CALCULUS II, Second Semester
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CALCULUS I, Second Semester

VI. Transcendental Functions

6.1 Inverse Functions

The functions e^x and $\ln x$ are inverses to each other in the sense that the two statements

$$y = e^x \ , \qquad x = \ln y$$

are equivalent. In general, two functions f, g are said to be *inverse to each other* when the statements

$$y = f(x) \ , \qquad x = g(y) \tag{6.1}$$

are equivalent for x in the domain of f, and y in the domain of g. Often we write $g = f^{-1}$ and $f = g^{-1}$ to express this relation. Another way of giving this citerion is

$$f(g(x)) = x \qquad g(f(x)) = x \ .$$

Example 6.1. Find the inverse function for $f(x) = 3x - 7$. We write $y = 3x - 7$ and solve for x as a function of y:

$$x = \frac{y+7}{3} \ . \tag{6.2}$$

The equations $y = 3x - 7$ and $x = (y + 7)/3$ are equivalent for all x and y, so (6.2) gives us the formula for the inverse of f: $f^{-1}(y) = (y + 7)/3$. Since it is customary to use the variable x for the independent variable, we should write:

$$f^{-1}(x) = \frac{x+7}{3} \ .$$

Example 6.2. Find the inverse function for

$$f(x) = \frac{x}{x+1} \ .$$

We let $y = x/(x + 1)$, and solve for x in terms of y:

$$yx + y = x \qquad \text{so that} \qquad y = x(1 - y) \ , \tag{6.3}$$

so that

$$x = \frac{y}{1-y} \ .$$

Thus

$$f^{-1}(x) = \frac{x}{1-x} \ .$$

Notice that -1 is excluded from the domain of f, and 1 is excluded from the domain of f^{-1}. In fact, we see that these substitutions in equations (6.3) lead to contradictions.

We have to be careful, in discussing inverses, to clearly indicate the domain and range, otherwise we have ambiguities and make mistakes.

Example 6.3. x^2 and $\sqrt{x}$ appear to be inverses since $(\sqrt{x})^2 = x$. But since the symbol $\sqrt{\ }$ gives the positive root, $\sqrt{x^2} = |x|$ which is not x when x is negative.This ambiguity is clarified by specifying the domains of the functions. So, for $x \geq 0$, $\sqrt{x}$ is the inverse of x^2, but for $x \leq 0$, $-\sqrt{x}$ is the inverse of x^2. Finally, $\sqrt{x}$ is only defined for nonnegative numbers.

We illustrate this graphically in figure 6.1.

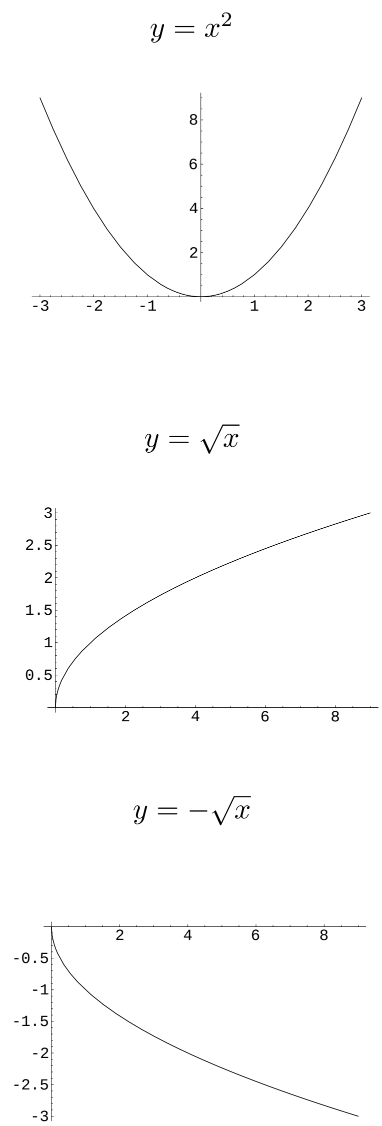


Figure 6.1

In the first graph each horizontal line $y = y_0$ intersects the graph in two points for $y_0 > 0$, and in no points for $y_0 < 0$. So the domain of an inverse function can contain no negative numbers, and for positive numbers, there are 2 choices of inverse, one for the function x^2, x nonnegative, and the other for x^2, x nonpositive.

In general, this provides a graphical criterion for a function to have an inverse:

Proposition 6.1. Let $y = f(x)$ for a function f defined on the interval $a \leq x \leq b$. Let $f(a) = \alpha,\ f(b) = \beta$. If, for each γ between α and β the line $y = \gamma$ intersects the graph in one and only one point, then f has an inverse defined on the interval between α and β.

For if (c, γ) is the point of intersection of the graph with the line $y = \gamma$, define $f^{-1}(\gamma) = c$.

For a continuous function, we know, from the Intermediate Value Theorem of Chapter 2, that each such line $y = \gamma$ intersects the graph in at least one point. Thus for continuous functions, we can restate the proposition as

Proposition 6.2. Let $y = f(x)$ for a continuous function f defined on the interval $a \leq x \leq b$. Let $f(a) = \alpha,\ f(b) = \beta$. If the condition

$$x_1 \neq x_2 \qquad \text{implies} \qquad f(x_1) \neq f(x_2) \tag{6.4}$$

then f has an inverse defined on the interval between α and β.

For a differentiable function, it follows from Rolle's theorem of chapter that condition (6.4) holds if $f'(x) \neq 0$ for all $a \leq x \leq b$.

Proposition 6.3. Let $y = f(x)$ for a differentiable function f defined on the interval $a \leq x \leq b$. Let $f(a) = \alpha,\ f(b) = \beta$. If $f'(x) \neq 0$ in the interval, then f has an inverse defined on the interval between α and β.

Example 6.4. Let $f(x) = x^2 - x$. Find the domains for which f has an inverse, and find the inverse function.

First, differentiate: $f'(x) = 2x - 1$. Thus $f'(x) < 0$ for $x < 1/2$, and $f'(x) > 0$ for $x > 1/2$, so we should be able to find inverses for f on each of the domains $(-\infty, 1/2), (1/2, \infty)$.To find the formula for the inverse, let $y = x^2 - x$ and solve for x in terms of y. To do this, we write the equation as $x^2 - x - y = 0$, and use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1 + 4y}}{2} \ .$$

How convenient: we're looking for two possible inverses, and here we have two choices. Notice first that because of the square root sign, the domain of y must be $y \geq -1/4$. We conclude that, in the domains $x \geq 1/2,\ y \geq -1/4$ the following statements are equivalent:

$$y = x^2 - x \ , \qquad x = \frac{-1 + \sqrt{1 + 4y}}{2}$$

and thus the inverse to $f(x) = x^2 - x$ on this domain is

$$f^{-1}(x) = (-1 + \sqrt{1 + 4x})/2 \ . \tag{6.5}$$

Similarly, in the domains $x \leq 1/2,\ y \geq -1/4$ the following statements are equivalent:

$$y = x^2 - x \ , \qquad x = \frac{-1 - \sqrt{1+4y}}{2}$$

and thus the inverse to $f(x) = x^2 - x$ on this domain is $f^{-1}(x) = (-1 - \sqrt{1-4x})/2$.

Example 6.5. Let

$$f(x) = \frac{e^x - e^{-x}}{2} \ .$$

This function is called the *hyperbolic sine*. The hyperbolic sine has an inverse function defined for all real numbers. First of all $f'(x) = (e^x + e^{-x})/2 > 0$ for all x, so f has an inverse function. Secondly,

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty$$

so the range of f, and thus the domain of its inverse, is all real numbers. We now find a formula for the inverse function. Let $y = f^{-1}(x)$, so that

$$x = f(y) = \frac{e^y - e^{-y}}{2} \ .$$

Multiply both sides of the equation by $2e^x$, giving

$$2xe^y = e^{2y} - 1 \qquad \text{or} \qquad e^{2y} - 2xe^y - 1 = 0 \ .$$

Using the quadratic formula we find

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1} \ .$$

Since this is positive for all x, we must have $e^y = x + \sqrt{x^2+1}$, and finally

$$y = \ln(x + \sqrt{x^2+1})$$

is the inverse hyperbolic sine.

Proposition 6.4. Suppose that f and g are inverse to each other in their respective domains. Let $y = g(x)$. Then

$$g'(x) = 1/f'(y) \ . \tag{6.6}$$

To see this, differentiate the relations $x = f(y),\ y = g(x)$ implicitly with respect to x:

$$1 = f'(y)\frac{dy}{dx} \ , \quad \frac{dy}{dx} = g'(x) \ ,$$

so

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(y)} \ .$$

Example 6.6. Let us illustrate this proposition with the exponential and logarithmic functions. Recall that $y = \ln x$ is defined as being equivalent to $x = e^y$. Differentiate that equation with respect to x implicitly .

$$1 = e^y \frac{dy}{dx} \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{e^y} \ .$$

Since $e^y = x$, we obtain the formula for the derivative of the logarithm:

$$\frac{d}{dx} \ln x = \frac{1}{x} \ .$$

Example 6.7. Let $y = f^{-1}(x)$ be the function defined on the domain $x \geq 2$ which is inverse to $f(x) = x^2 - x$ (recall example 6.4). We find the derivative of $f^{-1}(x)$ .First, write:

$$y = f^{-1}(x) \quad \text{is equivalent to} \quad x = y^2 - y \ .$$

Differentiate implicitly:

$$1 = 2y\frac{dy}{dx} - \frac{dy}{dx} \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{2y-1} \ .$$

or

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{2f^{-1}(x) - 1} \ . \tag{6.7}$$

Since we have an explicit formula for $f^{-1}(x)$ (see equation (6.5)), we may substitute that in (6.7) to obtain

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{\sqrt{1+4x}} \ .$$

Of course, in the above example the inverse functions are explicit, and so we can make a substitution for $f^{-1}(x)$ on the left side of (6.7), but that may not always be the case.

Example 6.8. Suppose that g is the inverse to the function $f(x) = x^2 - 4x - 44$ for $x > 2$. Find $g'(1)$.

Note, since the parabola has its vertex where $x = 2$, the function f does have an inverse in $x > 2$. Let $y = g(x)$. Since g is inverse to f, $x = f(y) = y^2 - 4y - 44$ and $f'(y) = 2y - 4$, so

$$g'(x) = \frac{1}{2y-4} \ .$$

To calculate $g'(1)$ we find the value of y corresponding to $x = 1 :\ 1 = y^2 - 4y - 44$ has the solutions $-9, 5$. Since f is restricted to values greater than 2, we must have $g(1) = 5$. Now $f'(y) = 2y - 4$, so

$$g'(1) = \frac{1}{f'(5)} = \frac{1}{2(5) - 4} = \frac{1}{6} \ .$$

Problems 6.1

1. Find the function inverse to

$$f(x) = \frac{2x+1}{x-3} \ .$$

2. Find the inverse function, and its domain, for

$$f(x) = \frac{e^x + e^{-x}}{2} .$$

If possible, find a formula for f^{-1}.

3. Find $g'((e + e^{-1})/2)$ where g is the inverse to the function of problem 2.

4. Show that $f(x) = x^3 + 3x + 1$ has an inverse. Find

$$\frac{d}{dx} f^{-1}(x)\big|_{x=1} .$$

5. Let $f(x) = x \ln x$ for $x > 1$. Show that f has an inverse g. Noting that $f(e^2) = 2e^2$, find $g'(2e^2)$.

6.2 Inverse Trigonometric Functions

In this section we use the ideas of the preceding section to define inverses for the trigonometric functions, and calculate their derivatives. Since the trigonometric functions are periodic, we will have to restrict the domain of definition in order to obtain a well-defined inverse.

We start with the tangent function. Recall that $\tan x$ is strictly increasing on the interval $(\pi/2, \pi/2)$ and takes every value between $-\infty$ and ∞, and then repeats itself in intervals of length π. Thus, if we restrict the domain of the tangent to the interval $(\pi/2, \pi/2)$, it has an inverse there, defined for all real numbers.

Definition 6.1. The function $y = \arctan x$ is defined on the interval $(-\infty, \infty)$, taking values in $-\pi/2, \pi/2]$ .by the condition $x = \tan y$.

The inverse tangent (or *arctangent*) is sometimes denoted by $y = \tan^{-1}(x)$. See figure 6.2 for the graph of the inverse tangent.

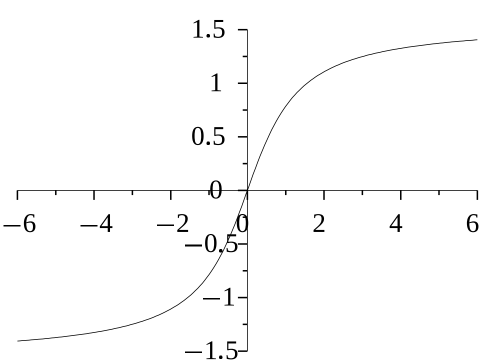


Figure 6.2

Proposition 6.5. $$\frac{d}{dx}\arctan x = \frac{1}{(1+x^2)}\ , \qquad \int \frac{1}{(1+x^2)}dx = \arctan x + C$$

To see this, we start with the equation $x = \tan y$ that defines y as the arctangent of x. We get:

$$1 = \sec^2 y \frac{dy}{dx}\ .$$

Now, since $\sec^2 y = \tan^2 y + 1$, we can replace $\sec^2 y$ by x^2+1, obtaining

$$1 = (x^2+1)\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{x^2+1}\ ,$$

which is just the first equation. The second is a restatement in terms of integrals.

Similarly, we define $y = \arcsin x$ by the condition $x = \sin y$. However, since the sine function is periodic, the equation $\sin y = x$ has many solutions for x between -1 and 1. But, if we insist that y be between $-\pi/2$ and $\pi/2$, there is only one solution. So, to pick a definite inverse for the sine function, we specify that its domain is the interval $[-1,1]$, and its range (set of values) is $[-\pi/2,\pi/2]$. Then, with this specification, it is true that the equation $\sin y = x$ has one and only one solution. That solution we call the *inverse sine function*, denoted $\arcsin x$ or $\sin^{-1} x$.

Definition 6.2. The function $y = \arcsin x$ is defined on the interval $(-1,1)$, taking values in $-\pi/2,\pi/2]$ .by the condition $x = \sin y$. See figure 6.3 for a graph of $y = \arcsin x$.

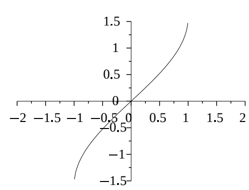


Figure 6.3

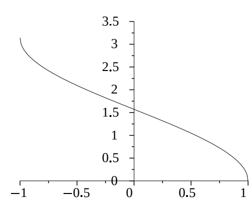


Figure 6.4

Proposition 6.6. $$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}\ , \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

Differentiate $x = \sin y$ implicitly:

$$1 = \cos y \frac{dy}{dx}\ .$$

Now, since $\sin^2 y + \cos^2 y = 1$, writing this as $x^2 + \cos^2 y = 1$, and thus replace $\cos y$ by $\sqrt{1-x^2}$:

$$1 = \sqrt{1-x^2}\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \ .$$

We took the positive root for, in the chosen domain for $\arcsin x$, it is increasing.

Turning to the cosine, since $\cos(-x) = \cos(x)$, it is not possible to define an inverse if we take the domain of cos to be any interval about 0. However, we note that since the cosine function is strictly decreasing between 0 and π, we can define an inverse on the interval $[-1, 1]$ taking values between 0 and π: this is the *inverse cosine*, denoted $\arccos x$. (See figure 6.4 for the graph).

Definition 6.3. The function $y = \arccos x$ is defined on the interval $(-1, 1)$, taking values in $(0, \pi]$, .by the condition $x = \cos y$.

Proposition 6.7.
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \ , \qquad \int \frac{1}{\sqrt{1-x^2}}dx = -\arccos x + C$$

The verification is the same as that of proposition 6.6, except that this time, since the arccosine is decreasing, we take the negative square root. Note that, for any acute angle α, its complementary angle is $\pi/2 - \alpha$, thus $\sin\alpha = \cos(\pi/2 - \alpha)$. Letting $x = \sin\alpha$, so that $\alpha = \arcsin x$, this tells us that $\arccos x = \pi/2 - \alpha = \pi/2 - \arcsin x$, explaining the coincidence in the formulas of propositions 6.6 and 6.7.

Example 6.9. Find
$$\int \frac{xdx}{x^4+1} \ .$$

Make the substitution $u = x^2, \ du = 2xdx$. This gives us

$$\frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C \ .$$

Example 6.10. Find, for any constant a:

$$\int \frac{dx}{x^2+a^2} \ .$$

Make the substitution $x = au, \ dx = adu$. The integral becomes

$$\int \frac{adu}{a^2u^2+a^2} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\arctan u + C = \frac{1}{a}\arctan(\frac{u}{a}) + C \ .$$

Problems 6.2

1. $\tan(\arccos x) =$

2. $\frac{1}{x^2} - \tan^2(\arccos x) =$

3. Show that $\arcsin x + \arccos x$ is constant.

4. Differentiate : $g(x) = \arcsin(\ln x)$.

5. Differentiate : $y = \arccos\sqrt{x}$

6. Find the equation of the line tangent to the curve $y = \arctan x$ at the point $(\sqrt{3}, \pi/3)$.

7. Find all points at which the tangent line to the curve $y = \arcsin x$ has slope 4.

8. What is the maximum value of the derivative of $f(x) = \arccos x$?

9. $$\int \frac{xdx}{\sqrt{1-x^4}} =$$

10. Show that $f(x) = \sec x$ has an inverse in the interval $(0, \pi/2)$. The inverse is denoted $y = \sec^{-1} x$ (called the *arcsecant*). Find the formula for the derivative of the arcsecant.

11. $$\int \frac{dx}{\sqrt{a^2-x^2}} =$$

12. The curve $$y = \frac{1}{\sqrt{1+x^2}} \qquad 1 \le x \le \sqrt{3}$$

is rotated around the x-axis. Find the volume of the enclosed solid.

6.3 First Order Linear Differential Equations

Definition 6.4. A *first order linear differential equation* is a differential equation of the type

$$\frac{dy}{dx} + P(x)y = Q(x) \ . \tag{6.8}$$

It is said to be *homogeneous* if the function $Q(x)$ is 0.

The equation is of "first order" since it involves only the first derivative, and linear since the equation expresses the first derivative of the unknown function y as a linear function of y.

If P and Q are constant functions we can easily solve the differential equation by separation of variables.

Example 6.11. To solve, say
$$\frac{dy}{dx} = 2y - 3$$
we rewrite the equation in the form $(2y-3)^{-1}dy = dx$. These differentials integrate to the relation
$$\frac{1}{2}\ln(2y-3) = x + C \quad \text{or} \quad \sqrt{2y-3} = Ke^x \ .$$

Squaring both sides and soving for y, we ge the general solution

$$y = \frac{Ke^{2x} + 3}{2} \ . \tag{6.9}$$

For example, to find the solution with initial value $y(0) = 5$, we first solve for K:

$$5 = \frac{Ke^{2(0)} + 3}{2} \ ,$$

so $K = 7$, and the particular solution is $y = (7e^{2x} + 3)/2$.

