrix}a(\mathbf{x})\mathbf{I} & c(\mathbf{x})\mathbf{I}\\ c(\mathbf{x})\mathbf{I} & b(\mathbf{x})\mathbf{I}\end{pmatrix}\begin{pmatrix}-\nabla V(\mathbf{x})\\ -\nabla\widetilde{V}(\mathbf{x})\end{pmatrix}, \quad \nabla\cdot\mathbf{j}=0, \quad \nabla\cdot\widetilde{\mathbf{j}}$$

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix}a(\mathbf{x})\mathbf{I} & c(\mathbf{x})\mathbf{I}\\ c(\mathbf{x})\mathbf{I} & b(\mathbf{x})\mathbf{I}\end{pmatrix}, \quad \beta\mathbf{I}\geq\mathbf{M}(\mathbf{x})\geq\alpha\mathbf{I} \text{ for some } \beta>\alpha>0$$

Following the ideas of Straley, Milgrom and Shtrikman suppose there is a matrix $\mathbf{W}$ such that

$$\mathbf{W}\mathbf{M}(\mathbf{x})\mathbf{W}^T = \begin{pmatrix}a'(\mathbf{x})\mathbf{I} & 0\\ 0 & b'(\mathbf{x})\mathbf{I}\end{pmatrix}$$

$$\begin{pmatrix}V(\mathbf{x})\\ \widetilde{V}(\mathbf{x})\end{pmatrix} = \mathbf{W}^T\begin{pmatrix}f(\mathbf{x})\\ 0\end{pmatrix} \text{ for all } \mathbf{x}\in\partial\Omega \quad \Rightarrow \quad W_{21}[\mathbf{n}\cdot\mathbf{j}(\mathbf{x})]+W_{22}[\mathbf{n}\cdot\widetilde{\mathbf{j}}(\mathbf{x})]=0 \quad \text{for all } \mathbf{x}\in\partial\Omega$$

Another simple example: in two dimensions suppose

$$c(\mathbf{x}) = 0, \quad b(\mathbf{x}) = \alpha^2/a(\mathbf{x})$$

Following ideas of Keller, Dykhne, Matheron and Mendelson, we have the boundary field equality

$$\boxed{\mathbf{n}\cdot\widetilde{\mathbf{j}}(\mathbf{x}) = \alpha\mathbf{t}\cdot\nabla V(\mathbf{x}) \text{ when } \mathbf{t}\cdot\nabla\widetilde{V}(\mathbf{x}) = -\alpha^{-1}\mathbf{n}\cdot\mathbf{j}(\mathbf{x})}$$

$\mathbf{n}$ normal to $\partial\Omega$, $\mathbf{t}$ tangential to $\partial\Omega$,

It's due to the fact that the equations are satisfied with

$$\widetilde{V}(\mathbf{x}) = -\alpha^{-1}\mathbf{R}_\perp\mathbf{j}(\mathbf{x}), \quad \widetilde{\mathbf{j}}(\mathbf{x}) = -\alpha\mathbf{R}_\perp\nabla V(\mathbf{x}),$$

where

$$\mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Key point:

These new boundary field equalities that in some sense generalize the divergence theorem, do not result from "integration by parts" but rather from algebraic properties tied with the operator Γ that is associated with the differential constraints satisfied by the fields on the left and right of the constitutive law.

There are "hidden identities" that go beyond integration by parts and still allow one to deduce exact identities satisfied by the fields at the boundary of a region Ω

Generalized viewpont of boundary field inequalities One eliminate $\mathbf{L}(\mathbf{x})$ from the the constitutive law $\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x})$ and just view the constaint on $\mathbf{L}(\mathbf{x})$ that $\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ as a constraint on the field pairs $(\mathbf{J}(\mathbf{x}), \mathbf{E}(\mathbf{x}))$ that is independent of $\mathbf{x}$.

