ON THE COMPUTATIONS OF CHARACTERISTIC CYCLES OF HARISH-CHANDRA MODULES

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INTRODUCTION

Suppose X is an irreducible Harish-Chandra module. By far the most important invariant attached to X is its characteristic cycle. The point of these notes is two-fold. In Section 1, I define the characteristic cycle and through some carefully chosen examples illuminate some subtleties. In the second section, I focus on how one might use Atlas to make partial computations of characteristic cycles. The first real test case will be the split real form of F4. In the final section I include the case of $\mathrm{Sp}(4,\mathbb{R})$.

1. CHARACTERISTIC CYCLES: DEFINITION AND EXAMPLES

To get started, I list the notation that I will use below and when it isn't obvious what the notation means I add parenthetic explanations: G (complex reductive algebraic group defined over $\mathbb{R}$), $G_\mathbb{R}$ (real point of G), $K_\mathbb{R}$, K, $\mathfrak{g}_\mathbb{R}$, $\mathfrak{g}$, $\mathrm{U}(\mathfrak{g})$, $\mathfrak{k}_\mathbb{R}$, $\mathfrak{k}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathcal{N}(\mathfrak{g}^*)$ (nilpotent cone in $\mathfrak{g}^*$), $\mathcal{N}(\mathfrak{g}/\mathfrak{k}) := \mathcal{N}(\mathfrak{g}^*) \cap (\mathfrak{g}/\mathfrak{k})^*$, $\mathfrak{B}$ (the variety of Borel subalgebras in $\mathfrak{g}$), $\mathcal{D}_\mathfrak{B}$ (sheaf of algebraic differential operators on $\mathfrak{B}$), $T^*\mathfrak{B}$ (cotangent bundle to $\mathfrak{B}$), $T^*_Q\mathfrak{B}$ (conormal bundle in $T^*\mathfrak{B}$ to a K orbit Q on $\mathfrak{B}$), μ (moment map for Hamiltonian action of G on $T^*\mathfrak{B}$).

Next I'll pause to describe a few of theses objects more explicitly as well as introduce more notation. First we may obviously identify

$$T^*\mathfrak{B} = \{(\mathfrak{b}, \xi) \mid \mathfrak{b} \in \mathfrak{B} \text{ and } \xi \in (\mathfrak{g}/\mathfrak{b})^*\}.$$

In terms of these identification, the moment map concretely may be written as

$$\begin{aligned} T^*\mathfrak{B} &\longrightarrow \mathfrak{g}^* \\ (\mathfrak{b}, \xi) &\longrightarrow \xi \end{aligned}$$

from which one observes that the image is contained in the nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$. Next we may identify

$$T^*_Q\mathfrak{B} = \{(\mathfrak{b}, \xi) \mid \mathfrak{b} \in Q \text{ and } \xi \in (\mathfrak{g}/\mathfrak{b} + \mathfrak{k})^*\},$$

and so for any $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ in Q

$$\mu(T^*_Q\mathfrak{B}) = K \cdot (\mathfrak{g}/\mathfrak{b} + \mathfrak{k})^* \subset \mathcal{N}(\mathfrak{g}/\mathfrak{b})^*$$

It is not hard to see that this image contains a dense orbit of K. We denote this orbit by $\mu(Q)$. Given such an orbit $\mathcal{O}_K$, we define

$$\mu^{-1}(Q) := \{Q \in K\backslash\mathfrak{B} \mid \mu(Q) = \mathcal{O}_K\}.$$

This set is always nonempty, and we obtain an interesting partition

$$K\backslash\mathfrak{B} = \coprod_{\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*} \mu^{-1}(\mathcal{O}_K). \tag{1.1}$$

This decomposition is a kind of generalized Robinson-Schensted correspondence: if $G_\mathbb{R} = \mathrm{GL}(n, \mathbb{C})$, then $K\backslash\mathfrak{B}$ naturally identifies with the symmetric group S_n, $K\backslash\mathcal{N}(\mathfrak{g}/\mathfrak{b})^*$ is parametrized by partitions

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of n, and the subsets $\mu^{-1}(\mathcal{O}_K)$ consist of those elements of S_n whose Robinson-Schensted tableaus have the shape corresponding to a fixed partition of n.

Fix an irreducible Harish-Chandra module X with trivial infinitesimal character. Set $\mathcal{X} := \mathcal{D}_{\mathfrak{B}} \otimes_{\mathrm{U}(\mathfrak{g})} X$; this makes sense since $\mathrm{U}(\mathfrak{g})$ acts by global differential operators on X. Roughly speaking $\mathcal{X}$ is a K-equivariant sheaf of $\mathcal{D}_{\mathfrak{B}}$ modules on $\mathfrak{B}$. Since X is irreducible its support on $\mathfrak{B}$ is the closure of a single K orbit on $\mathfrak{B}$ which we denote $\mathrm{supp}_\circ(X)$.