The acute reader will object that the integral of $(2y - 3)^{-1}dy$ is $(1/2)\ln|2y - 3|$, and if we follow through with this, this seems to lead to the alternative solution

$$y = \frac{3 - Ke^{2x}}{2} \ . \tag{6.10}$$

However, this is the same as (6.9), just with a different choice for the constant K. If we use (6.10) with the same initial conditions $y(0) = 5$, we find this $K = -7$, giving the same final answer. For this reason it is often the case that the absolute value is ignored.

Now, we note that the homogeneous equation (the case $Q(x) = 0$) is separable:

Example 6.12. Solve $y' - 2xy = 0,\ y(2) = 1$.

We separate the variables: $y^{-1}dy = 2xdx$ and integrate:

$$\ln y = x^2 + C \ .$$

Substituting the initial condition allows us to solve for C : $\ln 1 = 4 + C$, so $C = -4$. Thus the particular solution is given by

$$\ln y = x^2 - 4$$

which exponentiates to

$$y = e^{x^2 - 4} \ .$$

Now, to solve the general equation, we make a crucial observation:

Proposition 6.8. Given the differential equation, $y' + P(x)y = Q(x)$, suppose that v solves the homogeneous equation: $v' + Pv = 0$. Then, making the substitution $y = uv$ leads to a simple integration for the unknown function u.

Let's make the substitution in the given equation. Since $y' = uv' + u'v$, we have

$$uv' + u'v + Puv = Q \ , \qquad \text{or} \qquad u'v + u(v' + Pv) = Q \ , \qquad \text{or} \qquad u'v = Q \ ,$$

since $v' + Pv = 0$. But then $u' = Qv^{-1}$, and we find u by integration.

This leads to a method for solving the general first order differential equation

$$y' + Py = Q \ .$$

1. Find a solution v of the corresponding homogeneous equation.
2. Make the substitution $y = uv$, leading to an integration to find the new unknown function u.

Example 6.13. *Solve* $$\frac{dy}{dx} = \frac{y+1}{x} \ , \quad y(1) = 2 \ .$$

The homogeneous equation is $y' - x^{-1}y = 0$. which has the solution $y = Kx$. Try $y = ux$ in the given equation. This leas us to the equation $u'x = x^{-1}$, or $u' = x^{-2}$, which has the solution $u = -x^{-1} + C$. Thus the general solution is

$$y = ux = (\frac{-1}{x} + C)x = -1 + Cx \ .$$

Now solve for C using the initial conditions $y(1) = 2$: $2 = -1 + C$, so $C = 3$ and the solution is $y = 3x - 1$.

Now the solution of the homogeneous equation $y' + Py = 0$ is $e^{-\int P dx}$. With the substitution $y = ue^{-\int P dx}$, the terms involving an undifferentiated u disappear precisely because $e^{-\int P dx}$ solves the homogeneous equation. For this reason $e^{-\int P dx}$ is called an *integrating factor*. This method is called that of *variation of parameters*; the idea being to first find the general solution of an easier equation, and then trying that in the original equation, but with the constant replaced by a new unknown function. This method is very productive in solving very general types of differential equations.

Example 6.14. Solve $y' - 2xy = x, \ y(0) = 2$. First, as in example 6.12, solve the homogeneous equation $y' - xy = 0$, leading to
$$y = Ke^{x^2} \ .$$
Now substitute $y = ue^{x^2}$ into the original equation to obtain

$$u'e^{x^2} = x \quad \text{or} \quad u' = xe^{-x^2} \ .$$

This integrates to
$$u = -\frac{1}{2}e^{-x^2} + C \ ,$$
so that our general solution is $y = ue^{x^2}$ with this u:

$$y = \big(-\frac{1}{2}e^{-x^2} + C\big)e^{x^2} = -\frac{1}{2} + Ce^{x^2} \ .$$

Notice that the constant function $-1/2$ (found by taking $C = 0$) is a solution of the differential equation. However, this doesn't satisfy our initial conditions: $y(0) = 2$. Those give us $C = 2$, so the solution we seek is
$$y = -\frac{1}{2} + 2e^{x^2} \ .$$

Example 6.15. Find the general solution to $xy' - y = x^2$.

We first must put this in the form (6.8):

$$\frac{dy}{dx} + \frac{y}{x} = x \ .$$

The solution to the homogeneous equation is $y = Kx$. So, we try $y = ux$, and obtain the equation

$$u'x = x \ ,$$

which has the general solution $u = x + C$. Thus the general solution to the original problem is

$$y = ux = (x + C)x = x^2 + Cx \ .$$

Remember the steps to solve the equation $y' + P(x)y = Q(x)$:

1. Solve the homogeneous equation $y' + P(x)y = 0$, obtaining $y = e^{-\int P dx}$.

2. Try the solution $y = ue^{-\int P dx}$, leading to the equation for u : $u'e^{-\int P dx} = Q(x)$, or $u' = Q(x)e^{\int P dx}$.

Solve for u, and put that solution in the equation $y = ue^{-\int P dx}$. If an initial value is specified, now solve for the unknown constant.

This can, of course, be summarized in a formula:

Proposition 6.9 The general solution of the first order linear differential equation

$$y' + Py = Q$$

is

$$y = e^{-\int P dx}\left(\int Qe^{\int P dx} dx + C\right) \ .$$

We strongly advise students to remember the method rather than this formula.

A useful fact to know about linear first order equations is that if we know one particular solution, then we only have to solve the homogeneous equation to find all solutions.

Proposition 6.10. Suppose that y_p is a solution of the differential equation $y' + Py = Q$. Then every solution is of the form

$$y = y_p + Ke^{-\int P dx} \ ;$$

that is, every solution is of the form $y_p + y_h$, where y_h is a solution of the homogeneous equation.

For suppose that y is any solution of the equation: $y' + Py = Q$. Then $(y - y_p)' + P(y - y_p) = (y + Py) - (y_p + Py_p) = Q - Q = 0$ so solves the homogeneous equation.

Example 6.16. Find the solution of the equation $y' - 2y + 5 = 0$ such that $y(0) = 1$.

Now the constant function $y_p = 5/2$ solves the equation, since $y_p' = 0$. The general solution of the homogeneous equation is $y = Ke^{2x}$, so the general solution of the original equation is of the form $y = (5/2) + Ke^{2x}$. Substituting $y = 1,\ x = 0$, we find $1 = 5/2 + K$, so $K = -3/2$, and the particular solution we want is

$$y = \frac{1}{2}(5 - 3e^{2x}) \ .$$

Example 6.17. A body falling through a fluid is subject to the force due to gravity as well as a resistance, due to the viscosity of the fluid, proportional to its velocity. (Here we are assuming that the density of the body is much higher than the density of the fluid, and that its shape is not relevant). Let $x(t)$ represent the distance fallen at time t and $v(t)$ its velocity. The hypothesis leads to the equation

$$\frac{dv}{dt} = -kv + g$$

for some constant k (g is the acceleration of gravity), called the coefficient of resistance of the fluid. Notice that the constant $v = g/k$ is a solution of the equation. This is called the "free fall velocity", and for any falling body it will accelerate until it reaches this maximum velocity. By proposition 6.10, the general solution is

$$v(t) = \frac{g}{k} + Ke^{-kt} \ ,$$

for some constant k.

Example 6.18. Suppose a heavy spherical object is throuwn from an airplane at 10000 meters, and that the coefficient of resistance of air is $k = 0.05$. Find the velocity as a function of time. What is the free fall velocity? Approximately how long does it take to reach the ground?

Here $g = 9.8$ m/sec^2, so the free fall velocity is $v_p = 9.8/(.05) = 196$ meters/sec. The general solution to the problem is

$$v(t) = 196 + Ke^{-(.02)t} \ .$$

At $t = 0$, $v = 0$, so $0 = 196 + K$, and our solution is

$$v(t) = 196(1 - e^{-(.02)t}) \ .$$

To answer the last question, we have to find distance fallen as a function of time, by integrating the above:

$$x(t) = 196(t + 50e^{-(.02)t}) + C \ .$$

At $t = 0$, $x = 0$; this gives $C = -196(50)$, and the solution for our particular object:

$$x(t) = 196(t + 50(e^{-(.02)t} - 1)) \ .$$

Now we want to solve for t when $x = 10,000$. For large t, the exponential term is negligible, so T, the time to reach ground, is approximately given by the solution of

$$10,000 = 196(T - 50)$$

so $T = 101$ seconds.

Problems 6.3

1. Solve the initial value problem $xy' + y = x, \ \ y(2) = 5$.

2. Solve the initial value problem: $y' = x(5 - y), \ \ \ y(0) = 1$.

3. Solve the initial value problem $(x + 1)y' = 2y, \ y(1) = 1$.

4. Solve the initial value problem $xy' - y = x^3, \ \ y(1) = 2$.

5. Solve the initial value problem $y' - 2xy = e^{x^2}, \ y(0) = 4$.

6. Solve the initial value problem:

$$4y' + 3y = e^x \ , \qquad y(0) = 7 \ .$$

7. Solve the initial value problem:

$$xy' - 3y = x^2 \ , \qquad y(1) = 4 \ .$$

8. Solve the initial value problem $y' - 2xy = e^{x^2}, \ y(0) = 4$.

9. Solve the initial value problem: $y' + y = e^x, \ \ y(0) = 5$.

10. Solve the initial value problem : $\qquad y' + \frac{y}{x} = x, \quad y(1) = 2$.

VII. Terchniques of Integration

Integration, unlike differentiation, is more of an art-form than a collection of algorithms. Many problems in applied mathematics involve the integration of functions given by complicated formulae, and practitioners consult a *Table of Integrals* in order to complete the integration. There are certain methods of integration which are essential to be able to use the Tables effectively. These are: substitution, integration by parts and partial fractions. In this chapter we will survey these methods as well as some of the ideas which lead to the tables. After the study of this material, students should be able to easily use any set of Integral Tables.

7.1 Substitution

This was introduced in section 4.1 (recall Proposition 4.5). To integrate a differential $f(x)dx$ which is not known to us, we seek a function $u = u(x)$ so that the given differential can be rewritten as a differential $g(u)du$ whose integral is known to us. Then, if $\int g(u)du = G(u) + C$, we know that $\int f(x)dx = G(u(x)) + C$. Finding and employing the function u often requires some experience and ingenuity as the following examples show.

Example 7.1. $\int x\sqrt{2x+1}dx = ?$

Let $u = 2x + 1$, so that $du = 2dx$ and $x = (u - 1)/2$. Then

$$\int x\sqrt{2x+1}dx = \int \frac{u-1}{2}u^{1/2}\frac{du}{2} = \frac{1}{4}\int (u^{3/2} - u^{1/2})du == \frac{1}{4}(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}) + C$$

$$= \frac{1}{30}u^{3/2}(3u-5) + C = \frac{1}{30}(2x+1)^{3/2}(6x-2) + C = \frac{1}{15}(2x+1)^{3/2}(3x-1) + C \ ,$$

where at the end we have replaced u by $2x + 1$.

Example 7.2. $\int \tan x dx =?$

Since this isn't on our tables, we revert to the definition of the tangent: $\tan x = \sin x/\cos x$. Then, letting $u = \cos x,\ du = -\sin x dx$ we obtain

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\ln u + C = -\ln\cos x + C = \ln\sec x + C \ .$$

Example 7.3. $\int \sec x dx =?$

This is tricky, and there are several ways to find the integral. However, if we are guided by the principle of rewriting in terms of sines and cosines, we are led to the following:

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x} \ .$$

Now we can try the substitution $u = \sin x,\ du = \cos x dx$. Then

$$\int \sec x dx = \int \frac{du}{1-u^2} \ .$$

This looks like a dead end, but a little algebra pulls us through. The identity

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$

leads to

$$\int \frac{du}{1-u^2}dx = \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)du = \frac{1}{2}(\ln(1+u) - \ln(1-u) + C\ .$$

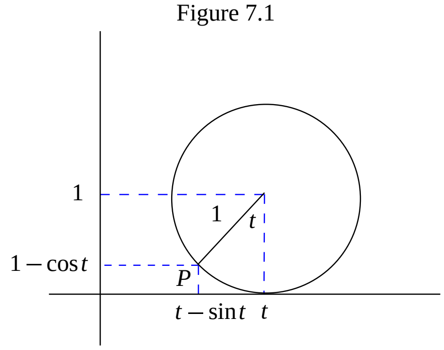
Using $u = \sin x$, we finally end up with

$$\int \sec x dx = \frac{1}{2}(\ln(1+\sin x) - \ln(1-\sin x) + C = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C\ .$$

Example 7.4. As a circle rolls along a horizontal line, a point on the circle traverses a curve called the *cycloid*. A *loop* of the cycloid is the trajectory of a point as the circle goes through one full rotation. Let us find the length of one loop of the cycloid traversed by a circle of radius 1.

Let the variable t represent the angle of rotation of the circle, in radians, and start (at $t = 0$) with the point of intersection P of the circle and the line on which it is rolling. After the circle has rotated through t radians, the position of the point is as given as in figure 7.1.

Figure 7.1



The point of contact of the circle with the line is now t units to the right of the original point of contact (assuming no slippage), so

$$x(t) = t - \sin t\ , \quad y(t) = 1 - \cos t\ .$$

To find arc length, we use $ds^2 = dx^2 + dy^2$, where $dx = (1-\cos t)dt,\ dy = \sin t dt$. Thus

$$ds^2 = ((1-\cos t)^2 + \sin^2 t)^2 dt^2 = (2 - 2\cos t)^2 dt^2$$

so $ds = \sqrt{2(1-\cos t)}dt$, and the arc length is given by the integral

$$L = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}dt\ .$$

To evaluate this integral by substitution, we need a factor of $\sin t$. We can get this by multiplying and dividing by $\sqrt{1+\cos t}$:

$$\sqrt{1-\cos t} = \frac{\sqrt{1-\cos^2 t}}{\sqrt{1+\cos t}} = \frac{|\sin t|}{\sqrt{1+\cos t}} \ .$$

By symmetry around the line $t = \pi$, the integral will be twice the integral from 0 to π. In that interval, $\sin t$ is positive, so we can drop the absolute value signs. Now, the substitution $u = \cos t,\ du = -\sin t dt$ will work. When $t = 0,\ u = 1$, and when $t = \pi, u = -1$. Thus

$$L = -2\sqrt{2}\int_1^{-1} u^{-1/2} du = 2\sqrt{2}\int_{-1}^{1} u^{-1/2} du = 2\sqrt{2}(2u^{1/2})\Big|_{-1}^{1} = 8\sqrt{2} \ .$$

Problems 7.1. Evaluate the Integrals.

1. $$\int_0^2 \frac{x}{1+x^4} dx$$

2. $$\int \frac{dx}{(1+x)\sqrt{x}}$$

3. $$\int \frac{2+x}{1+x} dx$$

4. $$\int \frac{xdx}{1+4x^2} =$$

5. $$\int_0^2 \frac{e^x}{1+e^{2x}} dx$$

6. $$\int \frac{\arccos x}{\sqrt{1-x^2}} dx$$

7. $$\int \frac{(\ln x + 1)^2}{x} dx$$

8. $$\int \cos^3 x \sin^2 x dx$$

9. $$\int_0^2 (x^2+3x-1)^2(2x+3)dx$$

10. $$\int_0^2 \frac{dx}{x^2+4x+5}$$

11. $$\int_0^2 \frac{xdx}{1+4x^2} =$$

12. $$\int_0^2 \frac{dx}{1+4x^2} =$$

13. $$\int \frac{e^x dx}{e^{2x}+1} =$$

14. $$\int \frac{dx}{e^x+e^{-x}} =$$

15. $$\int \frac{dx}{\sqrt{5-4x-x^2}} =$$

16. $$\int \tan^2 x dx =$$

17. $$\int \tan^3 x dx =$$

18. $$\int \frac{dx}{x^2-6x+13} =$$

7.2 Integration by Parts

Sometimes we can recognize the differential to be integrated as a product of a function that is easily differentiated and a differential that is easily integrated. For example, if the problem is to find

(7.1) $$\int x \cos x dx$$

then we can easily differentiate $f(x) = x$, and integrate $\cos x dx$ separately. When this happens, the integral version of the product rule, called *integration by parts*, may be useful, because it interchanges the roles of the two factors.

Recall the product rule: $d(uv) = udv + vdu$, and rewrite it as

(7.2) $$udv = d(uv) - vdu$$

In the case of (7.1), taking $u = x,\ dv = \cos x dx$, we have $du = dx,\ v = \sin x$. Put this all in (7.2):

$$x \cos x dx = d(x \sin x) - \sin x dx \ ,$$

and we can easily integrate the right hand side to obtain

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C \ .$$

7.1 Proposition (Integration by Parts). For any two differentiable functions u and v:

$$\int u dv = uv - \int v du \ . \tag{7.3}$$

To integrate by parts:

1. First identify the parts by reading the differential to be integrated as the product of a function u easily differentiated, and a differential dv easily integrated.

2. Write down the expressions for $u,\ dv$ and du, v.

3. Substitute these epxressions in (7.3).

4. Integrate the new differential vdu.

Example 7.5. Find $\int xe^x dx$.

Let $u = x,\ dv = e^x dx$. Then $du = dx,\ v = e^x$. (7.3) gives us

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C \ .$$

Example 7.6. Find $\int x^2 e^x dx$.

The substitution $u = x^2,\ dv = e^x dx,\ du = 2x dx,\ v = e^x$ doesn't immediately solve the problem, but reduces us to example 7.5:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2(xe^x - e^x + C) = x^2 e^x - 2xe^x + 2e^x + C \ .$$

Example 7.7. To find $\int \ln x dx$, we let $u = \ln x,\ dv = dx$, so that $du = (1/x)dx,\ v = x$, and

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C \ .$$

This same idea works for $\arctan x$: Let

$$u = \arctan x,\ dv = dx \quad du = \frac{dx}{1+x^2},\ v = x \ ,$$

and thus

$$\int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + C \ ,$$

where the last integration is accomplished by the new substitution $u = 1 + x^2,\ du = 2xdx$.

