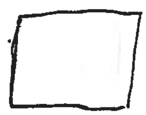
13.1 & 13.2 Double Integrals over Rectangles & Iterated Integration.

Question: What is signed volume?

Estimating signed volume

Let $z = f(x,y)$ be defined on closed retangle R.

Let $A_1, A_2, \ldots\ldots A_n$ be subrectangles

$A_1 \cup A_2 \cup \ldots . A_n =$  .

An estimate of the signed volume between the x-y plane and $z = f(xy)$ over R is:

$$\boxed{\sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k) A_k.}$$

Question: How do we get from the estimate to finding the actual signed volume?

Calculating Signed Volume

Let $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$.

Draw a picture of R in the x-y plane.

Assume $f(x,y)$ is continuous over R. the signed volume between the x-y plane and $z = f(x,y)$ over R is:

$$V = \iint_R f(x,y)\, dA$$

$=$ ____________________

write the iterated integral.

13.1 & 13.2 Cont

Ex 1 $\iint_R (y-x+4)\,dA$ $R=\{(x,y): 0\le x\le 4,\ 0\le y\le 4\}$

a.) sketch the solid whose volume is given by the integral.

b. Estimate the volume by dividing R up into four 2unit x 2unit rectangles.

c.) Calculate the exact volume.

13.1 & 13.2 Cont

Ex 2 Evaluate $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y} \, dy \, dx$.

Ex 3 Evaluate $\int_0^1 \int_0^2 \frac{y}{1+x^2} \, dy \, dx$.

13.3 Double Integrals over Nonrectangular Regions

Suppose S is a simple, closed curve region and $f(x,y)$ a continuous function over S.

If we integrate first with respect to x, then with respect to y, what is the signed volume?

$$V = \int_{\square}^{\square}\int_{\square}^{\square} f(x,y) \_\_\_\_ \_\_\_\_$$

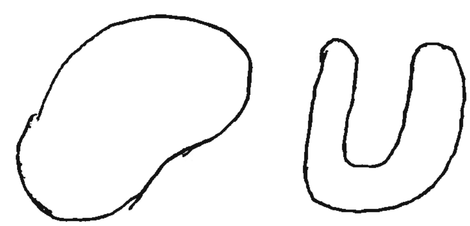
What are these limits?

What can these limits be?

All the regions over which we will integrate are <u>simple</u>.*

(Note, simple refers to the region, not the curve.)

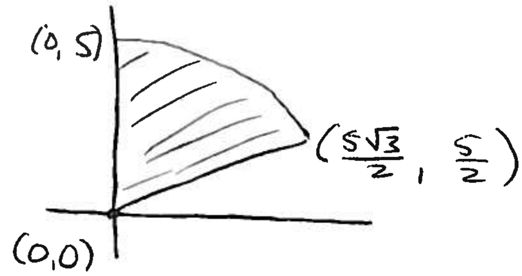
A region is simple if you can draw a line through it & never go in, then out, then back in the shape.



* Sometimes x-simple, or y-simple is enough.

Ex 1 Calculate $\int_1^2 \int_0^{x^2} \frac{y^2}{x} \, dy \, dx$.

13.3

Ex 2. $f(x,y) = x + y$, $S =$ 

a.) determine your limits of integration & your integral(s) if you set them up as "dxdy" and as "dydx".

b.) Calculate the volume between $f(x,y)$ & the x-y plane over the region S.

13.3 Cont

Ex 3 a) Sketch the solid in the first octant bounded by the coordinate planes $2x + y - 4 = 0$ and $8x + y - 4z = 0$. Then calculate its volume with iterated integration.

13.4 Double Integrals in Polar Coordinates

$$\text{Volume} = \iint_S f(r,\theta)\, \underbrace{r\, d\theta\, dr}_{dA}$$

↑
of a shape
over a surface S

or

$$\iint_S f(r,\theta)\, r\, dr\, d\theta$$

Ex 1 Calculate

a.

