The Mathematical Work of David A. Vogan, Jr.

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To David, with gratitude and respect

Abstract Over four decades David Vogan's groundbreaking work in representation theory has changed the face of the subject. We give a brief summary here.

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It is difficult to give a complete overview in a few short pages of the impact of the work of David Vogan, but it is easy to identify the starting point: his 1976 MIT Ph.D. Thesis [V76], completed at the age of 21 under the direction of Bertram Kostant, was a striking advance in the subject. It paved the way for an algebraic classification of irreducible (not necessarilty unitary) representations for a reductive Lie group G at a time when the existing approaches to such classification problems (in the work of Harish-Chandra, Langlands, Knapp-Zuckerman, and others) were heavily analytic. David's classification (published as an announcement in [V77] and partially in [V79d]) was later streamlined and extended with Zuckerman using Zuckerman's new technology of cohomological induction, which complemented the Lie algebra cohomology techniques developed in David's thesis. A full exposition, including an influential list of problems, appeared in [V81a], completed in 1980.

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Harish-Chandra developed the theory of non-unitary representations in large part to study unitary representation, and indeed unitary representation theory emerged as the central thread in virtually all of David's work. Along the way, David naturally proved many results about non-unitary representations, but rarely without considerations of their relevance for unitarity. This is beautifully laid out in [V83a]; in [V87], an exposition of David's Weyl Lectures at the Institute in 1986; and in [V87], the notes from David's 1986 Plenary ICM address.

As he was completing his algebraic classification, David tackled a description of irreducible characters, which he solved in [V83] and [LV83] with Lusztig before the completed manuscript [V81a] went to press. David's approach to computing irreducible characters involved expressing them as explicit linear combinations of characters of certain standard parabolically induced representations. Since the latter, in principle, can be computed from Harish-Chandra's results on the discrete series and the tractable effect of parabolic induction on characters, this gives the irreducible characters. The problem is therefore parallel to expressing an irreducible highest weight module as a virtual sum of Verma modules, the subject of the Kazhdan-Lusztig conjectures, and so the irreducible character problem for Harish-Chandra modules became informally known as the "Kazhdan-Lusztig algorithm for real groups". David constructed this algorithm, modulo a judicious technical conjecture about the semisimplicity of certain modules arising from wall-crossing translation functors, in [V79c].

A substantial part of [V79a, V79c] (and its exposition in [V81a]) relied on understanding how irreducible characters behave under coherent continuation. This was also the starting point for the earlier work with Speh [SV80] which addressed fundamental problems of the reducibility of standard modules, clearly of importance to both the classification and irreducible character problem (and to unitarity questions). The intricate arguments in [SV80] were simplified in [V79a] which further made a key connection with dimensions of certain Ext groups in the category of Harish-Chanda modules. This latter connection became an extremely powerful tool in the computation of irreducible characters, culminating in the Kazhdan-Lusztig algorithm for real groups in [V79c]. (When applied to the setting of highest weight modules, the algorithm of [V79c] in fact reduced to the original algorithm of Kazhdan-Lusztig.) The technical conjecture mentioned above, though entirely algebraic in its formulation, was ultimately only surmounted by geometric methods in positive characteristic [LV83] and partly based on the new localization theory of Beilinson-Bernstein and its connection to the cohomological methods in David's classification [V83]. (Particular cases of the Lusztig-Vogan geometry and related settings are considered in the contributions of Graham-Li and McGovern in this volume.) David gave a very accessible "roadmap" to the Kazhdan-Lusztig conjectures for real groups in [V83b], including complete recursion formals for the polynomials of [LV83] which came to be know as Kazhdan-Lusztig-Vogan, or KLV, polynomials.

Thus, in the span of just of a few spectacularly productive years, the irreducible character problem was solved by a dazzling array of new techniques. There was much more to be done with these powerful new ideas.

Because one is often interested in extracting less complete (but more accessible) information about irreducible representations than their characters, David was led to consider various invariants of Harish-Chandra modules. His influential paper [V78a] (see also [V80b]) lays the foundation of Gelfand-Kirillov dimension for Harish-Chandra modules and classifies those representations which are generic in the sense they admit Whitaker models. The contribution of Wallach to this volume discusses GK dimension for smaller discrete series.