Example 7.8. These ideas lead to some clever strategies. Suppose we have to integrate $e^x \cos x dx$. We see that an integration by parts leads us to integrate $e^x \sin x dx$, which is just as hard. But suppose we integrate by parts again? See what happens:

Letting $u = e^x,\ dv = \cos x dx,\ du = e^x dx, v = \sin x$, we get

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx \ . \tag{7.4}$$

Now integrate by parts again: letting $u = e^x,\ dv = \sin x dx,\ du = e^x dx, v = -\cos x$, we get

$$\int e^x \sin x dx = e^x \cos x + \int e^x \cos x dx \ .$$

Inserting this in (7.4) leads to

$$\int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \cos x dx \ .$$

Bringing the last term over to the left hand side and dividing by 2 gives us the answer:

$$\int e^x \cos x dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C \ .$$

Example 7.9. If a calculation of a definite integral involves integration by parts, it is a good idea to evaluate as soon as integrated terms appear. We illustrate with the calculation of

$$\int_1^4 \ln x dx$$

Let $u = \ln x dx,\ dv = dx$ so that $du = dx/x, v = x$, and

$$\int_1^4 \ln x dx = x \ln x\Big|_1^4 - \int_1^4 dx = 4\ln 4 - x\Big|_1^4 = 4\ln 4 - 3 \ .$$

Example 7.10. $$\int_0^{1/2} \arcsin x dx = \ ?$$

We make the substitution $u = \arcsin x,\ dv = dx,\ du = dx/\sqrt{1-x^2},\ v = x$. Then

$$\int_0^{1/2} \arcsin x dx = x \arcsin x\Big|_0^{1/2} - \int_0^{1/2} \frac{xdx}{\sqrt{1-x^2}} \ .$$

Now, to complete the last integral, let $u = 1 - x^2,\ du = -2xdx$, leading us to

$$\int_0^{1/2} \arcsin x dx = \frac{1}{2}(\frac{\pi}{6}) + \frac{1}{2}\int_1^{3/4} u^{-1/2} du = \frac{\pi}{12} + u^{1/2}\Big|_1^{3/4} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \ .$$

Problems 7.2. Evaluate the Integrals.

1. $\int x(\sin x)dx$

2. $\int e^x x dx$

3. $\int x \ln(2x) dx$

4. $\int \frac{\ln(2x)}{x} dx$

5. $\int \tan^2 x dx$

6. $\int x(e^{2x} + 1)dx$

7. $\int x^2 \sin x dx$

8. $\int (\ln x)^2 dx$.

9. $\int x^2 \ln x dx$.

10. $\int \arccos x dx$.

11. If the region in the first quadrant bounded by the curves $y = 1,\ \ y = e^x$ and $x = 1$ is rotated about the y-axis, what is the volume of the resulting solid?

12. $\int \sec^3 x dx$.

7.3. Partial Fractions

The point of the partial fractions expansion is that integration of a rational function can be reduced to the following formulae, once we have determined the roots of the polynomial in the denominator.

7.3. Proposition. *a)*
$$\int \frac{dx}{x-a} = \ln|x-a| + C \ ,$$

b)
$$\int \frac{du}{u^2+b^2} = \frac{1}{b}\arctan(\frac{u}{b}) + C \ ,$$

c)
$$\int \frac{udu}{u^2+b^2} = \frac{1}{2}\ln(u^2+b^2) + C \ .$$

These are easily verified by differentiating the right hand sides (or by using previous techniques).

Example 7.11. Let us illustrate with an example we've already seen(for example, in example 7.3). To find the integral

$$\int \frac{dx}{(x-a)(x-b)}$$

we check that

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}(\frac{1}{x-a} - \frac{1}{x-b}) \ , \tag{7.5}$$

so that

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\big(\ln|x-a| - \ln|x-b|\big) + C = \frac{1}{a-b}\ln|\frac{x-a}{x-b}| + C \ .$$

The algebraic manipulation in (7.5) can be applied to any rational function. Any polynomial can be written as a product of factors of the form $x-r$ or $(x-a)^2+b^2$, where r is a real root and the quadratic terms correspond to the conjugate pairs of complex roots. The partial fraction expansion allows us to write the quotient of polynomials as a sum of terms whose denominators are of these forms, and thus the integration is reduced to Proposition 7.3.

Here is the partial fractions procedure.

1. Given a rational function $R(x)$, if the degree of the numerator is not less than the degree of the denominator, by long division, we can write

$$R(x) = Q(x) + \frac{p(x)}{q(x)}$$

where now $\deg p < \deg q$.

2. Find the roots of $q(x) = 0$. If the roots are all distinct (that is, there are no multiple roots), express p/q as a sum of terms of the form

$$\frac{p(x)}{q(x)} = \frac{A}{x-r} \ , \quad \frac{B}{(x-a)^2+b^2} \ , \quad \frac{Cx}{(x-a)^2+b^2} \ . \tag{7.6}$$

3. Find the values of $A,\ B,\ C,\ldots$. This is done putting the expression on the right hand side over a common denominator, and then equating coefficients of the numerators in the equation.

4. Integrate term by term using Proposition 7.3.

If the roots are not distinct, the expansion is more complicated; we shall resume this discussion later. For the present let us concentrate on the case of distinct roots, and how to find the coefficients $A, B, C, \ldots$ in (7.6).

Example 7.12. Integrate

$$\int \frac{xdx}{(x-1)(x-2)} .$$

First we write

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} . \tag{7.7}$$

Now multiply this equation by $(x-1)(x-2)$, getting

$$x = A(x-2) + B(x-1) .$$

If we substitute $x = 1$, we get $1 = A(1-2)$, so $A = -1$; now letting $x = 2$, we get $2 = B(2-1$, so $B = 2$, and (7.7) becomes

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2} .$$

Integrating, we get

$$\int \frac{xdx}{(x-1)(x-2)} = -\ln|x-1| + 2\ln|x-2| + C = \ln\frac{(x-2)^2}{|x-1|} + C .$$

So, this is the procedure for finding the coefficients of the partial fractions expansion when the roots are all real and distinct:

1. Write down the expansion with unknown coefficients.

2. Multiply through by the product of all the terms $x - r$.

3. Substitute each root in the above equation; each substitution determines one of the coefficients.

Example 7.13. Integrate

$$\int \frac{(x^2-3)dx}{(x^2-1)(x-3)} .$$

Here the roots are $\pm 1, 3$, so we have the expansion

$$\frac{x^2-3}{(x^2-1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-3} \tag{7.8}$$

leading to

$$x^2 - 3 = A(x-1)(x-3) + B(x+1)(x-3) + C(x+1)(x-1) \ .$$

Substitute $x = -1:\ 1 - 3 = A(-2)(-4)$, so $A = -1/4$.

Substitute $x = 1:\ 1 - 3 = B(2)(-2)$, so $B = 1/2$.

Substitute $x = 3:\ 9 - 3 = C(4)(2)$, so $C = 3/4$, and (7.8) becomes

$$\frac{x^2-3}{(x^2-1)(x-3)} = (-\frac{1}{4})\frac{1}{x+1} + (\frac{1}{2})\frac{1}{x-1} + (\frac{3}{4})\frac{1}{x-3} \ ,$$

and the integral is

$$\int \frac{(x^2-3)dx}{(x^2-1)(x-3)} = -\frac{1}{4}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{3}{4}\ln|x-3| + C \ .$$

Quadratic Factors

Example 7.14. $\displaystyle\int \frac{dx}{x^2-4x-5} = ?$

Here we can factor: $x^2 - 4x - 5 = (x+1)(x-5)$, so we can write

$$\frac{1}{x^2-4x-5} = \frac{A}{x+1} + \frac{B}{x-5}$$

and solve for A and B as above: $A = 1/6,\ B = -1/6$, so we have

$$\frac{1}{x^2-4x-5} = \frac{1}{6}(\frac{1}{x-5} - \frac{1}{x+1})$$

and the integral is

$$\int \frac{dx}{x^2-4x-5} = \frac{1}{6}\ln|\frac{x-5}{x+1}| + C \ .$$

Example 7.15. $\displaystyle\int \frac{dx}{x^2-4x+5} = ?$

Here we can't find real factors, because the roots are complex. But we can complete the square: $x^2 - 4x + 5 = (x-2)^2 + 1$, and now use Proposition (7.3 b):

$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = \arctan(x-2) + C \ .$$

Example 7.16. $\displaystyle\int \frac{(x+3)dx}{x^2-4x+5} = ?$

Here we have to be a little more resourceful. Again, we complete the square, giving

$$\frac{x+3}{x^2-4x+5} = \frac{x+3}{(x-2)^2+1} \ .$$

If only that $x+3$ were $x-2$, we could use Proposition 7.3c, with $u = x-2$. Well, since $x+3 = x-2+5$, there is no problem:

$$\int \frac{(x+3)dx}{x^2-4x+5} = \int \frac{(x-2)dx}{(x-2)^2+1} + \int \frac{5dx}{(x-2)^2+1} = \frac{1}{2}\ln((x-2)^2+1) + 5\arctan(x-2) + C \ .$$

Example 7.17. $$\int \frac{(2x+1)dx}{x^2-6x+14} = ?$$

First, we complete the square in the denominator: $x^2-6x+14 = (x-3)^2+5$. Now, write the numerator in terms of $x-3:\ 2x+1 = 2(x-3)+7$. This gives the expansion:

$$\frac{(2x+1)dx}{x^2-6x+14} = \frac{7}{(x-3)^2+5} + 2\frac{x-3}{(x-3)^2+5}$$

so, using Proposition 7.3:

$$\int \frac{(2x+1)dx}{x^2-6x+14} = 7\int \frac{dx}{(x-3)^2+5} + 2\int \frac{(x-3)dx}{(x-3)^2+5}$$
$$= \frac{7}{\sqrt{5}}\arctan\frac{x-3}{\sqrt{5}} + \ln((x-3)^2+5) + C \ .$$

Example 7.18. $$\int \frac{(x+1)dx}{x(x^2+1)} = ?$$

Here we have to expect each of the terms in Proposition 7.3 to appear, so we try an expression of the form

(7.9) $$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{B}{x^2+1} + \frac{Cx}{x^2+1} \ .$$

Clearing the denominators on the right, we are led to the equation

(7.10) $$x+1 = A(x^2+1) + Bx + Cx^2 \ .$$

Setting $x=0$ gives $1=A$. But we have no more roots to substitute to find B and C, so instead we equate coefficients. The coefficient of x^2 on the left is 0, and on the right is $A+C$, so $A+C=0$; since $A=1$, we learn that $C=-1$. Comparing coefficients of x we learn that $1=B$. Thus (7.9) becomes

$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{x^2+1} \ ,$$

and our integral is

$$\int \frac{(x+1)dx}{x(x^2+1)} = \ln|x| + \arctan x - \frac{1}{2}\ln(x^2+1) + C \ .$$

Example 7.19. $$\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = ?$$

The denominator is $x((x-2)^2+1)$, so we expect a partial fractions expansion of the form

$$\frac{x^2+1}{x(x^2-4x+5)} = \frac{A}{x} + \frac{B}{(x-2)^2+1} + \frac{C(x-2)}{(x-2)^2+1} \ . \tag{7.11}$$

Clearing of denominators, we obtain the equation

$$x^2+1 = A((x-2)^2+1) + Bx + C(x-2)x \ .$$

For $x=0$, we obtain $1 = A(5)$, so $A = 1/5$. Comparing coefficients of x^2 we obtain $1 = A + C$, so $C = -1/5$. Comparing coefficients of x we obtain $0 = -4A + B - 2C$, so $0 = -4/5 + B + 2/5$, so $B = 2/5$ and (7.11) becomes

$$\frac{x^2+1}{x(x^2-4x+5)} = (\frac{1}{5})\frac{1}{x} + (\frac{2}{5})\frac{1}{(x-2)^2+1} - (\frac{1}{5})\frac{x-2}{(x-2)^2+1} \ ,$$

which we can integrate to

$$\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = \frac{1}{5}\ln|x| + \frac{2}{5}\arctan(x-2) - \frac{1}{10}\ln(x^2-4x+5) + C \ .$$

Multiple Roots

If the denominator has a multiple root, that is, there is a factor $x-r$ raised to a power, then we have to allow for the possibility of terms in the partial fraction of the form $1/(x-r)$ raised to the same power. But then the numerator can be (as we have seen above in the case of quadratic factors) a polynomial of degree as much as one less than the power. This is best explained through a few examples.

Example 7.20. $$\int \frac{(x^2+1)dx}{x^3(x-1)} = ?$$

We have to allow for the possibility of a term of the form $(Ax^2+Bx+C)/x^3$, or, what is the same, an expansion of the form

$$\frac{x^2+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} \ . \tag{7.12}$$

Clearing of denominators, we obtain

$$x^2+1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3 \ .$$

Substituting $x=0$ we obtain $1 = C(-1)$, so $C=-1$. Substituting $x=1$, we obtain $2 = D$. To find A and B we have to compare coefficients of powers of x. Equating coefficients of x^3, we have $0 = A + D$, so $A = -2$. Equating coeffients of x^2, we have $1 = -A + B$, so $B = 1 + A = -1$. Thus the expansion (7.12) is

$$\frac{x^2+1}{x^3(x-1)} = -\frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x-1} \ ,$$

which we can integrate term by term:

$$\int \frac{(x^2+1)dx}{x^3(x-1)} = -2\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + 2\ln|x-1| + C \ .$$

If the denominator has a quadratic factor raised to a power, the situation becomes much more complicated. If the quadratic factor has real roots, we can solve by partial fractions; otherwise we need to turn to the methods of the next section.

Example 7.21. $$\int \frac{dx}{(1-x^2)^2} = \ ?$$

Noting that $1-x^2 = (1-x)(1+x)$ we seek an expansion of the form

$$\frac{1}{(1-x^2)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} \ . \tag{7.13}$$

Clearing of denominators:

$$1 = A(1-x)(1+x)^2 + B(1+x)^2 + C(1-x)^2(1+x) + D(1-x)^2 \ .$$

Evaluating at $x=1$, we get $B=1/4$; at $x=-1$, $D=1/4$. Equating constant terms: $1 = A+B+C+D$, and equating the coefficients of x^3 gives $-A+C=0$, so all coefficients are equal to $1/4$. Now we easily integrate

$$\int \frac{dx}{(1-x^2)^2} = \frac{1}{4}(-\ln(1-x) + \frac{1}{1-x} + \ln(1+x) - \frac{1}{1+x}) = \frac{1}{4}\ln(\frac{1+x}{1-x}) + \frac{1}{2}(\frac{x}{1-x^2}) + C \ . \tag{7.14}$$

Problems 7.3 Evaluate the Integrals.

1. $$\int \frac{dx}{x^2(x+2)}$$

2. $$\int \frac{2+x}{1+x}dx$$

3. $$\int_2^4 \frac{dx}{x(x-1)}$$

4. $$\int_1^2 \frac{x^2-4x+1}{x(x-4)^2}dx$$

5. $$\int_2^4 \frac{dx}{x^2-1}$$

6. $$\int_1^2 \frac{dx}{x^2(x+1)}$$

7. $$\int \frac{dx}{x(x-1)(x+2)}$$

8. $$\int_2^4 \frac{dx}{x(x-1)^2}$$

9. $$\int \frac{dx}{x^2(x-1)}$$

10. $$\int \frac{dx}{x(x^2+4x+5)} =$$

11. $$\int \frac{(x+1)dx}{x(x+3)} \ .$$

12. $$\int \frac{(x+1)dx}{x^2(x+3)} \ .$$

13. $$\int \frac{dx}{(x-1)(x+2)^2} \ .$$

14. $$\int \frac{(x^2-1)dx}{(x^2+1)(x+3)} \ .$$

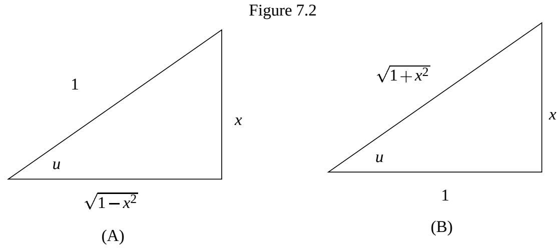
15. $$\int \frac{x^2dx}{(1-x^2)^2} \ .$$

7.4 Trigonometric Methods

Now, although the above techniques are all that one needs to know in order to use a Table of Integrals, there is one form which appears so often, that it is worthwhile seeing how the integration formulae are found. Expressions involving the square root of a quadratic function occur quite frequently in practice. How do we integrate $\sqrt{1-x^2}$ or $\sqrt{1+x^2}$?

When the expressions involve a square root of a quadratic, we can convert to trigonometric functions using the substitutions suggested by figure 7.2.

Figure 7.2



Example 7.22. To find $\int \sqrt{1-x^2}dx$, we use the substitution of figure 7.2A: $x = \sin u,\ dx = \cos u du,\ \sqrt{1-x^2} = \cos u$. Then

$$\int \sqrt{1-x^2}dx = \int \cos^2 u du \ .$$

Now, we use the half-angle formula: $\cos^2 u = (1+\cos 2u)/2$:

$$\int \sqrt{1-x^2}dx = \int \frac{1+\cos 2u}{2} du = \frac{u}{2} + \frac{\sin 2u}{4} + C \ .$$

Now, to return to the original variable x, we have to use the double angle formula: $\sin 2u = 2\sin u \cos u = x\sqrt{1-x^2}$, and we finally have the answer:

$$\int \sqrt{1-x^2}dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{4} + C \ .$$

Example 7.23. To find $\int \sqrt{1+x^2}dx$, we use the substitution of figure 7.2B: $x = \tan u,\ dx = \sec^2 u du,\ \sqrt{1+x^2} = \sec u$. Then

$$\int \sqrt{1+x^2}dx = \int \sec^3 u du \ .$$

This is still a hard integral, but we can discover it by an integration by parts (see problem 12 of section 7.2) to be

$$\int \sec^3 du = \frac{1}{2}(\sec u \tan u + \ln|\sec u + \tan u|) + C \ .$$

Now, we return to figure 7.2B to write this in terms of x: $\tan u = x,\ \sec u = \sqrt{1+x^2}$. We finally obtain

$$\int \sqrt{1+x^2}dx = \frac{1}{2}(x\sqrt{1+x^2} + \ln|\sqrt{1+x^2} + x|) + C \ .$$

Example 7.24.
$$\int x\sqrt{1+x^2}dx = ?$$

Don't be misled: always try simple substitution first; in this case the substitution $u = 1+x^2,\ du = 2xdx$ leads to the formula

$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int u^{1/2}du = \frac{2}{3}(1+x^2)^{3/2} + C \ .$$

Example 7.25.
$$\int x^2\sqrt{1-x^2}dx = ?$$

Here simple substitution fails, and we use the substitution of figure 7.2A:

$$x = \sin u,\ dx = \cos u du,\ \sqrt{1-x^2} = \cos u \ .$$

Then

$$\int x^2\sqrt{1-x^2}dx = \int \sin^2 u \cos^2 u du \ .$$

This integration now follows from use of double- and half-angle formulae:

$$\int \sin^2 u \cos^2 u du = \frac{1}{4}\int \sin^2(2u)du = \frac{1}{8}\int (1-\cos(4u))du = \frac{1}{8}(u - \frac{\sin(4u)}{4}) + C \ .$$

Now, $\sin(4u) = 2\sin(2u)\cos(2u) = 4\sin u \cos u(1-2\sin^2 u) = 4x\sqrt{1-x^2}(1-2x^2)$. Finally

$$\int x^2\sqrt{1-x^2}dx = \frac{\arcsin x}{8} + \frac{x\sqrt{1-x^2}(1-2x^2)}{2} + C \ .$$

Example 7.26. Let's do example 7.21 using these methods. We make the substitution of figure 7.2A: $x = \sin u,\ dx = \cos u du,\ \sqrt{1-x^2} = \cos u$, leading to

$$\int \frac{dx}{(1-x^2)^2} = \int \frac{\cos u du}{\cos^4 u} = \int \sec^3 u du \ ,$$

which we found in problem 12 of section 7.2 to be

$$\frac{1}{2}\sec u \tan u + \frac{1}{4}\ln\frac{1+\sin u}{1-\sin u} + C \ .$$

Substituting back from u to x, using figure 2a, we get (7.14).