For instance, if

- $\mathbf{E}$ consists of potential gradients,
- $\mathbf{J}$ consists of divergenge free fields (fluxes) that themselves may be expressed as curl's of additional potentials

Then collecting all potentials together as some grand potential $\mathbf{U}$,
The field constraints imply

$$\nabla\mathbf{U}(\mathbf{x}) \in \mathcal{A} \text{ for all } \mathbf{x} \in \Omega$$

where $\mathcal{A}$ is some non-linear manifold (determined by $\mathcal{M}$).

Then with appropriate nonlocal boundary conditions on the surface potential $\mathbf{U}(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$ we obtain the constraint that

$$\nabla\mathbf{U}(\mathbf{x}) \in \mathcal{C} \text{ for all } \mathbf{x} \in \Omega$$

for some appropriately defined subspace $\mathcal{C}$, and this in turns constrains the tangential derivatives of $\mathbf{U}$ at $\partial\Omega$: these are the boundary field equalities.

Note that if $\mathbf{N}$ is perpendicular to $\mathcal{C}$ then

$$0 = \mathrm{Tr}[\mathbf{N}(\nabla \mathbf{U}(\mathbf{x})] = \nabla \cdot (\mathbf{U}(\mathbf{x})\mathbf{N}^T)$$

So there are additional divergence free fields and additional associated boundary field equalities.

Formulation

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) \frac{\partial u_\beta(\mathbf{x})}{\partial x_j} \right) = f_\alpha(\mathbf{x}), \quad \alpha = 1, 2, \ldots, m,$$

Rewrite as

$$J_{i\alpha}(\mathbf{x}) = \sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) E_{j\beta}(\mathbf{x}) - h_{i\alpha}(\mathbf{x}), \quad E_{j\beta}(\mathbf{x}) = \frac{\partial u_\beta(\mathbf{x})}{\partial x_j}, \quad \sum_{i=1}^{d} \frac{\partial J_{i\alpha}(\mathbf{x})}{\partial x_i} = 0,$$

with

$$\sum_{i=1}^{d} \frac{\partial h_{i\alpha}(\mathbf{x})}{\partial x_i} = f_\alpha(\mathbf{x})$$

Can extend the formulation to plate equations,
wave equations at constant frequency in lossy media, etc.

E depends linearly on **h** and defines the (modified) infinite body Green's function in the inhomogeneous medium.

$$\mathbf{E}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathbf{G}(\mathbf{x}, \mathbf{x}')\mathbf{h}(\mathbf{x}')\, d\mathbf{x}',$$

Define the "polarization field"

$$\mathbf{P}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) - \mathbf{L}_0\mathbf{E}(\mathbf{x}) = [\mathbf{L}(\mathbf{x}) - \mathbf{L}_0]\mathbf{E}(\mathbf{x}) - \mathbf{h}(\mathbf{x})$$

Consider a point $\mathbf{x}^0$ and take $\mathbf{h}(\mathbf{x})$ to be proportional to a Dirac delta function localized at $\mathbf{x} = \mathbf{x}^0$:

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}^0\delta(\mathbf{x} - \mathbf{x}^0), \quad \text{with } \mathbf{h}^0 = -(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0,$$

$$\boxed{\mathbf{P}(\mathbf{x}) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0\delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0}$$

So $\mathbf{P}(\mathbf{x}) = \mathbf{T}(\mathbf{x}, \mathbf{x}_0)\mathbf{s}^0$ with

$$\mathbf{T}(\mathbf{x}, \mathbf{x}^0) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0).$$

$$\mathbf{T} = (\mathbf{L} - \mathbf{L}_0) - (\mathbf{L} - \mathbf{L}_0)\mathbf{G}(\mathbf{L} - \mathbf{L}_0) = (\mathbf{I} - \mathbf{K}\mathbf{\Psi})^{-1}\mathbf{K},$$