Connections to Kahzdan-Lustzig theory and Joseph's theory of primitive ideal cells, as well as weaker formulations of quantization, naturally led to fundamental questions in the theory of primitive ideals in enveloping algebras of complex semisimple Lie algebras initiated by Dixmier, Duflo, Joseph, and others. In [V79b], David studied primitive ideals directly by understanding their behavior under coherent continuation restricted to rank two root subsystems, generalizing the Borho-Jantzen τ-invariant which considered rank one subsystems. (The paper of Bonnafé and Geck in this volume takes up many of these ideas.) Later, working with Garfinkle, he proved analogous results for restricting to the root system of type D4 [GaV92], necessary for any systematic analysis of branched Dynkin diagrams. These turned out to be very powerful computational tools which were exploited to great effect in the work of Garfinkle and others. In [V80a], David related the ordering of primitive ideals to a preorder arising in the original paper of Kazhdan-Lusztig. He and Barbasch carried out the classification of primitive ideals in complex semisimple Lie algebras [BV82, BV83a]. Along the the way, they showed that representations of the Weyl group that arise in Joseph's Goldie rank construction are exactly the special ones in the sense of Lusztig, and related them to Fourier inversion of certain unipotent orbital integrals.

As he was developing his algebraic theory, David was naturally led to understand its relation with Langlands' original classification and the larger context of the Local Langlands Conjectures. The dual group makes a fundamental appearance in the technical tour de force [V82] where David uncovered an intricate symmetry in his Kazhdan-Lusztig algorithm for real groups: he proved that computing irreducible characters of real forms G of a complex connected reductive group $G_{\mathbb{C}}$ is dual, in a precise sense, to computing irreducible characters of real forms of the dual group $G_{\mathbb{C}}^{\vee}$. (When applied to the case of complex groups, it can be interpreted as the symmetry of the Kazhdan-Lusztig algorithm, noted in the original paper of Kazhdan-Lusztig, corresponding to the order-reversing symmetry of the Bruhat order.) The full significance of this deep and beautiful symmetry, now known as Vogan duality, was only fully realized later in [ABV92]. In order to get a perfectly symmetric statement, one must consider multiple real forms at the same time. This immediately leads to the question of when two collections of representation of multiple real forms should be considered equivalent, and eventually to the definition of strong real forms [ABV92], differing in subtle and interesting ways from the classical notion of real form. On the dual side, it led Adams, Barbasch, and Vogan (building on earlier ideas of Adams and Vogan [AV92a, AV92b]) to reformulate the space of Langlands parameters.

Like the space of classical Langlands parameters, the reformulated space of ABV parameters is a complex algebraic variety on which the complex dual groups acts. Unlike the space of Langlands parameters in the real case, the dual group orbits on the ABV space have closures with nontrivial singularities. The main result of [ABV92] is a refinement of the representation theoretic part of Local Langlands Conjecture for real groups where K-groups of representations of strong real forms are dual to K-groups of appropriate categories of equivariant perverse sheaves on the space of ABV parameters. This incredibly intricate correspondence is ultimately deduced from [V82]. Using it, [ABV92] makes precise (for real groups) and establishes a series of conjectures of Arthur and provide a different perspective on the Langlands-Shelstad theory of endoscopy. In the ABV theory, Arthur packets of representations are defined in terms of characteristic cycles of perverse sheaves on the space of ABV parameters. Such cycles are still mysterious and poorly understood. (Some related complications are on display in the new examples of Williamson in this volume.) For classical groups, Arthur has recently defined Arthur packets and established his conjectures from a different point of view, and comparing his results in the real case to those of [ABV92] is still to be understood. Meanwhile, Soergel (generalizing Beilinson-Ginzburg-Soergel) formulated a still-open conjecture extending the main result of [ABV92] (established on the level of K-groups) to a categorical statement. Using the real case as a model, David proposed refinements of the local Langlands conjectures in the much more difficult p-adic case as well ([V93a]). This has played an increasingly important role in recent years.

The paper [V84] is most often remembered for its a long sought-after proof that cohomological induction preserves unitarity under fairly general hypothesis. As a consequence, certain representations (so-called $A_{\mathfrak{q}}(\lambda)$ modules) constructed by Zuckerman from unitary characters are indeed unitary. Earlier, Vogan and Zuckerman [VZ84] had classified all unitary representations with nonzero relative Lie algebra cohomology as $A_{\mathfrak{q}}$ modules, except that it was still unproved that the modules they had classified were indeed unitary. For applications to the cohomology of locally symmetric spaces discussed in [VZ84] (and [V97b]), this was not important, but for unitary representation theory it was a central question at the time.

In later work, David explicitly clarified the role of the $A_{\mathfrak{q}}(\lambda)$ in the discrete spectrum of symmetric spaces [V88b], as well as how they appear as isolated representations [V07a]. Basic questions about explicitly constructing the unitary inner product geometrically on minimal globalizations are still open (as explained in [V08]). He returned to the Dirac operator methods of [VZ84] in an influential series of lectures [V97] that spawned an entire new area of investigation, still developing today (for example in the contribution of Huang to this volume).