Example 7.27. $$\int \frac{dx}{(1+x^2)^2} = ?$$

We use the substitution of figure 7.2B: $x = \tan u,\ dx = \sec u du,\ \sqrt{1+x^2} = \sec u$ This gives

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2 u du}{\sec^4 u} = \int \cos^2 u du = \frac{1}{2}(u + \sin u \cos u) + C = \frac{1}{2}(\arctan x + \frac{x}{1+x^2}) + C \ .$$

Problems 7.4

In this problem set, we not only have trigonometric substitutions, but also a variety of problems using methods from the entire chapter.

1. $$\int \frac{x^2 dx}{\sqrt{9-x^2}} \ .$$

2. $$\int \frac{x^2 dx}{\sqrt{9+x^2}} \ .$$

3. $$\int (x+1)x^{12}dx \ .$$.

4. $$\int x(x+1)^{12}dx$$

5. $$\int_1^e x^2 \ln(2x)dx\ .$$

6. $$\int \frac{xdx}{(1-x^2)^2}\ .$$

7. $$\int \frac{x^2dx}{(1+x^2)^2}\ .$$

8. $$\int \sqrt{x}(x+1)dx\ .$$

9. The curve $y = \cos x$ is revolved about the y-axis, for x running from o to $pi/2$. Find the volume of the resulting solid.

10. $$\int \frac{x^3dx}{(1+x^2)}\ .$$

VIII. Indeterminate Forms and Improper Integrals

8.1 L'Hôpital's Rule

In Chapter 2 we intoduced l'Hôpital's rule and did several simple examples. First we review the material on limits before picking up where Chapter 2 left off.

Suppose f is a function defined in an interval around a, but not necessarily at a. Then we write

$$\lim_{x\to a} f(x) = L$$

if we can insure that $f(x)$ is as close as we please to L just by taking x close enough to a. If f is also defined at a, and

$$\lim_{x\to a} f(x) = f(a)$$

we say that f is *continuous* at a . If the expression for $f(x)$ is a polynomial, we found limits by just substituting a for x; this works because polynomials are continuous.

But how do we calculate limits when the expression $f(x)$ cannot be determined at a? For example, the definition of the derivative:

$$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \ . \tag{8.1}$$

This is an example of an *indeterminate form of type* **0/0**: an expression which is a quotient of two functions, both of which are zero at a. As for (8.1), in case $f(x)$ is a polynomial, we found the limit by long division, and then evaluating the quotient at a (see Theorem 1.1). For trigonometric functions, we devised a geometric argument to calculate the limit (see Proposition 2.7).

For the general expression $f(x)/g(x)$ we have

Proposition 8.1 (l'Hôpital's Rule). If f and g have continuous derivatives at a and $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \ .$$

To see this we use the Mean Value Theorem, theorem 2.4. According to that theorem, we can write $f(x) - f(a) = f'(c)(x - a)$ for some c between x and a, and $gx) - g(a) = g'(d)(x - a)$ for some d between x and a. Since $f(a) = 0$ and $g(a) = 0$, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(c)(x-a)}{g'(d)(x-a)} = \lim_{x\to a} \frac{f'(c)}{g'(d)} \ .$$

But now, by assumption the derivatives f' and g' are continuous. So, since c and d lie between x and a, $f'(c)$ and $g'(d)$ have the same limits as $f'(x)$ and $g'(x)$ as $x \to a$.

Example 8.1. $$\lim_{x\to 2} \frac{x^3-3x+2}{\tan(\pi x)} =$$

After checking that the hypotheses are satisfied, we get

$$\lim_{x\to 2} \frac{x^3-3x+2}{\tan(\pi x)} =^{l'H} \lim_{x\to 2} \frac{3x^2-3}{\pi\sec^2(\pi x)} = \frac{12-9}{\pi} = \frac{3}{\pi} \ .$$

The second limit can be evaluated since both functions are continuous and the denominator nonzero.

Example 8.2. $$\lim_{x\to 0} \frac{x^2+2}{3x^2+1} =$$

Since neither the numerator nor denominator is zero at $x = 0$, we can just substitute 0 for x, obtaining 2 as the limit. However if we apply l'Hôpital's rule without checking that the hypotheses are satisfied, we get the wrong answer: 1/3.

Example 8.3. $$\lim_{x\to 0} \frac{\cos(3x)-1}{\sin^2(4x)} =$$

Both numerator and denominator are 0 at $x = 0$, so we can apply l'H (a convenient abbreviation for l'Hôpital's rule):

$$\lim_{x\to 0} \frac{\cos(3x)-1}{\sin^2(4x)} =^{l'H} \lim_{x\to 0} \frac{-3\sin(3x)}{8\sin(4x\cos(4x)} = -\frac{3}{8} \lim_{x\to 0} \frac{\sin(3x)}{\sin(4x)} \lim_{x\to 0} \frac{1}{\cos(4x)} \ .$$

The last limit is 1, and the other limit can be calculated by l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(4x)} =^{l'H} \lim_{x\to 0} \frac{3\cos(3x)}{4\cos(4x)} = \frac{3}{4} \ .$$

Thus the answer is $-9/32$.

l'Hôpital's rule also works when taking the limit as x goes to infinity, or the limits are infinite. We summarize all these rules:

Proposition 8.2. If f and g are differentiable functions, and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are both zero or both infinite. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \ .$$

The limit point a can be $\pm\infty$.

Example 8.4. $$\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{\ln(\pi/2-x)} =$$

The superscript "−" means that the limit is taken from the left; a superscript "+" means the limit is taken from the right. Since both factors tend to ∞, we can use l'Hôpital's rule:

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\ln(\pi/2-x)} =^{l'H} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{-(\pi/2-x)^{-1}} = -\lim_{x\to\frac{\pi}{2}^-} \frac{\pi/2-x}{\cos^2 x} \ .$$

Now, both numerator and denominator tend to 0, so again:

$$=^{l'H} -\lim_{x\to\frac{\pi}{2}^-} \frac{-1}{-2\cos x \sin x} = -\infty \ ,$$

since $\cos x \sin x$ is positive and tends to zero. We leave it to the reader to verify that the limit from the right is $+\infty$.

Example 8.5. $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} =$

This example is here to remind us to simplify expressions, if possible, before proceeding. If we just use l'Hopital's rule directly, we get

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} =^{l'H} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan x} \ ,$$

which tells us that the sought-after limit is its own inverse, so is ± 1. We now conclude that since both factors are positive to the left of $\pi/2$, then the answer is $+1$. But this would have all been easier to use some trigonometry first:

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}^-} \sin x = 1 \ .$$

Example 8.6. $\displaystyle\lim_{x\to+\infty} \frac{x^n}{e^x} =$

Both factors are infinite at the limit, so l'Hopital's rule applies. Let's take the cases $n = 1, 2$ first:

$$\lim_{x\to+\infty} \frac{x}{e^x} =^{l'H} \lim_{x\to+\infty} \frac{1}{e^x} = 0 \ ,$$

$$\lim_{x\to+\infty} \frac{x^2}{e^x} =^{l'H} \lim_{x\to+\infty} \frac{2x}{e^x} =^{l'H} 2 \lim_{x\to+\infty} \frac{1}{e^x} = 0 \ .$$

We see that for a larger integer n, the same argument will work, but with n applications of l'Hôpital's rule. We say that *the exponential function goes to infinity more rapidly than any polynomial.*

Example 8.7. $\displaystyle\lim_{x\to+\infty} \frac{x}{\ln x} =$

$$\lim_{x\to+\infty} \frac{x}{\ln x} =^{l'H} \lim_{x\to+\infty} \frac{1}{1/x} = \lim_{x\to+\infty} x = +\infty \ .$$

In particular, much as in example 8.6, one can show that polynomials grow more rapidly than any polynomial in $\ln x$.

Problems 8.1. Evaluate the limits.

1. $$\lim_{x\to 0} \frac{\cos x - 1}{x^2} =$$

2. $$\lim_{x\to 0} \frac{\sin x - x}{x(\cos x - 1)} =$$

3. $$\lim_{x\to \pi} \frac{(x-\pi)^3}{\sin x + x - \pi} =$$

4. $$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =$$

5. $$\lim_{x\to 1} \frac{\ln x}{\cos((\pi/2)x)} =$$

6. $$\lim_{x\to 0^+} (\frac{\cos(\sqrt{x}) - 1}{x}) =$$

7. $$\lim_{x\to 5} (\frac{5cos(\pi x) + x}{x^2 - 25}) =$$

8. $$\lim_{x\to \infty} \frac{x}{\sqrt{1+x^2}} =$$

9. $$\lim_{x\to \infty} \frac{x \ln x}{x^2 + 1} =$$

10. $$\lim_{x\to \infty} \frac{x(x+1)}{\sqrt{x^3 - 1}} =$$

8.2 Other inderminate forms

Many limits may be calculated using l'Hôpital's rule. For example: $x \to 0$ and $\ln x \to -\infty$ as $x \to 0$ from the right. Then what does $x \ln x$ do? This is called an *indeterminate form of type* $0 \cdot \infty$, and we calculate it by just inverting one of the factors.

Example 8.9.

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} =^{l'H} \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} \frac{x^2}{x} = -\lim_{x\to 0} x = 0\ .$$

Example 8.10. $\lim_{x\to\infty} x(\pi/2 - \arctan x) =$

This is of type $0 \cdot \infty$, so we invert the first factor:

$$\lim_{x\to\infty} x(\pi/2 - \arctan x) = \lim_{x\to\infty} \frac{\pi/2 - \arctan x}{1/x} =^{l'H} \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2}$$

$$= \lim_{x\to\infty} \frac{1}{1+x^{-2}} = 1 \ .$$

Another case, the *indeterminate form* $\infty - \infty$, is to calculate $\lim_{x\to a}(f(x) - g(x))$, where both f and g approach infinity as x approaches a. Although both terms become infinite, the difference could stay bounded, tend to zero, or also tend to infinity. In these cases we have to manipulate the form algebraically to bring it to one of the above forms.

Example 8.11. $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x}) =$

$$\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x}) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} =^{l'H} \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} =^{l'H} \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = 0 \ .$$

Example 8.12. $\lim_{x\to\infty} x - \sqrt{x^2+20} =$

Here we can change the subtraction of two positive functions to that of addition by remembering

$$x - \sqrt{x^2+20} = (x - \sqrt{x^2+20})\frac{x+\sqrt{x^2+20}}{x+\sqrt{x^2+20}} = \frac{x^2-(x^2+20)}{x+\sqrt{x^2+20}} = \frac{-20}{x+\sqrt{x^2+20}} \ ,$$

$$\lim_{x\to\infty} x - \sqrt{x^2+20} = \lim_{x\to\infty} \frac{-20}{x+\sqrt{x^2+20}} = 0 \ .$$

Finally, whenever the difficulty of taking a limit is in the exponent, try taking logarithms.

Example 8.13. $\lim_{x\to\infty} x^{1/x} =$

Let's take logarithms:

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} =^{l'H} \lim_{x\to\infty} \frac{1/x}{1} = 0 \ .$$

Now, exponentiate, using the continuity of exp:

$$\lim_{x\to\infty} x^{1/x} = \exp(\lim_{x\to\infty} \ln(x^{1/x})) = e^0 = 1 \ .$$

Problems 8.2: Find the limits.

1. $\lim_{x\to 1}(\frac{1}{\ln x} - \frac{1}{x-1})$

2. $\lim_{x\to\infty} \frac{\sqrt{1+x^2} - x}{x}$

3. $$\lim_{x\to\infty} x(\sqrt{1+x^2}-x)$$

4. $$\lim_{x\to\pi/2^+} (\tan x)(x-\pi/2)$$

5. $$\lim_{x\to 1^+} (x-1)\ln(\ln x)$$

8.3 Improper Integrals: Infinite Intervals

To introduce this section, let us calculate the area bounded by the x-axis, the lines $x=-a,\ x=a$ and the curve $y=(1+x^2)^{-1}$. This is

$$\int_{-a}^{a} \frac{dx}{1+x^2} = \arctan x\Big|_{-a}^{a} = 2\arctan a\ .$$

Since $\arctan a$ is always less than $\pi/2$, this area is bounded no matter how large we choose a. In fact, since $\lim_{a\to\infty}\arctan a=\pi/2$, the area under the total curve $y=(1+x^2)^{-1}$ adds up to $2(\pi/2)=\pi$. We can write this in the form

(8.2) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi\ ,$$

using the following definitions.

Definition 8.1. a) Suppose that $f(x)$ is defined and continuous for all $x\ge c$. We define

$$\int_{c}^{\infty} f(x)dx = \lim_{a\to\infty}\int_{c}^{a} f(x)dx$$

if the limit on the right exists. In this case we say the integral *converges*. If there is no limit on the right, we say the integral *diverges*.

b) In the same way, if $f(x)$ is defined and continuous in an interval $x\le c$, we define

$$\int_{-\infty}^{c} f(x)dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)dx$$

if the limit exists.

c) If $f(x)$ is defined and continuous for all x. Then

(8.3) $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx\ ,$$

if both integrals on the right side converge.

Note that it is insufficient to define (8.3) by the limit $\lim_{a\to\infty}\int_{-a}^{a} f(x)dx$, for this integral is always zero for an odd function, say $f(x) = x$, and it would not be appropriate to say that such an integral converges.

Example 8.14. $$\int_0^\infty e^{-x}dx = 1\ .$$

First we calculate the integral up to the positive number a:

$$\int_0^a e^{-x}dx = -e^{-x}\Big|_0^a = 1 - \frac{1}{e^a}\ .$$

Now, since $e^{-a} \to 0$ as $a \to \infty$, the limit exists and is 1.

Example 8.15. $$\int_1^\infty x^{-p}dx \quad \text{converges for} \quad p > 1.$$

We calculate the integral over a finite interval:

$$\int_1^a x^{-p}dx = \frac{1}{-p+1}x^{-p+1}\Big|_1^a = \frac{1}{-p+1}(a^{-p+1} - 1)\ .$$

Now, if $-p + 1 < 0$, $a^{-p+1} \to 0$ as $a \to \infty$, so our conclusion is valid, and in fact

(8.4) $$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad \text{for} \quad p > 1\ .$$

Also, if $p < 1$ then $-p + 1 > 0$, so a^{-p+1} becomes infinite with a, and thus

(8.5) $$\int_1^\infty \frac{dx}{x^p} \quad \text{diverges for} \quad p < 1\ .$$

The case $p = 1$ cannot be handled this way, because then $-p + 1 = 0$. But

Example 8.16. $$\int_1^\infty \frac{dx}{x} \quad \text{diverges}$$

We calculate over a finite interval:

$$\int_1^a \frac{dx}{x} = \ln x\Big|_1^a = \ln a\ ,$$

which goes to infinity as $a \to \infty$.

Sometimes we can conclude that the improper integral converges, even though we cannot calculate the actual limit. This is because of the following fact:

Proposition 8.3. Suppose that F is an increasing continuous function of x for all $x \geq c$, and suppose that F is bounded; that is, there is a positive number M such that $M \geq F(x)$ for all x. Then $\lim_{x\to\infty} F(x)$ exists.

This is an important fact, known as the *Monotone Convergence Theorem* the proof of which depends upon an axiomatic development of the real number system. To see why it is reasonable we consider the *least* upper bound M_0 of the set of values $F(x)$. The relevant fact about real numbers is that there always is a least upper bound for any nonempty bounded set of real numbers. There must be values $F(x)$ which come as close as we please to M_0, for if not, the values of F stay away from M_0, so this could not be the least upper bound. But now, because F is increasing, that means that eventually all values come that close to M_0.

Example 8.17. $\displaystyle\int_1^\infty e^{-x^2}dx$ converges.

In this range, $x^2 \geq x$, so $e^{-x^2} \leq e^{-x}$. So, for any a,

$$\int_1^a e^{-x^2}dx \leq \int_1^a e^{-x}dx \leq 1$$

by example 8.16. Thus the values of the integral are bounded by 1. But since the function is always positive, the integrals increase as a increases. Thus by Proposition 8.3, the limit exists.

This example generalizes to the following

Proposition 8.4. (Comparison Test). Suppose that f and g are continuous functions defined for all $x \geq c$, and suppose that for all x, $0 \leq f(x) \leq g(x)$. Then

a) If $\displaystyle\int_c^\infty g(x)dx$ converges, then $\displaystyle\int_c^\infty f(x)dx$ converges .

b) If $\displaystyle\int_c^\infty f(x)dx$ diverges, then $\displaystyle\int_c^\infty g(x)dx$ diverges .

Example 8.18. $\displaystyle\int_1^\infty \frac{|\cos x|dx}{x^{3/2}}$ converges.

Now, we don't know how to integrate this function, but we do know that $|\cos x| \leq 1$. Thus the integrand is always less than or equal to $x^{-3/2}$, and so, by example 8.17 and proposition 8.6, we can conclude that our integral converges.

Problems 8.3

In problems 1-6, determine whether or not the integral converges. If it does, try to find its value.

1. $\displaystyle\int_0^\infty xe^{-x^2}dx =$

2. $\displaystyle\int_0^\infty \frac{x^2}{x^3+1}dx =$

3. $$\int_0^1 \frac{dx}{x^{9/10}} =$$

4. $$\int_3^\infty \frac{dx}{x(\ln x)^2} =$$

5. $$\int_{1/5}^\infty \frac{\ln(5x)}{x^2} dx =$$

6. $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{3/2}} =$$

7. Find the area under the curve $y = (x^2 - x)^{-1}$, above the x-axis and to the right of the line $x = 2$.

8. The region in the first quadrant to the right of the line $x = 1$, and below the curve $y = 1/x$ is rotated about the x-axis. Show that the resulting solid has finite volume.

9. Find the area under the curve $y = (x^2 - x)^{-1}$, above the x-axis and to the right of the line $x = 2$.

10. The equiangular spiral is the curve given parametrically by the equations

$$x = e^{-t} \cos t \ , \quad y = e^{-t} \sin t \ , \qquad 0 \le t < \infty \ .$$

Show that this curve crosses the x axis infinitely often, but is of finite length.

8.4 Improper Integrals: Finite Asymptotes

Now, it is also possible, for a function which has a vertical asymptote, that the values approach the asymptote so fast that the area enclosed is finite.