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$$\mathbf{\Psi} = \mathbf{M} - \mathbf{\Gamma}$$

$$\begin{aligned}
\mathbf{T}(\mathbf{x}, \mathbf{x}^0) \quad = \quad & \delta(\mathbf{x} - \mathbf{x}^0)\mathbf{K}(\mathbf{x}^0) + \mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x} - \mathbf{x}^0)\mathbf{K}(\mathbf{x}^0) \\
& + \int_{R^d} \mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x} - \mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1 - \mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\; d\mathbf{y}_1 \\
& + \int_{R^d}\int_{R^d} \mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x} - \mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1 - \mathbf{y}_2)\mathbf{K}(\mathbf{y}_2)\widehat{\mathbf{\Psi}}(\mathbf{y}_2 - \mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\; d\mathbf{y}_1\; d\mathbf{y}_2 + \ldots,
\end{aligned}$$

Expand:

$$\mathbf{T}(\mathbf{x},\mathbf{x}^0) = \delta(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0) + \mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)$$
$$+\int_{R^d}\mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\,d\mathbf{y}_1$$
$$+\int_{R^d}\int_{R^d}\mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1-\mathbf{y}_2)\mathbf{K}(\mathbf{y}_2)\widehat{\mathbf{\Psi}}(\mathbf{y}_2-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\,d\mathbf{y}_1\,d\mathbf{y}_2+\ldots,$$

Upshot :

$$\mathbf{T}(\mathbf{x},\mathbf{x}_0)\text{ takes values in }\overline{\mathcal{K}}\text{ when }\mathbf{L}(\mathbf{x})\text{ takes values in }\mathcal{M}.$$

In the same way that one gets links between effective tensors so too can one get links between Green's functions of different physical problems (in inhomogeneous media)

Did not discuss how to get the “boundary field equalities” satisfied by the “Dirichlet to Neumann Map”.

The basic idea here (following Thaler and Milton, 2014, where for a body Ω containing 2-phases sharing the same shear modulus, the boundary field equalities give the volume fraction occupied by one phase in the body) is to choose nonlocal boundary conditions that mimic the body Ω embedded in an infinite medium with appropriate sources outside that ensure the appropriately defined polarization field takes values in the subspace $\overline{\mathcal{K}}$

For more details see arXiv:1712.03597,2018

Generally, to reveal the exact relations satisfied by the DtN map one applies not just one boundary condition but a succession of them.

In general,

- $\mathbf{E}(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$ take values in some N-dimensional tensor space $\mathcal{T}$
- $\mathbf{L}(\mathbf{x})$, $\mathbf{L}_0$, $\mathbf{K}(\mathbf{x})$ take values in $L(\mathcal{T})$ the N^2-dimensional space of linear maps $\mathcal{T} \to \mathcal{T}$
- $\mathcal{K}$ and $\overline{\mathcal{K}}$ are therefore subspaces of $L(\mathcal{T})$

Hence it does not really make sense to say $\mathbf{P}(\mathbf{x})$ takes values in $\overline{\mathcal{K}}$.

- Rather one should take a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N^2}$ for $\mathcal{L}(\mathcal{T})$.
- Consider polarization fields $\mathbf{P}_1(\mathbf{x}), \mathbf{P}_2(\mathbf{x}), \ldots, \mathbf{P}_{N^2}(\mathbf{x})$ associated with N^2 experiments, with appropriate sources outside Ω.