Next choose a filtration $\mathcal{X}^j$ on $\mathcal{X}$ compatible with the degree filtration on $\mathcal{D}_{\mathfrak{B}}$. (This choice can be avoided by working in a category where objects (roughly) are $\mathcal{D}_{\mathfrak{B}}$-modules together with filtrations. This is undoubtedly the right category to work in. But I will not go into this here.) The symbol calculus identifies $\mathrm{gr}\mathcal{D}_{\mathfrak{B}}$ with functions on $T^*\mathfrak{B}$. So $\mathrm{gr}\mathcal{X}$ naturally becomes an $(\mathcal{O}_{T^*\mathfrak{B}}, K)$ modules (where $\mathcal{O}_{T^*\mathfrak{B}}$ denotes the structure sheaf of $T^*\mathfrak{B}$), i.e. a K-equivariant sheaf on $T^*\mathfrak{B}$. Define $\mathrm{CV}(X)$ to be the support of this sheaf. While the sheaf itself depended on the choice of filtration, its support does not. Since $\mathcal{X}$ is a special kind of $\mathcal{D}_{\mathfrak{B}}$ module (arising as the localization of X), $\mathrm{CV}(X)$ has a special form: there is a subset $\mathrm{cv}(X) \subset K\backslash\mathfrak{B}$ such that

$$\mathrm{CV}(X) = \bigcup_{Q \in \mathrm{cv}(X)} \overline{T^*_Q\mathfrak{B}}.$$

By keeping track of the rank of $\mathrm{gr}\mathcal{X}$ along each irreducible component of $\mathrm{CV}(X)$, we obtain an integral linear combination,

$$\mathcal{CV}(X) = \sum_{Q \in \mathrm{cv}(X)} m_Q[\overline{T^*_Q\mathfrak{B}}],$$

called the characteristic cycle of X.

The next lemma provides some very weak information. It's proof is easy.

Lemma 1.2. *Retain the setting above. Then*

(a) $\mathrm{supp}_\circ(X) \in \mathrm{cv}(X)$*; moreover* $m_{\mathrm{supp}_\circ(X)} = 1$.
(b) *if* $Q \in \mathrm{cv}(X)$*, then* $Q \subset \overline{\mathrm{supp}_\circ(X)}$.

The next result, which is also very easy to prove, shows how to recover Vogan's invariant from the characteristic variety.

Proposition 1.3. *Fix X as above. Recall the associated variety of X,*

$$AV(X) = \bigcup_{\mathcal{O}_K \in \mathrm{av}(X)} \overline{\mathcal{O}_K};$$

here $\mathrm{av}(X)$ *is a subset of K orbits on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$. Then*

$$\mathrm{AV}(X) = \bigcup_{Q \in \mathrm{cv}(X)} \overline{\mu(Q)}.$$

Because of Lemma 1.2(b) and Proposition 1.3, it is natural to investigate how the closure order on $K\backslash\mathfrak{B}$ interacts with the closure order on $K\backslash\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$. More precisely if $Q' \subset \overline{Q}$ does this imply a relationship between the closures of $\mu(Q')$ and $\mu(Q')$? Here is a partial affirmative answer. Again the proof is very easy.

Proposition 1.4. *Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let α denote a simple root of $\mathfrak{h}$ in $\mathfrak{g}$. Write $\mathfrak{P}_\alpha$ for the variety of parabolic subalgebras of $\mathfrak{g}$ of type α and write π_α for the canonical projection from $\mathfrak{B}$ to $\mathfrak{P}_\alpha$. Fix an orbit $Q' \in K\backslash\mathfrak{B}$ and suppose that*

$$\dim(Q') = \dim(\pi_\alpha(Q')).$$

Let Q denote a (in fact, the unique) dense orbit of K on $\pi_\alpha^{-1}(\pi_\alpha(Q'))$. So, in particular, $Q' \subset \overline{Q}$. Then

$$\mu(Q) \subset \overline{\mu(Q')}. \tag{1.5}$$

The closure relations $Q' \subset \overline{Q}$ appearing in the proposition account for many — but not all — codimension one closure relations for $K\backslash\mathfrak{B}$. Remarkably for the other codimension one relations not accounted for by the π_α construction, the conclusion of (1.5) need *not* hold. This is closely connected to the next example which indicates how complicated things can get in even the simplest of cases.