Example 8.19. Consider $y = x^{-1/2}$ for x positive. For a slightly larger than 0,

$$\int_a^1 x^{-1/2} dx = 2x^{1/2}\Big|_a^1 = 2(1 - \sqrt{a}) \ .$$

Now, as $a \to 0^+$, this converges to 2. Thus it makes sense to say that $\int_0^1 x^{-1/2} dx = 2$, as we do with this definition.

Definition 8.2. Let $f(x)$ be defined and continuous for all x in an interval $(c, b]$. We define

$$\int_c^b f(x)dx = \lim_{a \to c+} \int_a^b f(x)dx$$

if the limit exists. Similarly if $f(x)$ is defined and continuous for all x in an interval $[b, c)$, we define

$$\int_b^c f(x)dx = \lim_{a \to c^-} \int_b^a f(x)dx \ .$$

Example 8.20. $\int_0^1 x^{-p} dx$ converges for $p < 1$.

We calculate the integral over an interval $(a, 1)$, with $a > 0$:

$$\int_a^1 x^{-p} dx = \frac{1}{-p+1} x^{-p+1} \Big|_a^1 = \frac{1}{-p+1}(1 - a^{-p+1}) \ .$$

Now, if $-p + 1 > 0$, $a^{-p+1} \to 0$ as $a \to 0$, so our conclusion is valid, and in fact

(8.6)
$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p} \quad \text{for} \quad p < 1 \ .$$

Also, if $p > 1$ then $-p + 1 < 0$, so a^{-p+1} becomes infinite as a goes to zero, and thus

(8.7)
$$\int_0^1 \frac{dx}{x^p} \quad \text{diverges for} \quad p > 1 \ .$$

As for the case $p = 1$, since

$$\int_a^1 \frac{dx}{x} = \ln x \Big|_a^1 = -\ln a \ ,$$

this integral diverges to infinity as $a \to 0$. However:

Example 8.21. $\int_0^1 \ln x dx$ converges .

By example 9 of chapter 7, for a positive and near 0,

$$\int_a^1 \ln x dx = (x \ln x - x)\Big|_a^1 = -1 - (a \ln a - a) \ .$$

By example 8.9, $\lim_{a \to 0+} a \ln a = 0$, so the limit exists and is equal to -1.

Problems 8.4. Determine whether or not the integral converges. If it does, try to find its value.

1. $$\int_0^{\pi/2} \frac{dx}{1 - \cos x} =$$

2. $$\int_0^1 \frac{dx}{(1-x)^{3/2}} =$$

3. $$\int_0^{1/2} \frac{dx}{\sqrt{x}(1-x)}$$

4. $$\int_0^2 \frac{dx}{\sqrt{x}} =$$

5. $$\int_0^1 \frac{dx}{(x-1)^2} =$$

6. $$\int_1^{10} \frac{dx}{x\sqrt{\ln x}} =$$

7.The region in the first quadrant above the line $y = 1$, and left of the curve $y = 1/x$ is rotated about the y-axis. Show that the resulting solid has finite volume.

IX. Sequences and Series

9.1 Sequences

The purpose of this chapter is to introduce a particular way of generating algorithms for finding the values of a function defined, not by a formula, but by its properties. For example, the trigonometric functions have been defined geometrically,and the exponential function as the solution of a particular differential equations. This type of definition, while uniquely identifying the function, does not give a way to calculate its values at specific points. Such a way is given by the technique of *Infinite Series*. Computer algorithms for determining the value of a function are based on the usual arithmetic operations; thus an exact determination can only be achieved for those functions expressed explicitly in terms of the arithmetic operations: the rational functions (quotients of polynomials). If a function is transcendental, its values can only be approximated. For example, we have seen that

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n \ .$$

This expression tells us that, if for any n, we calculate the expression on the right, these numbers will, for n large enough, be close to the "true" value of e^x. Now, it turns out that this is a very inefficient way to calculate e^x, and the expression as an infinite series (which we will discuss in depth later in this chapter)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{9.1}$$

is far better. Equation (9.1) is to be understood in this way: start with $E_0 = 1$. To get E_1 add $x/1!$ to E_0; now get E_2 by adding $x^2/2!$ to E_1, and so forth. That is, for every $n \geq 1$ add $x^n/n!$ to E_{n-1} to get E_n.Finally, if we take n large enough, we have a good approximation to e^x, and as n increases the approximation gets better. Of course, it is important to have estimates on how good this approximation is, as well as, in general, to have ways of discovering these approximating sums. That is what we study in this chapter, starting with the idea of convergence in the sense of "good approximation".

Definition 9.1. A *sequence* is a list of numbers, denoted $\{a_n\}$, where a_n is the nth term of the sequence.

A sequence may be defined by a specific formula or an algorithm for determining the members of the sequence successively.

Example 9.1. The formulae

$$a_n = n \ , n \geq 1 \ ; \quad b_n = \frac{n+1}{n-1} \ , n \geq 2 \ ; \quad c_n = 3 + 2n, n \geq 0 \tag{9.2}$$

define the sequences, respectively:

$$1, 2, 3, \ldots, n, \ldots \ ; \qquad \frac{3}{1}, \frac{4}{2}, \frac{5}{3}, \ldots, \frac{n+1}{n-1}, \ldots \ ; \qquad 3, 5, 7, 9, \ldots, 3 + 2n, \ldots$$

A sequence is said to be defined *recursively*, or by a *recursive algorithm* when we are told the first member (or members) of the sequence; and then given an expression for determining the nth number, once we have calculated the first $n-1$ numbers. For example, the data:

$$c_0 = 3 \ ; \quad \text{and for } n > 0, \ c_n = c_{n-1} + 2$$

defines the last sequence of (9.2). Similarly, the first sequence of (9.2) is given by the recursion $a_1 = 1,\ a_n = a_{n-1} + 1$.

The symbol $n!$ (read "n-factorial") is used to denote the product of the first n integers. This also has the recursive definition: $a_0 = 1$, and for $n > 0$, $a_n = na_{n-1}$. (Note that we have taken $0!$ to be 1).

We can also verify formulas or assertions about the positive integers by recursion. That is, suppose that $P(n)$ represents an assertion for the integer n. If we can verify that (A): $P(1)$ is true, and (B): the truth of $P(n)$ follows from the truth of $P(n-1)$, then we can assert that $P(n)$ is true for all n. For, (A) tells us that $P(1)$ is true, and so by (B) we conclude that $P(2)$ is also true, and so, by (B) again, $P(3)$ is true, and so also $P(4),\ P(5)$ and so on. For any integer n, with n applications of (B), we verify the truth of $P(n)$. For future reference we record this method as:

Proposition 9.1. (The Principle of Mathematical Induction). Let $P(n)$ represent an assertion about the positive integer n. If we can verify $P(1)$ and also show that the truth of $P(n-1)$implies the truth of $P(n)$, then $P(n)$ is true for all integers n.

Example 9.2. Consider the sequence defined recursively by $a_1 = 1,\ a_n = a_{n-1} + n$. Note that this equivalent to saying that a_n is the sum of the first n positive integers. Let's show that

$$a_n = \frac{n(n+1)}{2} \ .$$

Call this the assertion $P(n)$. Clearly $a_1 = 1(2)/2$, so $P(1)$ is true. Now, let's assume we know the truth of $P(n-1)$, and verify it for n:

$$a_n = a_{n-1} + n = \frac{(n-1)n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} \ .$$

Example 9.3. Define the sequence recursively by $c_0 = 1,\ c_n = 1 + rc_{n-1}$. Then

$$c_n = \frac{1 - r^{n+1}}{1 - r} \ .$$

The first case ($n = 0$) is certainly true:

$$c_0 = 1 = \frac{1 - r^{0+1}}{1 - r} \ .$$

Now, let's verify that the truth for $n - 1$ implies that for n:

$$c_n = 1 + rc_{n-1} = 1 + r\frac{1 - r^n}{1 - r} = \frac{1 - r + r - r^{n+1}}{1 - r} = \frac{1 - r^{n+1}}{1 - r} \ .$$

Of the sequences described in (9.2), the first and the third clearly grow without bound, but the second is bounded; in fact, if we rewrite the general term as

$$b_n = \frac{n+1}{n-1} = \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} \ ,$$

we see that the sequence b_n approaches 1 as n gets larger and larger. We say that b_n converges to 1, as in the following definition.

Definition 9.2. A sequence $\{a_1, a_2, \ldots, a_n, \ldots\}$ *converges* to a limit L, written

$$\lim_{n\to\infty} a_n = L \ ,$$

if, for every $\epsilon > 0$, there is an n_0 such that for all $n \geq n_0$ we have $|a_n - L| < \epsilon$.

This just says that we can be sure that a_n is as close to L as we need it to be, just by taking the index n large enough. We will rarely have to actually use this definition, relying more on understanding what it says, and known facts about limits. For example:

Proposition 9.2. If the general term a_n of a sequence can be expressed as $f(n)$ for a continuous function f, then if we know that $\lim_{x\to\infty} f(x) = L$, then we can conclude that $\lim_{x\to\infty} a_n = L$.

As an application, using results from the preceding chapter, we have

Proposition 9.3.

(*a*) $$\lim_{n\to\infty} n^p = \infty \ \text{ for } \ p > 0 \ ,$$

(*b*) $$\lim_{n\to\infty} \frac{1}{n^p} = 0 \ \text{ for } \ p > 0 \ ,$$

(*c*) $$\lim_{n\to\infty} A^{1/n} = 1 \ \text{ if } \ A > 0 \ .$$

Let p and q be polynomials.

(*d*) $$\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0 \text{ if } \deg p < \deg q, \quad \lim_{n\to\infty} \frac{p(n)}{q(n)} = \infty \text{ if } \deg p > \ \deg q \ .$$

(e) If the polynomials p and q have the same degree, then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \frac{a}{b} \ ,$$

where a and b are the leading coefficients of p and q.

(*f*) $$\lim_{n\to\infty} \frac{p(n)}{e^n} = 0 \ \text{ for any polynomial } p \ .$$

(*g*) $$\lim_{n\to\infty} \frac{p(n)}{\ln(n)^c} = \infty \ \text{ for any polynomial of positive degree and any positive } c \ .$$

These can all be derived by replacing n by x, and using limit theorems already discussed (such as l'Hôpital's rule).

Example 9.4. $$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 \text{ , by (e) above .}$$

Example 9.5. $$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \text{ ,}$$

since the numerator oscillates between -1 and 1, and the denominator goes to zero. We should not be perturbed by such oscillation, so long as it remains bounded. For example we also have

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0 \text{ ,}$$

since the term $\sin(n)$ remains bounded. The following propositions state the general rule for handling such cases.

Proposition 9.4. a) (Squeeze theorem) Given three sequences $a_n,\ b_n, c_n$, if

$$a_n \geq b_n \geq c_n \text{ for all n , and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \text{ ,}$$

then also

$$\lim_{n\to\infty} b_n = L \text{ .}$$

b) If $a_n = b_n c_n$, the sequence b_n is bounded, $c_n \geq 0$ and $\lim_{n\to\infty} c_n = 0$, then also $\lim_{n\to\infty} a_n = 0$.

Let's see why b) is true, using a). Let M be the bound of the $|b_n|$. Then

$$Mc_n \geq b_n c_n \geq -Mc_n$$

so a) applies and the conclusion follows.

In some cases where none of the above rules apply, we have to return to the definition of convergence.

Example 9.6. $$\text{For any } a > 0, \quad \lim_{n\to\infty} \frac{a^n}{n!} = 0 \text{ .}$$

To see why this is true, we think of the sequence as recursively defined: $a_1 = 1$, and each a_n is obtained by multiplying its predecessor by a/n. Now, eventually, that is, for n large enough, $a/n < 1/2$. Thus each term after that is less than half its predecessor. This now surely looks like a sequence converging to zero. To be more precise, let N be the first integer for which $a/N < 1/2$. Then for any $k > 0$,

$$\frac{a^{N+k}}{(N+k)!} < \frac{1}{2^k}\frac{a^N}{N!} \text{ .}$$

Now the sequence on the right is a fixed number ($a^N/N!$) times a sequence ($1/2^k$) which tends to zero. Thus our sequence converges to zero, also by the squeeze theorem (proposition 9.4a).

Note that in the above argument, we only had to show that the general term of our sequence is dominated by the general term of a sequence converging to zero *from some point on.* What happens to any finite collection of terms of a sequence is not relevant to the question of convergence. We shall use the word *eventually* to mean "from some point on", or more precisely, "for all n greater than some fixed integer N". We restate proposition 9.4, using the word "eventually":

Proposition 9.5. a) (Squeeze theorem) Given three sequences $a_n,\ b_n, c_n$, if eventually

$$a_n \geq b_n \geq c_n \text{ for all n , and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \ ,$$

then also

$$\lim_{n\to\infty} b_n = L \ .$$

b) Suppose that $a_n = b_n c_n$ eventually, that is, for all n larger than some N. If the sequence b_n is bounded, $c_n \geq 0$ and $\lim_{n\to\infty} c_n = 0$, then also $\lim_{n\to\infty} a_n = 0$.

Example 9.7. For any positve integer p, $\displaystyle\lim_{n\to\infty} \frac{n^p}{n!} = 0$.

The idea here is that the numerator is a product of p terms, whereas the denominator is a product of n terms, so grows faster than the numerator. To make this precise, write

$$\frac{n^p}{n!} = \frac{n \cdots n}{n(n-1)\cdots(n-p+1)} \frac{1}{(n-p)!} \ .$$

Now, if n is so large that $n/(n-p) < 2$, ($n > 2p$ will do), then the first factor is bounded by 2^p. Thus, for $n > 2p$, that is, eventually,

$$\frac{n^p}{n!} < 2^p \frac{1}{(n-p)!} \ .$$

Since $1/(n-p)! \to 0$ asn $n \to \infty$, the result follows from the squeeze theorem.

An important fact that we will need is the following.

Proposition 9.6. A bounded monotonically increasing sequence converges.

Let's make sure that the terms involved are clear. A sequence a_n is *bounded* if there is a number M such that $M \geq a_n$ for all n. A sequence is *monotonically increasing* if, for all n, $a_n \leq a_{n+1}$.

Proposition 9.6 follows from the fact about real numbers that any bounded nonempty set has a least upper bound. So, for a the least upper bound of the given sequence $\{a_n\}$, we have $\lim_{nrightarrow\infty} a_n = a$. For if c is any number less than a, it is not an upper bound of the sequence, so there is an$N$ such that $c < a_N < a$. But now, since the sequence is monotonically increasing, for every $n \geq N$, we have $c < a_n < a$.

Finally, we note that the limit of a sum is the sum of the limits:

Proposition 9.7. If $a_n = b_n + c_n$, and the sequences b_n and c_n converge, then so does the sequence a_n, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n \ .$$

Problems 9.1

Find the limits.

1. $$\lim_{n\to\infty} \frac{n}{(\ln n)^{15}}$$

2. $$\lim_{n\to\infty} \frac{n^k}{n!}$$

3. $$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2$$

4. $$\lim_{n\to\infty} \frac{(2n-1)^2}{n^2-3n+1}$$

5. $$\lim_{n\to\infty} \frac{(1+n)^n}{n!}$$

6. Show part c) of proposition 9.3:

$$\lim_{n\to\infty} A^{1/n} = 1 \quad \text{if} \quad A > 0 \ .$$

7. Find $\lim_{n\to\infty} n^{1/n}$.

8. Find $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$.

9. Define the sequence a_n recursively by

$$a_1 = 1 \ , \quad a_n = \frac{1}{2}(10 + a_{n-1}) \ .$$

Show that a_n converges to 10.

10. Let $a_n = r^n$ where

$$r = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad r = \frac{1-\sqrt{5}}{2} \ .$$

Show that

$$a_{n+2} = a_{n+1} + a_n \quad \text{for all } n \geq 2 \ .$$

9.2 Series

For many sequences, in fact, the most important ones, the general term is formed by adding something to its predecessor; that is, the sequence is formed by the recursion $s_n = s_{n-1} + a_n$, where a_n is from another sequence. Such a sequence is called a *series*. Explicitly, the terms of the series are

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_n, \cdots \ .$$

It is useful to use the summation symbol:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \ .$$

Definition 9.3. The *series*

$$\sum_{k=0}^{\infty} a_k$$

is to be considered as the limit of the sequence

$$s_n = \sum_{k=0}^{n} a_k \ .$$

If the limit L of the sequece $\{s_n\}$ exists, the series is said to *converge*, and L is called its *sum*. If the limit does not exist, the series *diverges*. The terms of the sequence s_n are called the *partial sums* of the series.

Example 9.8. $$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \ .$$

Let's look at a few partial sums:

$$\frac{1}{2} \ , \quad \frac{3}{4} \ , \quad \frac{7}{8} \ , \quad \frac{15}{16} \ , \quad \ldots$$

We see that, at least for the first four terms

$$s_n = \frac{2^n - 1}{2^n} \ . \tag{9.3}$$

Let's now see that this is true for all n, using the principle of mathematical induction. Suppose we've verified (9.3) for all integers up to $n-1$; we now verify this for n. By definition and (9.3) for s_{n-1}:

$$s_n = s_{n-1} + \frac{1}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} + \frac{1}{2^n} \ .$$

Putting this all over the denominator 2^n, we obtain

$$s_n = \frac{2^n - 2 + 1}{2^n} = \frac{2^n - 1}{2^n} \ ,$$

which is just (9.3) for s_n.

Now, by (9.3):

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1 \ .$$

Remember that the index is a way of relating the partial sums of the series to the general term from which it is defined, so if we change that relation consistently, we don't change the series. For example,

$$\sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} a_{k+1} = \sum_{m=9}^{\infty} a_{m-8}$$

and so forth. Each representation comes about by replacing the index with a new index. For example, if we substitute n for k, we get the first equality; if we substitute $k+1$ for n we get the second equality, and if we replace $k+1$ by $m-8$, we get the last one. It is often useful to make a change of index as the next examples show.

Example 9.9. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 2 \ .$

For

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 + 1 = 2 \ .$$

Example 9.10. $\displaystyle\sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}} \ .$

First, change the index by $k = m + n$, and then factor out 2^{-n}:

$$\sum_{k=n}^{\infty} \frac{1}{2^k} = \sum_{m=0}^{\infty} \frac{1}{2^{m+n}} = 2^{-n} \sum_{m=0}^{\infty} \frac{1}{2^m} = 2^{-n} \cdot 2 = 2^{-n+1} \ .$$

Proposition 9.8 (Geometric Series) :

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1 \ ,$$

$$\sum_{k=0}^{\infty} x^k \quad \text{diverges for } |x| \geq 1 \ .$$

To show this, we obtain (by a clever little observation) a formula for the partial sums

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n \ .$$

Note that

$$s_{n+1} = (1 + x + x^2 + \cdots + x^n) + x^{n+1} = s_n + x^{n+1} \quad \text{and}$$

(9.4) $$s_{n+1} = 1 + (x + x^2 + \cdots + x^{n+1}) = 1 + x s_n \ .$$

(Note that (9.4) is the recursive definition of the partial sums we've already seen in example 9.3). Equating these expressions for s_{n+1}, we obtain $s_n + x^{n+1} = 1 + x s_n$. Solving this for s_n:

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \ ,$$

so

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} \ ,$$

which equals $(1-x)^{-1}$ if $|x|<1$ and diverges if $|x|>1$.