Striking examples of complete classifications of unitary representations include David's description of the the unitary dual of $\mathrm{GL}(n,\mathbb{R})$ (obtained in 1984 and appearing in [V86a]), and later the unitary dual of G_2 [V94] (dedicated to Borel).The description of the unitary duals given in these cases was organized in terms of systematic procedures (like cohomological induction and construction of complentary series) applied to certain building blocks. The systematic role of cohomological induction was clarified later in his deep work with Salamanca-Riba [SaV98], [SaV01];

see also the exposition [V00b]. The groundwork for systematizing the construction of complementary series is [BV84]. But abstracting a general definition of the mysterious building blocks, which came to be known (somewhat imprecisely) as unipotent representations, proved to be more difficult and is a major theme in David's work. The overviews [V97a], [V00c], [V93b] contain many ideas. At the heart of the notion of unipotent is the connection to nilpotent coadjoint orbits in semisimple Lie algebras and modern approaches to geometric quantization pioneered by Kirillov and Kostant. The paper of Graham [GrV98] makes significant progress in the setting of complex groups.

Early on, David proved that the annihilator of an irreducible unitary Harish-Chandra module for a complex group was completely prime. He then formulated a conjectural kind of Nullstellensatz for such ideals in [V86b] in which finite algebra extensions of primitive quotients of the enveloping algebra play a crucial role. Such algebra extensions, sometimes called Dixmier algebras, were further studied in [V90] and the notion of induced ideal was extended to them. Many beautiful facets of David's conception of the orbit method are explained in [V88a].

One of the fundamental ways that nilpotent orbits appear in the representation theory of real groups is through the asymptotics of the character expansion discovered in [BV80]. In this construction, each Harish-Chandra module gives rise to a real linear combination of real nilpotent coadjoint orbits. Nilpotent orbits also arise through David's construction of the associated cycle of a Harish-Chandra module [V91], a positive integral combination of complex nilpotent coadjoint orbits for symmetric pairs (the setting originally investigated by Kostant-Rallis). The Barbasch-Vogan conjecture, proved by Schmid and Vilonen, asserted that the two kinds of linear combinations coincide perfectly under the Kostant-Sekiguchi bijection. David uses these invariants to define conditions on a class of unipotent representation in [V91]. A beautiful example of the explicit desiderata in a case of great interest is given in [AHV98].

A weaker version of the quantization of nilpotent orbits instead focuses on constructing Harish-Chandra modules with prescribed annihilator. Barbasch and Vogan long ago identified a set of interesting infinitesimal characters and sought to understand Harish-Chandra modules annihilated by maximal primitive ideals with those infinitesimal characters, conjecturing that such Harish-Chandra modules were unitary. (This generalizes the study of minimal representations [V81b], where the maximal primitive ideals are the Joseph ideals.) In [BV85], Barbasch and Vogan discovered that many of their interesting infinitesimal characters — conjecturally those arising as the annihilators of unitary representations with automorphic applications — fit perfectly into the framework of the ideas proposed by Arthur. The paper [BV85] gives strikingly simple character formulas for these so-called special unipotent representation in the setting of complex groups (and the ideas make sense for real groups too). An aspect of the proof of the character formulas relied on ways to count special unipotent representations using the decomposition of the coherent continuation representation into cells. The theory of Kazhdan-Lusztig cells for complex groups was extended to real groups with Barbasch in [BV83b].

The theory of [BV85] was generalized in the important closing chapter of [ABV92], where it can be understood entirely in terms of the principle of Langlands functoriality. A central theme in David's work is the extent to which functoriality extends to organize all unitary representations, not just automorphic ones. (A very accessible introduction to this set of ideas is contained in [V01]). The Shimura-type lifting in [ABPTV97] can also be understood in these terms as part of an aim to extend functoriality to certain nonalgebraic groups.

The interaction between the philosophy of the orbit method and the construction of associated varieties was developed further in [V00a] culminating in a general (still unrealized) approach toward proving the unitarity of the special unipotent representations defined in [BV85] and [ABV92]. One facet of this involved relating the $K_{\mathbb{C}}$ eqivariant K-theory of nilpotent cone in the symmetric space setting to the tempered dual of G. Vogan conjectured a precise relationship (independently and earlier conjectured by Lusztig for complex groups) that was later proved in special cases by Achar and by Bezrukavnikov. The article of Achar in this volume provides an up-to-date look at this direction.