Example 1.6. Suppose $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$. Then, as remarked above, we may identify $K\backslash\mathfrak{B}$ with S_n. Write Q_w for the orbit parametrized by $w \in S_n$ and X_w for the unique Harish-Chandra module with trivial infinitesimal character supported on the closure of Q_w. It follows by combining results of Springer, Steinberg, and Joseph that for a fixed irreducible Harish-Chandra module X with trivial infinitesimal character

$$\mu(\mathrm{supp}_\circ(X)) = \mathrm{av}(X);$$

that is, the associated variety of X is irreducible and, in fact, simply the closure of the moment map image of the conormal bundle to the support of X. As mentioned above, the computation can be made very explicit: $\mu(Q_w)$ is the orbit parametrized by the partition corresponding to the shape of the Robinson-Schensted tableau corresponding to w.

Kazhdan-Lusztig conjectured that indeed $\mathrm{cv}(X) = \{\mathrm{supp}_\circ(X)\}$ for any irreducible X; i.e. $\mathrm{cv}(X_w) = \{Q_w\}$. This turns out to be true for all $n \leq 7$, but remarkably it fails for $n = 8$. More precisely, let s_i denote the transposition of i and $i + 1$ in S_8. Consider the reduced expressions

$$w = s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_7 s_6 s_5 s_4 s_3; \text{ and}$$
$$w' = s_1 s_3 s_4 s_3 s_5 s_4 s_3 s_7.$$

Then Kashiwara and Saito proved that

$$\mathrm{cv}(X_w) = \{X_w, X_{w'}\}.$$

(Notice that $Q_{w'} \subset \overline{Q}_w$ — obviously w' is a subexpression of the reduced expression for w — and this must be the case by Lemma 1.2(b).) This is certainly a very surprising result indicating that the characteristic variety of a Harish-Chandra module is a very, very delicate invariant.

How were Kashiwara and Saito led to such an example? I can only guess, but the the elements w and w' have a quite striking property. First notice that $\tau(w) \subset \tau(w')$; this is a necessary condition by the equivariance results described below. Next we can compute $\mu(Q_w)$ and $\mu(Q_{w'})$ using the Robinson-Schensted correspondence to find

$$\mu(Q_w) = \mathcal{O}_\lambda \text{ where } \lambda \text{ is the partition } 8 = 4 + 2 + 2.$$

while

$$\mu(Q_{w'}) = \mathcal{O}_{\lambda'} \text{ where } \lambda' \text{ is the partition } 8 = 3 + 3 + 1 + 1.$$

The conclusion is that

$$Q_{w'} \subset \overline{Q}_w$$

but

$$\mu(Q_{w'}) = \mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}_\lambda} = \overline{\mu(Q_w)}.$$

This is the *opposite* of the conclusion of (1.5). So the pair Q_w and $Q_{w'}$ are natural candidates to appear in a reducible characteristic variety. Even so, I know of no conceptual "explanation" of the Kashiwara-Saito example. It is very mysterious.

Here is another painful example.

Example 1.7. Consider the categories of Harish-Chandra modules for $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ at trivial infinitesimal character. In fact, these categories are equivalent, but much more is true: their Grothendieck groups carry isomorphic Hecke algebra representations. (Another way to say this is that for complex groups, the Kazhdan-Lusztig algorithm only depends on the Weyl groups which of course are isomorphic in this case.) Thus, for most algebraic purposes, the two categories are indistinguishable. It is thus perhaps surprising that *characteristic cycle computations in these* *categories are very different.* Even if one could compute all characteristic cycles of irreducible Harish-Chandra modules for $\mathrm{Sp}(2n, \mathbb{C})$, it would tell you very little about the corresponding computations for $\mathrm{SO}(2n+1, \mathbb{C})$. Again this indicates the subtleties of this invariant.

The Kashiwara-Saito example is especially deflating and indicates that one really has no hope of computing characteristic cycles in any generality. Here is a weakened invariant which seems to be weakened just enough to be computable, and yet still retain a lot of interesting information.

Definition 1.8. Fix a Harish-Chandra module X with trivial infinitesimal character. Define

$$\mathrm{lt}(X) = \{Q \in \mathrm{cv}(X) \mid \dim(\mu(Q)) \geq \dim(\mu(Q')) \text{ for all } Q' \in \mathrm{cv}(X)\};$$

that is, $\mathrm{lt}(X)$ consists of those orbits in $\mathrm{cv}(X)$ whose moment map images have the largest dimension. Define the leading term of the characteristic variety of X to be

$$\mathrm{LT}(X) = \bigcup_{Q \in \mathrm{lt}(X)} \overline{T_Q^* \mathfrak{B}},$$

and the leading term cycle of X to be

$$\mathcal{LT}(X) = \sum_{Q \in \mathrm{lt}(X)} m_Q [\overline{T_Q^* \mathfrak{B}}].$$

Here is the first indication that the leading term cycle is a tractable invariant.