Define $\mathbb{P}(\mathbf{x})$: $\mathcal{T} \to \mathcal{T}$ via $\mathbb{P}(\mathbf{x})\mathbf{e}_i = \mathbf{P}_i(\mathbf{x})$

It is $\mathbb{P}(\mathbf{x})$ that takes values in $\overline{\mathcal{K}}$ for appropriately chosen sources outside Ω

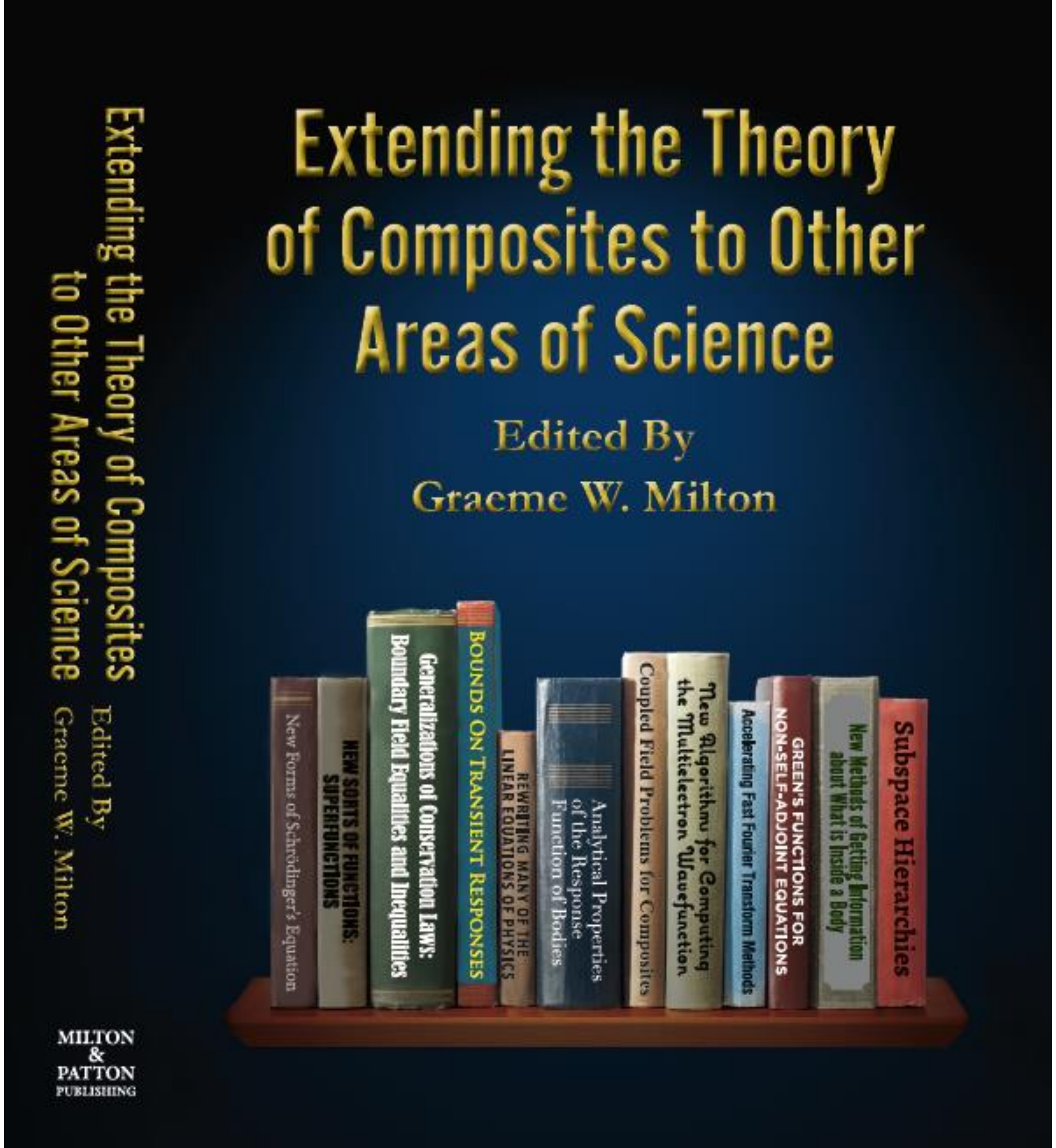
Thank you!

Thank you!

Thank you!

Thank you!

Thank you!



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