We look at the cases $x=\pm 1$ separately. For $x=1$, $s_n = n$, so the series diverges. For $x=-1$, the sequence s_n is the sequence $1,0,1,0,1,0,\ldots$, so cannot converge to any particular number.

Example 9.11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k(k+1)} = 1 \ .$

We first use the fact that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \ .$$

Thus the partial sum s_n can be calculated:

$$s_n = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \cdots + (\frac{1}{n}-\frac{1}{n+1})$$

$$= 1 + (-\frac{1}{2}+\frac{1}{2}) + (-\frac{1}{3}+\frac{1}{3}) + (-\frac{1}{4}+\frac{1}{4}) - \cdots + (-\frac{1}{n}+\frac{1}{n}) - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \ ,$$

which converges to 1 as n goes to infinity. This is an example of a *telescoping series*.

We now observe that if a series converges, its general term must go to zero.

Proposition 9.9. If $\displaystyle\sum_{k=0}^{\infty} a_k$ converges, then $\displaystyle\lim_{n\to\infty} a_k = 0$.

To see this, let $s_n = \sum_{k=0}^{n} a_k,\ t_n = \sum_{k=0}^{n-1} a_k$. Then, since these are both sequences of the partial sums of the series, but indexed differently, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. Thus $\lim_{n\to\infty}(s_n - t_n) = 0$. But $s_n - t_n = a_n$.

Be careful: there are many series whose general term goes to zero which do not converge.

Proposition 9.5 for sequences translates to the following for series:

Proposition 9.10. If $a_n = b_n + c_n$, and the series $\sum b_n$ and $\sum c_n$ converge, then so does the series $\sum a_n$, and

$$\sum a_n = \sum b_n + \sum c_n \ .$$

Absolute Convergence

There are new difficulties when we have to consider series including negative as well as positive terms. For example, although the series $\sum 1/n$ diverges (as we'll see below, example 9.16), if we alternately change signs, the series converges.

Example 9.12. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{converges.}$$

To see this, we start by looking at the sequences of even partial sums and odd partial sums separately. Since

$$s_{2(n+1)} = s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} > s_{2n}$$

the sequence of even partial sums is increasing. Similarly,

$$s_{2(n+1)+1} = s_{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} < s_{2n+1}$$

tells us that the sequence of odd partial sums is decreasing. Now

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1} > s_{2n} \ , \tag{9.5}$$

that is, the odd partial sums are all greater than all the even partial sums. So both sequences are monotonic and bounded, and thus converge. But, they converge to the same limit, as we see by taking the limits in the expression (9.5):

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} = \lim_{n\to\infty} s_{2n} \ ,$$

since $1/(2n+1 \to 0$. Since the sequences of even partial sums and that of odd partial sums converge to the same limit, the full sequence also converges, and to the same limit.

This argument actually generalizes to any *alternating series*, a series whose terms alternate in sign.

Proposition 9.11. If a_n is a decreasing sequence, and $\lim_{n\to\infty} a_n = 0$ then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Definition 9.4 Given a sequence a_n, we say the series $\sum a_n$ *converges absolutely* if, for the series formed of the absolute values $|a_n|$, we have convergence: $\sum |a_n| < \infty$.

Proposition 9.12. If a series converges absolutely, it converges. That is,

$$\text{if} \ \sum |a_n| < \infty \ , \quad \text{then} \ \sum a_n \ \text{converges.}$$

To see that, let s_n be the nth partial sum of the sequence, p_n the sum of all the positive terms making up s_n, and q_n the sum of the absolute values of all the negative terms. Then

$$s_n = p_n - q_n \ .$$

Both sequences p_n and q_n are increasing, and bounded by $\sum |a_n|$, so converge, to, say p, q respectively. Then

$$\sum a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} p_n - \lim_{n\to\infty} q_n = p - q \ .$$

Problems 9.2

Does the series converge? If it does, try to find the sum.

1. $$\sum_{n=1}^{\infty} \frac{5^n}{8^{n+1}}$$

2. $$\sum_{n=1}^{\infty} \frac{5^n}{8^n + 1}$$

3. $$\sum_{k=1}^{\infty} \frac{1}{(2k)(2k+2)}$$

4. $$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+21}$$

5. $$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Do these series converge:

6. $$\sum_{0}^{\infty} (-1)^{3n+1} \frac{n^2}{n^3 - (-1)^n} \ .$$

7. $$\sum_{0}^{\infty} \frac{2^n + 3^{n+1}}{6^n} \ .$$

8. Let a_n be a sequence of positive numbers. Show that if $\sum a_n$ converges then $\sum a_n^2$ converges.

9.3 Tests for Convergence

Throughout this section, unless otherwise specified, we will be considering series, all of whose terms are positive. For such a series, the sequence of partial sums is increasing. If they remain bounded, then, by proposition 9.6, the sequence of partial sums will converge.

Proposition 9.13. If $a_k \geq 0$ for all k, and there is an $M > 0$ such that

$$\sum_{k=0}^{n} a_k \leq M \text{ for all n },$$

then

$$\sum_{k=0}^{\infty} a_k \text{ converges .}$$

Because of this proposition, for a series with positive terms, the statements $\sum a_k$ converges, $\sum a_k$ diverges, are usually written simply as

$$\sum_{k=0}^{\infty} a_k < \infty \quad \text{(converges) ,} \qquad \sum_{k=0}^{\infty} \mathrm{a_k} = \infty \quad \text{(diverges) .} \tag{9.6}$$

Here is an important application of this proposition:

Proposition 9.14. (Comparison Test). Given two sequences $a_k,\ b_k$ with $0 \leq a_k \leq b_k$. Then

(a) $$\text{if } \sum b_k < \infty \text{ , then } \sum \mathrm{a_k} < \infty \text{ ,}$$

(b) $$\text{if } \sum a_k = \infty \text{ , then } \sum \mathrm{b_k} = \infty \text{ .}$$

As for (a), the sequence of partial sums of $s_n = \sum_0^n a_k$ is bounded by $\sum_0^\infty b_k$, so converges by Proposition 9.13. In the second case, since the sequence of partial sums $\sum a_k$ has no bound, neither does the sequence of partial sums of $\sum b_k$.

It is important to observe that it is not necessary that the inequalities in the hypothesis of proposition 9.14 hold for all k, only that they eventually hold. That is because the issue of convergence series is determined by the end of the series, and not affected by any finite number of terms.

Example 9.13. $$\sum \frac{1}{r^k(r+1)} < \infty \text{ if } 0 < \mathrm{r} < 1.$$

Since $r^{k+1} < r^k(r+1)$,

$$\frac{1}{r^k(r+1)} < \frac{1}{r^{k+1}} \text{ ,}$$

so the comparison test applies.

Example 9.14. $$\sum \frac{k}{r^k} < \infty \text{ if } \mathrm{r} > 1.$$

Now, here the trouble is that the numerator grows without bound - but it doesn't grow as fast a power. So, what we do is borrow something from the denominator to compensate for the numerator. We note that eventually $k/r^{k/2} < 1$; in fact, this is true as soon as $k > 2\ln k/\ln r$ (which eventually happens, since $k/\ln k \to \infty$). Then for all k larger than this number

$$\frac{k}{r^k} = \frac{k}{(\sqrt{r})^k}\frac{1}{(\sqrt{r})^k} < \frac{1}{(\sqrt{r})^k} \ .$$

Since $r > 1$, we also have $\sqrt{r} > 1$, and so the series

$$\sum \frac{1}{(\sqrt{r})^k}$$

converges, and thus, by comparison, our original series converges.

Example 9.15. $$\sum_{n=0}^{\infty} \frac{1}{n^2} < \infty \ .$$

Now,

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \ ,$$

so our series is dominated by a telescoping series which converges (see example 9.11 above).

A very useful application of the comparison test is the following.

Proposition 9.15 (The Integral Test). Suppose that f is a nonnegative, nonincreasing function defined on an interval $[M, \infty)$. Suppose the a_n is a sequence such that for $n > M$, $a_n = f(n)$. Then

(*a*) $$\text{if } \int_M^\infty f(x)dx < \infty \quad \text{then} \quad \sum_{\mathrm{n=0}}^{\infty} \mathrm{a_n} < \infty \ ,$$

(*b*) $$\text{if } \int_M^\infty f(x)dx = \infty \quad \text{then} \quad \sum_{\mathrm{n=0}}^{\infty} \mathrm{a_n} = \infty \ .$$

Let

$$b_n = \int_n^{n+1} f(x)dx \ .$$

Then, since the function is nonincreasing, $f(n) \geq b_n \geq f(n+1)$; that is $a_n \geq b_n \geq a_{n+1}$. Now, use the comparison theorem. For example, if $\int f(x)dx < \infty$, then $\sum b_n$ converges, so by comparison $\sum a_{n+1}$ also converges.

Example 9.16 (The harmonic series).

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \ .$$

We apply the integral test using the function $f(x) = 1/x$. Since

$$\int_1^\infty \frac{dx}{x} = \infty \ ,$$

as we saw in chapter 8, the result follows.

If we apply example 8.17 to series via the integral test we have a result which is very useful for comparisons:

Proposition 9.16. Let p be a positive number.

(*a*) $$\sum_{n=1}^\infty \frac{1}{n^p} < \infty \quad \text{if } p > 1$$

(*b*) $$\sum_{n=1}^\infty \frac{1}{n^p} = \infty \quad \text{if } p \le 1$$

This follows from the facts (example 8.17):

$$\int_1^\infty \frac{dx}{x^p} < \infty \quad \text{if } p > 1 \quad \text{and} \quad = \infty \quad \text{if } p \le 1 \ .$$

Example 9.17. $$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p} \ .$$

The function $f(x) = 1/x(\ln x)^p$ is decreasing. We integrate using the substitution $u = \ln x$:

$$\int_2^A \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln A} \frac{du}{u^p} \ .$$

We know (again from example 8.17) that this converges if $p > 1$, and otherwise diverges. Thus, by the integral test,

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p} < \infty \quad \text{if p} > 1 \ ,$$

and otherwise diverges.

We now turn to a tool to test for convergence when we cannot realize the general term of the series in the form $f(n)$ for some function f. For example, if the expression for a_n involves the factorial, we proceed to the following.

Proposition 9.17. (Ratio Test). Given the series $\sum a_n$, consider

$$\lim \frac{a_{n+1}}{a_n} = L \ ,$$

if the limit exists. If $L < 1$, the series converges; if $L > 1$, the series diverges. For the case $L = 1$, we can draw no conclusion.

Suppose that $L < 1$. Then there a number r with $L < r < 1$ such that eventually $a_{n+1}/a_n < r$. That is, there is an integer N such that $a_{n+1}/a_n < r$ for all $n \geq N$. We conclude

$$a_{N+1} < a_N r \ , \quad a_{N+2} < a_{N+1} r < a_N r^2 \ , \quad a_{N+3} < a_{N+2} r < a_N r^3 \ ,$$

and so forth. Thus, we have, for all $k \geq 1$, $a_{N+k} < a_N r^k$, so by comparison with the geometric series, our series converges.

If on the other hand, $L > 1$, there is a number r, $L > r > 1$, such that eventually $a_{n+1}/a_n > r$. Following the same argument but with the inequalities reversed, we conclude that for all $k \geq 1$, $a_{N+k}/a_N r^k$, so we have divergence by comparison with the geometric series. We can conclude nothing if $L = 1$. This is the case for the all the series of the type $\sum 1/n^p$, and as we have seen, for some p we get convergence, and divergence for other p.

Example 9.18.

$$\sum_{n=1}^{\infty} \frac{a^n}{n!} \ .$$

We try the ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \frac{a}{n+1} \to 0$$

as $n \to \infty$, so the ratio test gives us convergence.

Example 9.19.

$$\sum_{n=1}^{\infty} n^2 x^n \quad \text{converges for} \quad -1 < \mathrm{x} < 1 \ .$$

Here we use the ratio test for the absolute values;

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} = (\frac{n+1}{n})^2 |x| \to |x| \ .$$

Thus, we get convergence for x of absolute value less than 1.

Example 9.20.

$$\sum_{n=1}^{\infty} \frac{2^n n^3}{3^n} \ .$$

Try the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)^3}{3^{n+1}} \frac{3^n}{2^n n^3} = \frac{2}{3} (\frac{n+1}{n})^3 \to \frac{2}{3}$$

so we have convergence.

Example 9.21.

$$\sum_{n=1}^{\infty} r^n \ .$$

Here the ratio test gives

$$\frac{a_{n+1}}{a_n} = r \ ,$$

so we conclude that the series converges if $r < 1$, and diverges if $r > 1$. This may seem to be a simplification of proposition 9.8, but in fact it is a fraud. The argument is circular, for we have used proposition 9.8 to derive the ratio test.

Notice that we didn't really need to know that the limit of a_{n+1}/a_n exists, only that eventually these ratios are either less than some number less than 1 to conclude convergence, or greater than some number greater than 1, for divergence.

Problems 9.3

For each problem, determine whether or not the series converges or diverges. Give your reasoning.

1. $$\sum_{n=1}^{\infty} \frac{n+1}{n^3}$$

2. $$\sum_{n=2}^{\infty} \frac{(n+1)^2}{n^3 \ln n}$$

3. $$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

4. $$\sum_{n=1}^{\infty} \frac{n^e}{e^n}$$

5. $$\sum_{n=1}^{\infty} \frac{n^{5/2}}{n^4 - n^3 + n^2 + 1}$$

6. $$\sum_{n=1}^{\infty} \frac{n!n}{(2n)!}$$

7. $$\Sigma_{n=1}^{\infty} \frac{n^2+1}{n^3\sqrt{n}}$$

8. $$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

9. $$\sum_{n=1}^{\infty} \frac{2^n n^3}{n!}$$

10. For what positive integers k (if any) does the following series converge? Give your reasoning.

$$\sum_{n=1}^{\infty} \frac{k!(n-k)!}{n!}$$

9.4 Power series

Definition 9.5. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n \ . \tag{9.7}$$

The point c is called the *center* of the power series.

A power series defines a function on the set of points for which it converges by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \ .$$

The series provides an effective way of approximately evaluating the function f; our goal in these last sections is to show that the transcendental functions we've come across do have a power series representation. We can use the ratio test to determine the question of convergence. We take the ratio of successive terms of (9.7):

$$\frac{|a_{n+1}||x-c|^{n+1}}{|a_n||x-c|^n} = \frac{|a_{n+1}|}{|a_n|}|x-c| \to L|x-c| \ ,$$

if the limit $L = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ exists. In this case the series converges absolutely for $|x-c| < 1/L$, and diverges for $|x-c| > 1/L$. It can be shown that, in general, even if the limit of the ratio of successive coefficients doesn't exist, there is an interva, say of radius R, l centered at c in which the power series converges absolutely, and diverges outside that interval. R may be zero, in which case the series converges only for $x = c$, or we may have $R = \infty$ in which case the series converges for all real numbers. For other values of R, what happens at the endpoints of the interval needs to be determined independently. R is called the *radius of convergence* of the power series.

Proposition 9.18. Given the power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \ ,$$

there is a number R, $0 \le R \le \infty$ such that we get absolute convergence for all x, $|x-c| < R$, and divergence for all x, $|x-c| > R$. We have this value of R:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R} \ ,$$

if the limit exists.

The first example of a power series representation is that of the geometric series:

Example 9.22. $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1$$

has the radius of convergence $R = 1$ (recall proposition 9.8).

Example 9.23. $$\sum_{n=0}^{\infty} n^k x^n \quad \text{converges for} \quad |x| < 1$$

for any number k. We use the ratio test. The ratio of successive coefficients

$$\frac{(n+1)^k}{n^k} = (\frac{n+1}{n})^k \to 1$$

as $n \to \infty$.

Example 9.24. $$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{has radius of convergence} \quad \mathrm{R} = \infty \ .$$

Using the ratio test:

$$\frac{1}{(n+1)!} \Big/ \frac{1}{n!} = \frac{1}{n+1} \to 0 \ ,$$

so $R = \infty$, and the series converges for all x. On the other hand, the ratio test shows us that the series

$$\sum_{n=0}^{\infty} n! x^n$$

has radius of convergence $R = 0$, so converges only for $x = 0$.

Power series, like the geometric series, converge quite rapidly. To illustrate this, consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \ .$$

By example 9.24, this converges, and as we shall see in example 9.29, the sum is e. We now see how close to e the sum of the first k terms brings us. The difference between e and this sum is the sum of the remaining terms

$$\sum_{n=k}^{\infty} \frac{1}{n!} = \sum_{m=0}^{\infty} \frac{1}{(m+k)!} \ ,$$

by the substitution $n = m + k$. Now $(m+k)! \geq m!k!$, since $(m+k)!$ is $m!$ times k terms, each of which is greater than the corresponding term in $k!$. Thus

$$\sum_{n=k}^{\infty} \frac{1}{n!} \leq \sum_{m=0}^{\infty} \frac{1}{k!m!} = \frac{1}{k!} \sum_{m=0}^{\infty} \frac{1}{m!} = \frac{e}{k!} \ .$$

So, for example,

$$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}$$

is within 3/120 of e (using the simple estimate $e \leq 3$).

Newton thought of power series as "generalized polynomials" - that is, as polynomials, only longer. This is justified, because we can operate with power series just as we operate with poynomials: we can add, multiply, and substitute in them by doing so term by term.

Example 9.25.
$$\frac{x}{1-x}=\sum_{n=0}^{\infty} x^{n+1} \quad \text{for} \quad \mathrm{R}<1 \ .$$

For

$$\frac{x}{1-x}=(x)\frac{1}{1-x}=x(1+x+x^2+x^3+\cdots)=x+x^2+x^3+x^4+\cdots$$

Example 9.26.
$$\frac{1}{1-x^2}=\sum_{n=0}^{\infty} x^{2n} \ , \quad \frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } |\mathrm{x}|<1 \ .$$

To see the first, we note that $1/(1-x^2)$ is obtained from $1/(1-x)$ by substituting x^2 for x. Thus, the power series representation is obtained in the same way. In the second, we have substituted $-x^2$ for x.

Example 9.27. Find a power series expansion for $1/(5-2x)$ centered at the origin. What is its radius of convergence?

To solve a problem like this, we have to relate the function to another function, whose power series we know. In this case that would be $1/(1-x)$. Now $5-2x=5(1-(2/5)x)$, so our function is obtained from $1/(1-x)$ by first replacing x by $(2/5)x$, and then dividing by 5. We follow the same instructions with the power series.