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\, dr\, d\theta$$

b.)

$$\int_0^{\sin\theta} \int_0^{\frac{\pi}{2}} r\, d\theta\, dr$$

13.4

Ex 2 Sketch the region over which you are integrating, convert to polar coordinates and evaluate.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2y^2+y^4)\,dy\,dx$$

13.4 Cont

Ex 3 Consider the solid inside the paraboloid $z = 4 - x^2 - y^2$, outside the cylinder $x^2 + y^2 = 1$, and above the x-y plane.

Sketch the solid & calculate the volume.

13.6 Surface Area

Let $f(x,y)$ be continuous over S (S is a region of the function's domain). Then the surface area of $f(x,y)$ over S is:

$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

Ex1 Make a sketch & find the area of the part of the surface $z = \sqrt{4-y^2}$ in the first octant that is directly above the circle $x^2+y^2=4$ in the xy-plane.

Ex2 Make a sketch & find the area of the surface $z = \frac{x^2}{4} + 4$ that is cut off by the planes $x=0$, $x=1$, $y=0$, & $y=2$.

13.6 Cont

Ex3 Make a sketch & find the area of the surface that is the part of the cylinder $x^2+y^2=ay$ inside the sphere $x^2+y^2+z^2=a^2$, where $a>0$.

13.7 Triple Integrals in Cartesian Coordinates

$$\iiint_S f(x,y,z)\,dV = \int_{a_1}^{a_2} \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dV$$

Ex 1 a) Sketch the solid $S = \{(x,y,z) : 0 \le x \le 5,\ z^2 \le y \le 9,\ 0 \le z \le 3\}$

b.) Write an iterated integral for $\iiint_S xyz\,dV$ where S is the set in A.

13.7 Cont

Ex2 Find the volume of the solid bounded by the cylinder $y = x^2 + 2$ and the planes $y = 4$, $z = 0$, $3y - 4z = 0$, using a triple iterated integral.

13.7 Cont.

Ex3 Write the integral $\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x,y,z)\, dx\, dz\, dy$

with the order $dy\, dx\, dz$.

13.8 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical

$$\iiint_S f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Ex1) Describe the region of integration & evaluate the integral.

$$\int_0^{\pi}\int_0^{\sin\theta}\int_0^{2} r\,dz\,dr\,d\theta$$

13.8 Cont

Ex2. Sketch the region bounded above by the plane $z = y + 4$, below by the xy-plane, and laterally by the right circular cylinder having radius 4 & whose axis is the z-axis.

Spherical

$$\iiint\limits_S f(x,y,z)\,dV = \int_{\varphi_1}^{\varphi_2}\int_{g_1(\varphi)}^{g_2(\varphi)}\int_{\psi(\theta,\varphi)}^{\psi(\theta,\varphi)} f\,\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

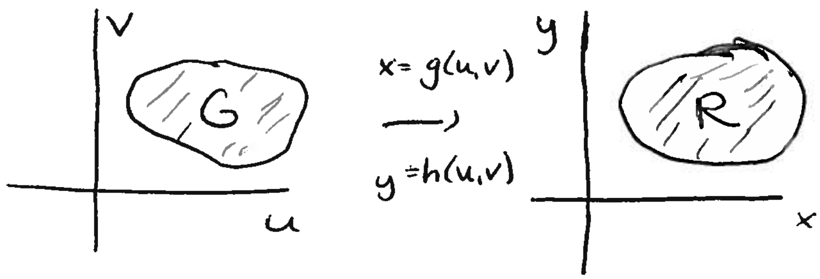
Convert from rectangular to spherical

Ex 3 Change this integral to spherical Coordinates & evalulte it.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dy\,dx$$

13.9 Change of Variables (Jacobian Method)

Idea



abs value

$$\iint_G f(g(u,v),\ h(u,v))\,|J(u,v)|\,du\,dv = \iint_R f(x,y)\,dx\,dy$$

$$|J(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

determinant

<u>Ex1</u> Find the image of the rectangle with corners $(0,0)$, $(3,0)$, $(3,1)$, $(0,1)$ under the transformation $x = 2u + 3v$, $y = u - v$. Then find the Jacobian of the transformation.

Ex2 Use a transformation to evaluate $\iint_R (2x-y)\cos(y-2x)\,dA$ over R. R is the triangle with vertices $(0,0)$, $(0,2)$ $(1,0)$.