Example 1.9. Suppose $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{C})$ and let X be an irreducible Harish-Chandra module for X. Then

$$\mathrm{lt}(X) = \{\mathrm{supp}_{\circ}(X)\}.$$

Thus,

$$\mathcal{LT}(X) = 1 \cdot [\overline{T^*_{\mathrm{supp}_{\circ}(X)} \mathfrak{B}}].$$

So the leading term defines the mysterious Kashiwara-Saito example out of existence.

Here is one of the basic results from a recent preprint of mine. It's proof is not really that difficult; it is more a matter of assembling the pieces properly.

Theorem 1.10. *Fix an irreducible Harish-Chandra module X with trivial infinitesimal character and let Q denote the support of the localization $\mathcal{X}$. Suppose (for simplicity) that for $\xi \in \mu(Q)$, the component group of the centralizer of ξ in K surjects onto the component group of the centralizer of ξ in G. Then given the computation of the leading term $\mathcal{LT}(X)$, there is an algorithm to determine the annihilator and associated cycle of X. Conversely, given the computation of* $\mathrm{Ann}(X)$ *and* $\mathcal{AV}(X)$, *there is an algorithm to determine* $\mathrm{LT}(X)$.

The theorem is really meant only as philosophical encouragement: the algorithms in the theorem are ineffective inasmuch as they depend on certain explicit calculations of Springer representations which are (at present) unavailable. But since the Atlas can compute annihilators, for instance, one should view the computation of $\mathcal{LT}(X)$ as potentially tractable. Here is yet another deflating example.

Example 1.11. The first case of an irreducible Harish-Chandra module X with a reducible leading term occurs in $\mathrm{Sp}(6, \mathbb{R})$. It transpires that X is cohomologically induced (in the good range) from the trivial representation on $U(1, 0)$ tensored with a certain lowest weight module X' for $\mathrm{Sp}(4, \mathbb{R})$. It turns out to be very easy to see that X' has a reducible characteristic cycle (but an irreducible leading term). Using a relatively simple relationship between characteristic cycles and cohomological induction, one deduces the reducibility of $\mathrm{LT}(X)$ from the reducibility of $\mathcal{CV}(X')$. Since $\mathrm{LT}(X')$ is indeed irreducible, this is bad news: the computation of leading terms is *not closed under induction.*

These examples leave the water murky. One would need to make further serious computations to see what we might expect in general.

2. TOWARD COMPUTING CHARACTERISTIC CYCLES

By far the most powerful tool in computing characteristic cycles is the Weyl group equivariance result of Tanisaki which we now recall. Let

$$T_K^*\mathfrak{B} = \bigcup_{Q \in K\backslash\mathfrak{B}} \overline{T_Q^*\mathfrak{B}},$$

the conormal (or generalized Steinberg variety) for K orbits on $\mathfrak{B}$. Then the topological construction of Springer representations due to Kazhdan-Lusztig may be adapted to prove that the top Borel-Moore homology

$$\mathrm{H}_{top}^{\infty}(T_K^*\mathfrak{B}, \mathbb{Z})$$

is a Weyl group representation. Of course the fundamental classes of the irreducible component $\overline{T_Q^*\mathfrak{B}}$ of $T^*\mathfrak{B}$ are a basis for this space,

$$\mathrm{H}_{top}^{\infty}(T_K^*\mathfrak{B}, \mathbb{Z}) \simeq \bigoplus_{Q \in K\backslash\mathfrak{B}} [\overline{T_Q^*\mathfrak{B}}].$$

Since the construction of the characteristic cycle given in the last section transparently descends to the Grothendieck group $\mathrm{K}\mathcal{HC}$ of Harish-Chandra modules with trivial infinitesimal character, we may thus view the characteristic cycle as a map

$$CC \ : \ \mathrm{K}\mathcal{HC} \longrightarrow \mathrm{H}_{top}^{\infty}(T_K^*\mathfrak{B}, \mathbb{Z}).$$

We have already discussed that the range is a representation of W. Of course W also action on the domain by the coherent continuation action. Tanisaki (building on earlier work of Kashiwara and Tanisaki in the complex case) prove that CC is W-equivariant.