Start with :
$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n \ .$$

Replace x by $(2/5)x$:
$$\frac{1}{1-(2/5)x}=\sum_{n=0}^{\infty}(\frac{2}{5}x)^n \ .$$

Divide by 5 and clean up :
$$\frac{1}{5-2x}=\frac{1}{5}\sum_{n=0}^{\infty}(\frac{2}{5}x)^n=\sum_{n=0}^{\infty}\frac{2^n x^n}{5^{n+1}} \ .$$

We can calculate the radius of convergence using proposition 9.18, or we can reason as follows: since the series we started with converges for $|x|<1$, our final series converges for $|(2/5)x|<1$, or $|x|<5/2$.

Finally, we can also integrate and differentiate power series term by term:

Proposition 9.19. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \ ,$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \ ,$$

and both have the same radius of convergence, R.

Example 9.28. $$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \ .$$

We know that the derivative of the arc tangent is $1/(1+x^2)$. Now, in example 9.26, we have already found the power series representation of that function, so we obtain the power series representation of $\arctan x$ by integrating term by term.

Example 9.29. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \ .$$

Let $f(x) = \sum_{n=0}^{\infty} x^n/n!$ Then, differentiating term by term, we find

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ ,$$

where the last equation is obtained by replacing the index n by $n+1$. Thus $f'(x) = f(x)$, so satisfies the differential equation, $y' = y$, defining the exponential function. Since $f(0) = 1$ also, it is the exponential function.

Example 9.30. $$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \quad \text{for all } x \ .$$

Just replace x in example 9.29 by $-x^2$.

Problems 9.4

In problems 1-5 find the radius of convergence of the series:

1. $$\sum_{n=1}^{\infty} \frac{2^n}{(n+1)!} x^n$$

2. $$\sum_{n=1}^{\infty} \frac{n}{3^n} x^n$$

3. $$\sum_{n=0}^{\infty} n(n-1)(n-2)(\frac{x}{3})^n$$

4. $$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

5. $$\sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{n!} x^n$$

6. Let
$$f(x) = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{n!} x^n .$$
Find a formula for the function f.

7. We know that for $r > 0,\ r < 1$,
$$\sum_{n=0}^{\infty} r^k = \frac{1}{1-r} .$$
Show that the error made in summing just the first $k+1$ terms is at most $r^{k+1}/(1-r)$.

8. Does the series converge or diverge? Give your reasoning.

a) $$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4 - n^2 + n}.$$

b) $$\sum_{n=1}^{\infty} \frac{n!}{e^n}$$

c) $$\sum_{n=1}^{\infty} \frac{e^{\cos(n\pi)}}{n^2}$$

9. a) Let $f(x) = \sum_{n=0}^{\infty} (2^n - 1)x^n$. What is the radius of convergence of the series?

b). Write $f(x)$ in closed form (that is, as an algebraic expression).

9.5 Taylor series

Finally we tackle the question: how do we find the power series representation of a given function? Recalling that the purpose of the power series is to have an effective way to approximate the values of a function by polynomials, we turn to that question: what is the best way to so approximate a function? We start with a function f that has derivatives of all orders defined in an interval about the origin. To begin with, we recall the definition of the derivative in this context:
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = f'(0) .$$

If we rewrite this as

$$\lim_{x\to 0} \frac{f(x) - (f(0) + f'(0)x)}{x} = 0 ,$$

we see that the linear function $y = f(0) + f'(0)x$ approximates $f(x)$ to first order: $f(0) + f'(0)x$ is closer to $f(x)$ than x is to zero, and by an order of magnitude. We now ask, can we find a quadratic polynomial which approximates f to second order? Let $y = a + bx + cx^2$ be such a polymomial. Then we want

$$\lim_{x\to 0} \frac{f(x) - (a + bx + cx^2)}{x^2} = 0 .$$

We calculate this limit using l'Hôpital's rule. First of all, for l'Hôpital's rule to apply, we have to have $a = f(0)$. Then

$$\lim_{x\to 0} \frac{f(x) - (f(0) + bx + cx^2)}{x^2} =^{l'H} \lim_{x\to 0} \frac{f'(x) - (b + 2cx)}{2x} .$$

We can apply l'Hôpital's rule again, if we have $b = f'(0)$:

$$\lim_{x\to 0} \frac{f(x) - (f'(0) + 2cx)}{2x} =^{l'H} \lim_{x\to 0} \frac{f''(x) - 2c}{2} = 0$$

if $c = f''(0)/2$. We conclude that the polynomial

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

approximates f to second order: this is closer to $f(x)$ than x is to 0 by two orders of magnitude. Furthermore, it is the unique quadratic polynomial to do so.

We can repeat this procedure as many times as we care to, concluding

Proposition 9.19. The polynomial which approximates f near 0 to nth order is

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!} .$$

Of course we can make the same argument at any point, not just the origin. To summarize:

Definition 9.6. Suppose that f is a function with derivatives at all orders defined in an interval about the point c. The *Taylor polynomial of degree* n of f, centered at c is

$$(T_c^{(n)} f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k .$$

Proposition 9.20. The Taylor polynomial $T_c^{(n)} f$ is the polynomial of degree n which approximates f near c to nth order.

So, we can compute effective approximations to the values of $f(x)$ near c by these Taylor polynomials; but the question is, how effective is this? More precisely, what is the error? We use this estimate:

Proposition 9.21. Suppose that f is differentiable to order $n+1$ in the interval $[c-a, c+a]$ centered at the point c. Then the *error* in approximating f in this interval by its Taylor polynomial of degree n, $T_c^{(n)}f$ is bounded by

$$\frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \ , \tag{9.8}$$

where M_{n+1} is a bound of the values of $f^{(n+1)}$ over the interval $[c-a, c+a]$. To be precise, we have the inequality

$$|f(x) - T_c^n f(x)| \le \frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \ .$$

In the first section of the next chapter we will show how the error estimate is obtained, and see how to work with it. What we want now is to concentrate on the representation by series.

Definition 9.7. Let f be a function which is differentiable to all orders in a neighborhood of the point c. The **Taylor series** for f centered at c is

$$T_c f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If c is the origin, this series is called the **Maclaurin series** for f.

Proposition 9.22. Suppose that f is a function which has derivatives of all orders in the interval $(c-a, c+a)$. Let M_n be a bound for the nth derivative of f in the interval. If the sequence

$$\frac{M_n}{n!}|x-c|^n \to 0, \tag{9.9}$$

converges to zero for all x in the interval, then f is given by its Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

in $(c-a, c+a)$.

This gives us another way of seeing that e^x has the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ ,$$

since the nth derivative of e^x is still e^x, and its value at $x=0$ is 1. By a parallel calculation we obtain the power series representation of e^x centered at any point:

Example 9.31. For c any point, the function e^x has the Taylor series representation centered at c:

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n \ .$$

We do have to verify that the remainders converge to zero, that is, the terms (9.9) converge to zero. Since e^x is an increasing function, its maximum in the interval $[a-c, a+c]$ is at $x = a+c$, so we can take $M_n = e^{a+c}$. Then, for the exponential function we have

$$\lim_{n\to\infty} \frac{M_n}{n!}|x-c|^n = e^{a+c} \lim_{n\to\infty} \frac{|x-c|^n}{n!} = 0$$

by example 9.6.

It is useful to make the following observation

Proposition 9.22. Suppose that f has a power series representation:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \ . \tag{9.10}$$

Then, this is its Taylor series. More precisely:

$$a_n = \frac{f^{(n)}(c)}{n!} \ .$$

This is easy to see; if we differentiate (9.10) k times we obtain:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k)a_n(x-c)^{n-k} \ .$$

Now, let $x = c$: only the first term remains since all terms but the first have the factor $x - c$. Thus we obtain $f^{(k)}(c) = k!a_k$,

So, if we have found a power series representative of a function, then that is automatically the Taylor series for the function.

Example 9.33. Find the Maclaurin series for the function $f(x) = 1 - x + 5x^2 - x^3$. Since a polynomial is already expressed as a sum of powers of x, that expression is a power series, and thus the Maclaurin series for the polynomial.

Example 9.34. Find the Taylor series centered at $c = 1$ for the function $f(x) = 1 - x + 5x^2 - x^3$. We have to find the values of the derivatives of f at $c = 1$:

$$f(1) = 4 \ ,$$

$$f'(x) = -1 + 10x - 3x^2 \ , \quad \text{so} \quad \mathrm{f}'(1) = 6 \ ,$$
$$f''(x) = 10 - 6x \ , \quad \text{so} \quad \mathrm{f}'(1) = 4 \ ,$$
$$f'''(x) = -6 \ , \quad \text{so} \quad \mathrm{f}'(1) = -6 \ ,$$

and all higher derivatives are zero. Thus the Taylor series is

$$f(x) = 4 + 6(x-1) + \frac{4}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 = 4 + 6(x-1) + 2(x-1)^2 - (x-1)^3 \ .$$

We can find the Maclaurin series for many functions, so long as we know how to differentiate them. Following is a list of some important Maclaurin series.

Proposition 9.23.

(a)
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

(b)
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(c)
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

(d)
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(e)
$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

We have already seen how to get (a), (b and (e). For the trigonometric functions, we proceed as follows. First, the cosine:

$$\begin{aligned} f(0) &= 1\ , \\ f'(x) &= -\sin x\ , \quad \text{so} \quad \mathrm{f}'(1) = 0\ , \\ f''(x) &= -\cos x\ , \quad \text{so} \quad \mathrm{f}'(1) = -1\ , \\ f'''(x) &= \sin x\ , \quad \text{so} \quad \mathrm{f}'(1) = 0\ . \\ f^{(iv)}(x) &= \cos x\ , \quad \text{so} \quad \mathrm{f}^{(\mathrm{iv})}(1) = 1\ . \end{aligned}$$

Thus, up to four terms we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\ .$$

But, now, since we have returned to $\cos x$, the cycle $\{1, 0, -1, 0\}$ repeats itself again and again. We conclude that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots .,$$

which can be rewritten as (c) of proposition 9.23.

As another example, we calculate the Taylor series for $\ln x$ for x near 1, using the fact that $\ln x$ is the integral of $1/x$. Start with the geometric series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \text{for} \quad |t| < 1\ .$$

Substitute $x = 1 - t$:

$$\frac{1}{x} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n \quad \text{for} \quad |x-1| < 1 \ .$$

Integrate for the final result:

$$\ln x = \sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n=1}}{n+1} \quad \text{for} \quad |x-1| < 1 \ .$$

Problems 9.5.

1. Find the Taylor series centered at the origin for the function

$$F(x) = \int_0^x \frac{dt}{1-t^4} \ .$$

2. Find the Taylor series centered at the origin for the antiderivative (indefinite integral) of

$$f(x) = \frac{e^{-x^2} - 1}{x} \ .$$

3. Find the Taylor series centered at the origin for the function

$$\int_0^x \frac{1+t^2}{1-t^2} dt \ .$$

4. Find the Taylor series centered at the origin for the function

$$\frac{1}{(1-x^2)^2} .$$

5. Find the Taylor expansion of x^3 centered at the point -1.

6. Find the Taylor series centered at the origin for the function

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

7. Find the first 5 coefficients of the Maclaurin series for $f(x) = e^x \cos x$.

8. Expand $f(x) = 1 + x - 3x^2 + x^9$ in a Maclaurin series.

9. For the Maclaurin series expansion:

$$\frac{t}{2-t^2} = \sum_{n=0}^{\infty} a_n t^n$$

find the values of a_0, a_1, a_2, a_3.

10. Since the concept of convergence of a power series depends only on the notion of the distance between two numbers a, b, given by the absolute value $|a - b|$, we can consider series defined for complex numbers:

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad z = x + iy$$

with x and y real numbers, $i = \sqrt{-1}$ and $|z| = \sqrt{x^2 + y^2}$. With this definition we see (with the same proof) that the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges for all z. This we call the *complex exponential* e^z. Show that, for real numbers x:

$$e^{ix} = \cos x + i \sin x \ .$$

X. Numerical Methods

10.1 Taylor Approximation

Suppose that f is a function defined in a neighborhood of a point c, and suppose that f has derivatives of all orders near c. In section 5 of chapter 9 we introduced the Taylor polynomials for f:

Definition 10.1. The *Taylor polynomial of degree* n of f, centered at c is

$$(T_c^{(n)} f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \ .$$

We saw, in section 9.5, that the Taylor polynomial of degree n is for the best approximation near c to f by a nth degree polynomial. We recall that fact:

Proposition 10.1. The Taylor polynomial $T_c^{(n)} f$ is the polynomial of degree n or less that approximates f near c to nth order.

This statement is not very useful without some estimate of the error bar in this approximation. This was stated in the preceding chapter without proof as:

Proposition 10.2. Suppose that f is differentiable to order $n+1$ in the interval $[c-a, c+a]$ centered at the point c. Then the *error* in approximating f in this interval by its Taylor polynomial of degree n, $T_c^{(n)} f$ is bounded by

$$\frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \ , \tag{10.1}$$

where M_{n+1} is a bound of the values of $f^{(n+1)}$ over the interval $[c-a, c+a]$. To be precise, we have the inequality

$$|f(x) - T_c^n f(x)| \le \frac{M_{n+1}}{(n+1)!}|x-c|^{n+1} \quad \text{for all } x \text{ between } c-a \text{ and } c+a \ .$$

Before demonstrating how to use this estimate, let us see how it comes about. To simplify the notation, we shall take c to be the origin. First, a preliminary step:

Lemma. Suppose that $f(0) = 0, f'(0) = 0, \ldots, f^{(n)}(0) = 0$. Then

$$|f(x)| \le \frac{M_{n+1}}{(n+1)!}|x|^{n+1} \ .$$

First we show the case $n = 1$. We have $|f''(s)| \le M_2$ for all s, $0 \le s \le x$. So, for any t, $0 \le t \le x$, we have

$$|f'(t)| = |\int_0^t f''(s)ds| \le \int_0^t |f''(s)|ds \le M_2 \int_0^t ds \le M_2 t \ .$$

But now,

$$|f(x)| = |\int_0^x f'(t)dt| \le \int_0^x |f'(t)|dt \le \int_0^x M_2 t dt \le M_2 \frac{x^2}{2} \ .$$

Of course, the same argument works for x negative, we just have to be careful with the signs.

Now, we show that if we assume the lemma for $n-1$ we can show it for n, and then invoke the principle of mathematical induction. Suppose we have gotten to the $(n-1)$th case. Then the lemma applies (at $n-1$) to the derivative f'; so we know that

$$|f'(t)| \leq \frac{M_{n+1}}{n!}|t|^n \text{ for all } t \text{ in the interval } [-a, a] .$$

(We have M_{n+1} because the nth derivative of f' is the $(n+1)$th derivative of f). Now we argue independently on each side of 0: for $x > 0$:

$$|f(x)| = |\int_0^x f'(t)dt| \leq \int_0^x |f'(t)|dt \leq \int_0^x \frac{M_{n+1}}{n!}t^n dt \leq \frac{M_{n+1}}{n!}\frac{t^{n+1}}{n+1} = \frac{M_{n+1}}{(n+1)!}|x|^{n+1} .$$

The argument for $x < 0$ is the same; just be careful with signs.

Now that the lemma is verified, we go to the proposition itself. Let $g = f - T_c^{(n)}f$ Then g satisfies the hypotheses of the lemma. Furthermore, since $T_c^{(n)}f$ is a polynomial of degree n, its $(n+1)$th derivative is identically zero. Thus $g^{(n+1)}$ has the same bound, M_{n+1}. Applying the lemma to g, we have the desired result:

$$|f(x) - T_c^{(n)}f| \leq \frac{M_{n+1}}{(n+1)!}|x|^{n+1} .$$

If this error estimate converges to 0 as $n \to \infty$, then we saw that f is be represented by its *Taylor series*:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

in the interval $[c-a, c+a]$.

Before doing some examples, let's review what has to be done. To use the Taylor polynomials to find an approximation to the value to a function within some error bound, we first have to find bounds M_n for the successive derivatives of the function. Then we have to calculate the values of

$$\frac{M_n}{n!}|x-c|^n \qquad (10.2)$$

for successive values of n until we have found one which is within the desired error. Then we calculate using the Taylor polynomial of degree $n-1$.

Example 10.1. Find $\sqrt{e}$ to within an error of 10^{-4}.

This is $e^{1/2}$, so we look at the function $f(x) = e^x$. Since $f^{(n)}(x) = e^x$ for all n, and the value $x = 1/2$ is within 1 of 0, we can use the Maclaurin series for e^x and the bounds $M_n = e^1$. Since 3 is more manageable than e, we take $M_n = 3$. Now we estimate the error (10.2) at stage n which we'll call $E(n)$. We have, in this example

$$E(n) = \frac{3}{n!}(\frac{1}{2})^n .$$

$$n = 1: \quad E(1) = \frac{3}{2}$$

$$n = 2: \quad E(2) = \frac{3}{2}\frac{1}{4}$$

$$n = 3: \quad E(3) = \frac{3}{6}\frac{1}{8} = \frac{3}{48}$$

$$n = 4: \quad E(4) = \frac{3}{24}\frac{1}{16} = \frac{3}{384}$$

$$n = 5: \quad E(5) = \frac{3}{120}\frac{1}{32} = 7.8 \times 10^{-4}$$

$$n = 6: \quad E(6) = \frac{3}{720}\frac{1}{64} < 10^{-4}$$

Thus, we have our estimate to within 10^{-4} by taking the fifth Taylor polynomial:

$$T_0^5(e^x)(1/2) = 1 + \frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{6}\frac{1}{8} + \frac{1}{24}\frac{1}{16} + \frac{1}{120}\frac{1}{32} = 1.6487$$

Example 10.2. Find $\sin(\pi/8)$ to within an error of 10^{-3}.

Here we start with the Maclaurin series for $\sin x$:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

Since the derivatives of $\sin x$ alternate between $\pm \sin x$ and $\pm \cos x$, we may take $M_n = 1$ for all n. We need to choose n large enough that the Taylor error estimate $E(n)$ satisfies

$$E(n) = \frac{1}{n!}(\frac{\pi}{8})^n < \frac{1}{2}10^{-3} \ .$$

$$n = 1: \quad E(1) = \frac{\pi}{8} = .3927$$

$$n = 2: \quad E(2) = \frac{1}{2}(\frac{\pi}{8})^2 = .077$$

$$n = 3: \quad E(3) = \frac{1}{6}(\frac{\pi}{8})^3 = .010$$

$$n = 4: \quad E(4) = \frac{1}{24}(\frac{\pi}{8})^4 = .0009$$

so we need only go to $n = 3$. The estimate is

$$T_0^3(\sin x)(\pi/8) = \frac{\pi}{8} - \frac{1}{6}(\frac{\pi}{8})^2 = .3826$$

or $\sin(\pi/8) = .383$, correct to three decimal places.

Some Taylor series converge too slowly to get a reasonable approximation by just a few terms of the series. As a rule, if the series has a factorial in the denominator, this technique will work efficiently, otherwise, it will not.

Example 10.3. Use the Maclaurin series

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

to estimate $\ln(1+a)$ for $a > 0$ to within 4 decimal places.