The unitarity of $A_{\mathfrak{q}}(\lambda)$ in [V84] is a deep and important result, but it is the theory of signature characters of Harish-Chandra modules that David developed to tackle the problem that has proved to be even more influential. It immediately led Wallach to a shorter proof of the unitarity of the $A_{\mathfrak{q}}(\lambda)$ modules, for example, and was adapted to unramified representation of split p-adic groups by Barbasch and Moy. But for David it was part of an approach to determining the entire unitary dual of a reductive group. The paper [V84] proposes an algorithm (heavily rooted in his Kazhdan-Lusztig theory for real groups and the theory of the Jantzen conjecture) to determine if an irreducible representation specified in the Langlands classification is in fact unitarity. The algorithm was predicated on determining certain signs that, at the time, were inaccessible. Determining the signs in the algorithm of [V84] was finally surmounted in [ALTV12], giving a finite effective algorithm to locate the unitary dual of a reductive Lie group in the Langlands classification.

The paper [ALTV12] relies on relating classical invariant Hermitian forms on irreducible Harish-Chandra modules (the object of study in unitary representation theory) to forms with a different, more canonical invariance property. Once this latter invariance property was uncovered, its importance was immediately recognized in other settings (for example in the geometric setting of Schmid and Vilonen explained in their contribution to this volume and the analogous p-adic setting in the contribution of Barbasch and Ciubotaru). Translating between the two kinds of forms in [ALTV12] immediately leads one to certain extended groups which are not the real points of a connected reductive algebraic group, and which are outside the class of groups for which [V83] established a Kazhdan-Lusztig algorithm. In recent work, Lusztig and Vogan [LV14] (generalizing their earlier work [LV83, LV12]) provides the geometric foundations of Kazhdan-Lusztig theory for such extended groups. In particular, they define a Hecke algebra action on an appropriate Grothendieck group. This action characterizes the "twisted" Kazhdan-Lusztig polynomials in this setting. [LV14] gives explicit formulas for individual Hecke operators, but they depended on certain choices. The effect of these choice is completely understood in the contribution of Adams-Vogan in this volume. Meanwhile, Lusztig and Vogan in this volume provide an extension of the results of [LV12] to the setting of arbitrary Coxeter groups using the new theory of Elias and Williamson.

Over the last fifteen years, David has been deeply involved in the atlas project, the goal of which is to translate much of the mathematics described above into the computer software package `atlas` in the generality of the real points of *any* complex connected reductive algebraic group. This has involved his close collaboration with many people, but especially with Adams, du Cloux, and van Leeuwen. David's Conant Prize winning article [V07b] gives an overview into the first step, the implementation of the computation of irreducible characters and, in particular, the computation of the KLV polynomials for the split real form of E8. His paper [V07c] is devoted to algorithms at the heart of computing the K-spectrum of any irreducible Harish-Chandra module. At present the software is able to test the unitarity of any irreducible Harish-Chandra module specified in the Langlands classification. The results of [V84] imply that testing a finite number of such representations suffices to determine the entire unitary dual. The implementation of this will almost certainly be complete in the next year or two, a remarkable achievement that no one could have predicted was possible even just a decade ago. In many ways, it is the culmination of David's seminal contributions to unitarity representation theory.

The above captures a sliver of the mathematics developed in David's papers. It says little of influential expositions that have, by now, educated generations. It also says nothing of the immense amount of mathematical ideas David gave freely to others, nor of his selfless devotion to the profession of mathematics. But, we hope, it points to the breadth of his influence to date, as well as some of the exciting work left to be done.

Ph.D. Students

Pramod Achar, 2001
Jesper Bang-Jensen, 1987
Luis Casian, 1983
Chunyang Fang , 2007
Eugenio Garnica, 1992
William Graham, 1992
Jerin Gu, 2008
Ben Harris, 2011
Hongyu He, 1998
Marketa Havlickova, 2008
Jing-Song Huang, 1989
Joseph Johnson, 1983
Wentang Kuo, 2000
Adam Lucas, 1999
Christopher Malon, 2005
Eric Marberg, 2013
William McGovern, 1987
Hisayosi Matumoto, 1988
Diko Mihov, 1996
Monica Nevins, 1998
Alessandra Pantano, 2004
Dana Pascovici, 2000
Thomas Pietraho, 2001
Iwan Pranata, 1989
Susana Salamanca, 1986
James Schwartz, 1987
Peter Speh, 2012
Kian Boon Tay, 1994
Peter Trapa, 1998
Wai Ling Yee, 2004

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