Theorem 2.1. *The characteristic cycle functor is W-equivariant. Since the coherent continuation action is effectively computable, effectively computing characteristic cycles of Harish-Chandra modules is equivalent to computing the Kazhdan-Lusztig action of W on $\mathrm{H}_{top}^{\infty}(T_K^*\mathfrak{B}, \mathbb{Z})$ in the basis of fundamental classes of conormal bundle closures. (This, for instance, implies the effective computation of the top-dimensional Springer representation of W in the basis of fundamental classes of irreducible components of the Springer fiber; but it is much stronger.)*

I now remark on two additional powerful aids in computing characteristic cycles. The first is easiest to understand. Suppose X is cohomologically induced from $\mathfrak{s} = \mathfrak{g}' \oplus \mathfrak{u}'$ in the good range from a module X' for $G'_{\mathbb{R}}$. Suppose

$$CC(X') = \sum_{Q'} m_{Q'} [\overline{T_{Q'}^*\mathfrak{B}}].$$

Given an orbit Q', there is a unique orbit Q such that the projection of Q from $\mathfrak{B}$ to G/S fibers over its image with fiber isomorphic to Q'. Then

$$CC(X) = \sum_{Q'} m_{Q'} [\overline{T_Q^*\mathfrak{B}}].$$

In particular if X is of the form $A_{\mathfrak{s}}$, then

$$\mathcal{CV}(X) = 1 \cdot [\overline{T_{\mathrm{supp}_\circ(X)}^*\mathfrak{B}}].$$

This is very powerful in practice.

The next utilizes the known leading term calculations for type A. I phrase the result as follows. Fix two adjacent simple roots α and β of the same length. Suppose

$$s_\alpha \cdot [\overline{T_Q^*\mathfrak{B}}] = -[\overline{T_Q^*\mathfrak{B}}],$$

but

$$s_\beta \cdot [\overline{T_Q^*\mathfrak{B}}] \neq -[\overline{T_Q^*\mathfrak{B}}].$$

Then there is a unique dense orbit, say Q^β, in the preimage of the projection of Q to $\mathfrak{P}_\alpha$ (notation as in Proposition 1.4) and

$$s_\beta \cdot [\overline{T^*_Q \mathfrak{B}}] = [\overline{T^*_Q \mathfrak{B}}] + [\overline{T^*_{Q^\beta} \mathfrak{B}}] + \text{other terms}.$$

Moreover if $[\overline{T^*_{Q'} \mathfrak{B}}]$ is one of the other terms, both s_α and s_β map it to its negative. I prove this in a recent preprint. There are other more subtle (but quite powerful) restrictions also given in that preprint. They are too technical to discuss here however.

Apart from this list, I know of very few other general restrictions.

The question remains: how much can one compute given the above restrictions? This is a fascinating question.

3. EXAMPLES

Here I record the full Springer representation on the integral homology of the conormal variety for $\mathrm{Sp}(4, R)$. Since there are 11 orbits of K on the flag variety, the representation is 11 dimensional. I adhere to the Atlas labeling. Below I give two matrices. One is the simple reflection s_1 in the short simple root. It's ijth entry is the coefficient of the conormal bundle to the orbit Atlas labels $i-1$ in s_1 applied to the conormal bundle to the orbit Atlas labels $j-1$.

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

$$s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

I double-checked in Maple that $s_1^2 = s_2^2 = e$ and $(s_1 s_2)^2 = (s_2 s_1)^2$.

Compare with the coherent continuation representations on the 12 dimensional block containing the trivial representation (in the basis of irreducible characters ordered to be consisting with the above ordering so that the "extra" 12th representations is last).

$$
s_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

$$
s_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Again, I double-checked these in Maple.

Let T_i denote the closure of the conormal bundle to the dense K orbit in the support of X_i where X_i is the irreducible Harish-Chandra module with trivial infinitesimal character that Atlas labels i. So $i \mapsto T_i$ fails to be injective (only) because $T_{10} = T_{11}$ are both the zero section. Then $CC(X_i) = T_i$ except in the following cases:

$$
\begin{aligned}
CC(X_8) &= T_8 + T_2 \\
CC(X_9) &= T_9 + T_3 \\
CC(X_{11}) &= T_{11} + T_9 + T_8 + T_6.
\end{aligned}
$$

As a final note, for the real rank one form of F4 I checked that

$$
\mathcal{CV}(X) = 1 \cdot [\overline{T^*_{\mathrm{supp}_\circ(X)}\mathfrak{B}}]
$$

for all irreducible Harish-Chandra modules X with trivial infinitesimal character.

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