First we calculate the successive derivatives of $f(x) = \ln(1+x)$ to obtain the bounds M_n. We have

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

so we can take $M_n = (n-1)!$. Thus we need, for $x = a$:

$$E(n) = \frac{(n-1)!}{n!}a^n = \frac{a^n}{n} < 10^{-4} \ .$$

If $a = 1/10$, then the estimate occurs at $n = 4$, so the first three terms will do. But if $a = 1/2$, we don't get this inequality until $n = 12$, so we'll need 11 terms of the series.

Let's close this section with a more direct argument for the Taylor estimate, proposition 10.2. Again we assume that f is differentiable of all orders in the interval $[c-R, c+R]$, and that M_n is an upper bound of $|f^{(n)}(x)|$ on that interval. We have to treat the cases $x \le c$ and $x \ge c$ separately. Here we demonstrate the proposition for x in the interval $[c, c+R]$, the case of the left half interval is the same, but with more care given to signs. First, we show the case $n = 0$. By the fundamental theorem of the calculus,

$$f(x) - f(c) = \int_c^x f'(t)dt \ .$$

Since $f'(t) \le M_1$ in that interval,

$$f(x) - f(c) \le \int_c^x M_1 dt = M_1(x-c) \ .$$

We proceed now to the induction step: assume the theorem is true for all functions f and the integer $n-1$. Apply the theorem to the derivative f' of f and the integer $n-1$. Now, for every k, the k derivative of f' is the $(k+1)$st derivative of f, so, in particular, the bound on the nth derivative of f' is M_{n+1}. The induction hypothesis is this:

$$f'(x) - \big(f\text{"}(c) + f\text{"}(c)(x-c) + \frac{f^{(3)}(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n))}(c)}{(n-1)!}(x-c)^{n-1}\big) \le \frac{M_{n+1}}{n!}(x-c)^n \ .$$

Integrate this inequality from c to x, obtaining

$$f(x) - f(c) - \big(f\text{"}(c)(x-c) + f\text{"}(c)\frac{(x-c)^2}{2} + \frac{f^{(3)}(c)}{2!}\frac{(x-c)^3}{3} + \cdots + \frac{f^{(n)}(c)}{(n-1)!}\frac{(x-c)^n}{n}\big)$$

$$\le \frac{M_{n+1}}{n!}\frac{(x-c)^{n+1}}{n+1} \ ,$$

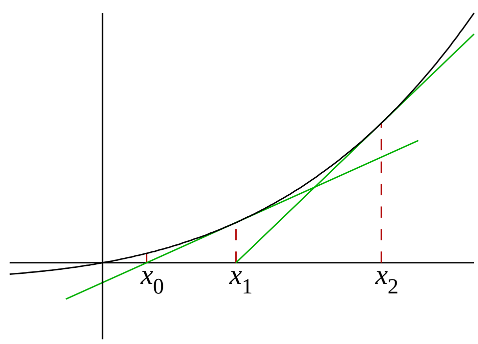
which is the desired result.

10.2 Newton's method

Suppose that we want to estimate the solution of the equation $f(x) = 0$. If this is a linear equation, there is no problem: we just look for the point at which the line crosses the x-axis. Newton's idea is that, since the tangent line approximates the graph, why not take as the estimate the point at which the tangent line crosses the x-axis? Well, to make sense of this, let's start at some value x_0 and calculate $y_0 = f(x_0)$. If that is 0, then we're through. If not, let x_1 be the point at which the tangent line at (x_0, y_0) crosses the x-axis. That is our first approximation. If it is not good enough, replace x_0 with x_1, and repeat the process, over and over again, until the result is good enough. Of course, the definition of "good enough" is to be determined by the degree of precision desired in the context of the problem at hand.

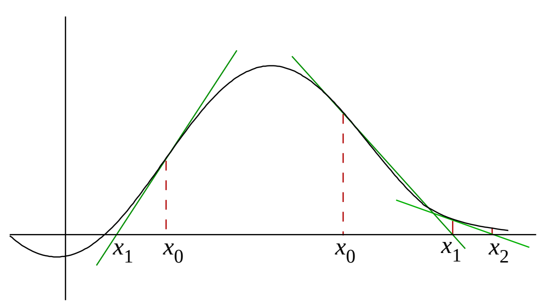
This process is illustrated in figure 10.1.

Figure 10.1



However, it doesn't always work so well, as figure 10.2 shows. The choice f x_0 on the left works, but the choice on the right sets in motin a search for a root that doesn't exist. The important point is to start with a decent guess for x_0, so that we start in a range of the function where the concavity of the curve forces convergence of these successive approximations.

Figure 10.2



Newton's method thus, is a technique for replacing an approximation by a better one. Suppose we start with the function $y = f(x)$, and have found an approximation $x = a$, with $f(a)$ (relatively) close to zero. The slope of the tangent line at $x = a$ is $f'(a)$, and the equation of the tangent line is $y - f(a) = f'(a)(x - a)$. This intersects the x-axis where $y = 0$, so we have

$$-f(a) = f'(a)(x-a) \quad \text{which has the solution} \quad x = a - \frac{f(a)}{f'(a)} \tag{10.3}$$

We now replace a by this value, and repeat the process. That is, we define a sequence of approximate solutions (hopefully converging to the root), using (10.3) as the recursion relation:

Newton's Method. Given the differentiable function $y = f(x)$, define a sequence recursively as follows:

$$a_0 = \quad \text{a good guess of the solution of} \quad f(x) = 0 \ ,$$

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \ . \tag{10.4}$$

If the guess is in an interval containing a root, and $f'(x) > 0$ or $f'(x) < 0$ in that interval, then the sequence a_n converges to a solution of $f(x) = 0$.

The criterion for convergence seems problematic, since we don't know the root of the equation - we're trying to find it. But, ordinarily the relation $y = f(x)$ is one we can graph, and from the graph we can see if the condition is satisfied, and make a good guess. How do we know when we are within an error e of the solution? This is a difficult question, and will be discussed in a course in Numerical Analysis. For our purposes, in the presence of the criterion, we can stop as soon as they are within e of each other.

Example 10.4. Find $\sqrt{8}$ correct to within 4 decimal places.

Here we want to find the root of the equation $f(x) = x^2 - 8 = 0$. Since $f'(x) = 2x$, the criterion is satisfied so long as $x > 0$ Let's start with $x_0 = 3$, since $3^2 = 9$ and 9 is close to 8. The next estimate is

$$x_1 = 3 - \frac{3^2 - 8}{2 \times 3} = 2.8333$$

We now use this in the recursion, and continue until we reach stability in the first 4 decimal places:

$$x_2 = 2.8333 - \frac{(2.8333)^2 - 8}{2 \cdot 2.8333} = 2.8284$$

$$x_3 = 2.8284 - \frac{(2.8284)^2 - 8}{2 \cdot 2.8284} = 2.8284 \ ,$$

which we take as the estimate accurate to 4 decimal places.

Before going on to other examples, we summarize the process: To solve $f(x) = 0$:
1. Graph $y = f(x)$ to find plausible intervals in which to work.
2. Calculate $f'(x)$, and determine an interval $[a.b]$ in which $f'(x)$ does not change sign, but for which the signs of $f(a)$ and $f(b)$ differ.

3.. Select a first estimate a_0 in $[a, b]$ so that $f(a_0)$ is small..
4. Calculate the recursion relation

$$x' = x - \frac{f(x)}{f'(x)} \ .$$

5. Find a_1 from a_0 by taking $x = a_0, \ x' = a_1$ in the recursion relation.
6. Repeat step 5 until the successive estimates are no further from each other than the desired estimate.

Of course, in practice this is all done on the computer. Furthermore, the way the function is defined may make it difficult, even impossible, to complete some of the steps. In such a case, pick a starting point as best you can and calculate a number of terms. If they don't seem to converge, try another starting point.

Example 10.5. Find the solutions of $f(x) = x^3 - 12x + 1 = 0$ to three decimal places.

First we graph the function so as to make an intelligent first estimate.

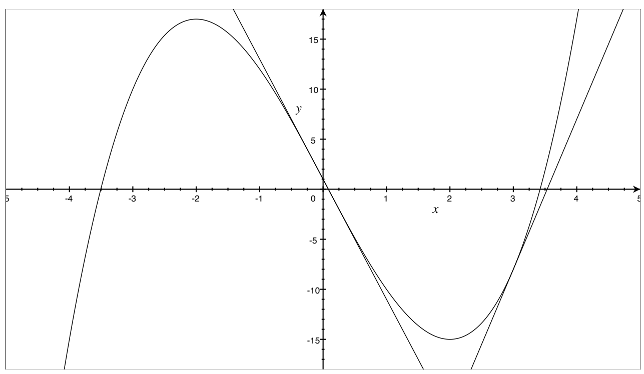


Figure 10.3

We have $f'(x) = 3x^2 - 12$, so the local maximum and minimum of $y = f(x)$ are at (-2,17), (2, -15). At $x = 0$, $y = 1$, and the graph is as shown in figure 10.3. There are three solutions; one close to 0, another larger than 2, and a third less than -2

First, find the solution near zero. We see from the graph that the derivative is well away from zero in a range including $x = 0$ and the solution, so we can take our first estimate to be $x_0 = 0$. Now we calculate successive estimates:

$$x_1 = 0 - \frac{1}{-12} = \frac{1}{12} = .08333$$

$$x_2 = .08333 - \frac{(.08333)^3 - 12(.08333) + 1}{3(.08333)^2 - 12} = .08338 \ ,$$

$$x_3 = .08338$$

so this solution is $x = .08338$ up to four decimal places.

Now, to find the solution larger than 2, it will not do to take $x_0 = 2$, since the derivative is 0 there. But if we take $x_0 = 3$, we have $f'(3) = 15$, a nice large number, so the recursion should work. We find

$$x_1 = 3 - \frac{3^3 - 12(3) + 1}{3(3^2) - 12} = 3 - \frac{-8}{15} = 3.5333 \ ,$$

$$x_2 = 3.5333 - \frac{(3.5333)^3 - 12(3.5333) + 1}{3(3.5333)^2 - 12} = 3.4267$$

$$x_3 = 3.4215 \;, \qquad x_4 = 3.4215 \;,$$

so this is our estimate to four decimal places. In the same way, starting at $x_0 = -3$, we find the third solution.

Example 10.6. Solve $e^x = x + 2$ to three decimal places.

Here $f(x) = e^x - x - 2,\ f'(x) = e^x - 1$. Since the derivative is increasing, and greater than 1 at $x = 1$, and $f(1) < 0$, while $f(2) > 0$, a good first estimate will be any number between 1 and 2. So, take $x_0 = 1$. The recursion is

$$x' = x - \frac{e^x - x - 2}{e^x - 1} = \frac{e^x(x-1) + 2}{e^x - 1} \;.$$

We now calculate the successive estimates:

$$x_1 = 1.16395 \;, \quad x_2 = 1.1464 \;, \quad x_3 = 1.1462 \;, \quad x_4 = 1.1462 \;,$$

so this is the desired estimate. Notice that in this range, the derivative is not very large, so that the convergence is slower than in the preceding examples.

10.3. Numerical Integration

We have learned techniques for calculating definite integrals which are based on finding antiderivatives of the function to be integrated. However, in many cases we cannot find an expression for the antiderivative, and these techniques will not lead to an answer. For example $f(x) = \sqrt{1 + x^3}$. No formula for the integral exists in any integral tables. In such a case, we have to return to the definition of the integral, and approximate the definite integral by the approximating sums. To explain this, we first review the definition of the definite integral.

10.4 Definition. Let $y = f(x)$ be a function defined on the interval $[a, b]$. The *definite integral* is defined as follows. A *partition* of the interval is any increasing sequence

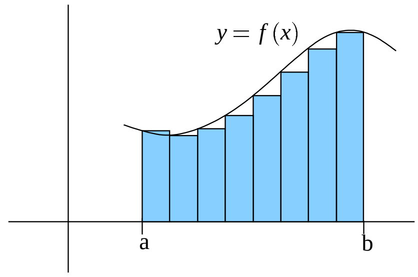
$$\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

of points in the interval. The corresponding *approximating sum* is

$$\sum_{1}^{n} f(x_i')\Delta x_i \tag{10.5}$$

where Δx_i is the length $x_i - x_{i-1}$ of the ith interval, x_i' is any point on that interval, and $\sum$ indicates that we add all these products together (see Figure 10.4).

Figure 10.4: Approximation to the area under a curve.



If these approximating sums approach a limit as the partition becomes increasingly fine (the lengths of the subdivisions go to zero), this limit is the *definite integral* of f over the interval $[a, b]$, denoted

$$\int_a^b f(x)dx \ .$$

Thus, we can approximate a definite integral by the sums (10.5). One way to accomplish this is: Pick an integer N, and divide the interval $[a, b]$ into N subintervals, all of size $(b-a)/N$. For each subinterval, evaluate the function at the right endpoint x_i, and form the sum

$$\sum_1^N f(x_i)\Delta x_i = \frac{(b-a)}{N}\sum_1^N f(x_i) \quad \text{(Approximating Rectangles)} \tag{10.6}$$

Example 10.7. Let's find an approximate value for $\int_0^1 \sqrt{1+x^3}dx$ this way. Let's divide the interval into 10 subintervals. Then the sum (7) is

$$\frac{1}{10}(\sqrt{1+(1/10)^3} + \sqrt{1+(2/10)^3} + \sqrt{1+(3/10)^3} + \cdots \sqrt{1+(10/10)^3}) =$$

$$= \frac{1}{10}(1.0005 + 1.0040 + 1.0134 + 1.0315 + 1.0607 + 1.1027 + 1.1589 + 1.2296 + 1.3149 + 1.4142) =$$

$$= 1.1330 \ .$$

Of course these calculations are tedious if done by hand, but, by computer - completely trivial. It is a good idea to try these using a spreadsheet, because there you get to follow the computation. If we take more subdivisions, we get a better approximation. For example, if we take $N = 100$ we get

$$\frac{1}{100}(\sqrt{1+(1/100)^3} + \sqrt{1+(2/100)^3} + \sqrt{1+(3/100)^3} + \cdots \sqrt{1+(100/100)^3}) = 1.113528 \ ,$$

an apparently better approximation.

Example 10.8. Just to see how well this method is working, let us use it to approximate $\int_1^2 x^{1.4}dx$, which we know to be $(2^{2.4}-1)/(2.4) = .1.782513\ldots$. Let's first take $N = 10$. The approximating sum is

$$\frac{1}{10}((1.1)^{1.4} + (1.2)^{1.4} + (1.3)^{1.4} + \cdots (2)^{1.4})$$

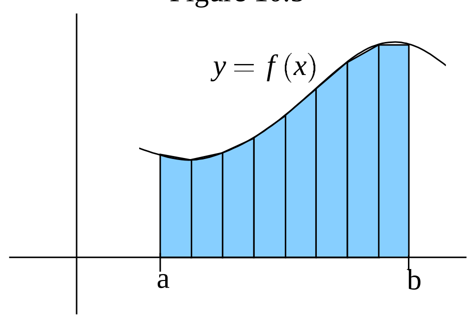
$$= \frac{1}{10}(1.1427 + 1.2908 + 1.4438 + 1.6017 + 1.7641 + 1.9309 + 2.1020 + 2.2771 + 2.4562 + 2.6390)$$

$$= 1.8648 \; .$$

For $N = 100$ we obtain the estimate 1.790712, which is better, but not great.

We can improve this method by improving the estimate in each subinterval. First, note that we have estimated the integral in each subinterval by the area of the rectangle of height at the right endpoint. If instead we estimate this area using the trapezoid whose upper side is the line segment joining the two endpoints (see figure 10.5), it makes sense that this is a better estimate.

Figure 10.5



This comes down to

$$(10.7) \qquad \frac{(b-a)}{2N}(f(a) + 2\sum_{1}^{N-1} f(x_i) + f(b)) \quad \text{(Trapezoid Rule)}$$

Going one step further, we might replace the upper curve by the best parabolic approximation. For N even, this leads to the rule

$$(10.8) \qquad \frac{(b-a)}{3N}(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{N-1}) + f(b)) \quad \text{(Simpson's Rule)}$$

Let us now do the above examples using these two rules:

Example 10.9. The calculation of $\int_0^1 \sqrt{1+x^3}dx$ using the Trapezoidal rule and $N = 10$ gives us

$$\frac{1}{20}(1 + 2\big(\sqrt{1+(1/10)^3} + \sqrt{1+(2/10)^3} + \sqrt{1+(3/10)^3} + \cdots \sqrt{1+(9/10)^3}\big) + \sqrt{2}) =$$

$$= \frac{1}{20}(1 + 2\big(1.0005 + 1.0040 + 1.0134 + \cdots + +1.2296 + 1.3149\big) + 1.4142) =$$

$$= 1.0123 \; .$$

Example 10.10. Let's compare the estimates of $\int_0^1 x^{1.4}dx$ with these two new methods.. First, with $N = 10$, and the trapezoid rule:

$$\frac{1}{20}(1 + 2\big((1.1)^{1.4} + (1.2)^{1.4} + (1.3)^{1.4} + \cdots (1.9)^{1.4}\big) + 2^{1.4} =$$

$$= \frac{1}{20}(1 + 2(1.1427 + 1.2908 + \cdots + 2.2771 + 2.4562\big) + 2.6390) =$$

$$= 1.78288 \quad \text{(Trapezoid Rule)} \ .$$

Now Simpson's rule and $N = 10$:

$$= \frac{1}{30}(1 + 4(1.1427) + 2(1.2908) + \cdots + 2(2.2771) + 4(2.4562) + 2.6390)$$

$$= 1.782513 \quad \text{(Simpson's Rule)} \ .$$

In general, these estimate of a definite integral get better as N gets larger, and the trapezoid rule is better than the rectangular sums, but not as good as Simpson's rule, simply because the local approximations to the curve are better. The question is, of course: how good are these rules: what is the error for a given N and a given rule? The following proposition gives the estimates (these are not easy to derive).

10.5 Proposition. Let f be a function defined on the interval $[a, b]$, and let M_n be a bound on the nth derivative of f in this interval. Then, the approximations to $\int_a^b f(x)dx$ using N subdivisions are correct to within the error $E(N)$ given by:

$$E(N) = \frac{(b-a)^3}{12N^2}M_2 \quad \text{(Trapezoid Rule)} \ ; \qquad E(N) = \frac{(b-a)^5}{180N^4}M_4 \quad \text{(Simpson's Rule)} \ .$$

Example 10.11. Let us calculate the error in the trapezoid estimate for $\int_0^1 \sqrt{1+x^3}dx$ given in example 10.9. We first have to find a bound on the second derivative. Differentiating twice, we have

$$f''(x) = \frac{3(2x + \frac{x^4}{2})}{2(1+x^3)^{3/2}} \ ,$$

which is bounded in [0,1] by 15/4. Since $N = 10$, the error is less than

$$\frac{1^3}{12(10^2)}\frac{15}{4} = .003125,$$

so the answer 1.0123 of example 10.9 is correct to two decimals.

Example 10.12. Now, let's calculate the error in the use of Simpson's rule in example 10, with $N = 10$. First we need to bound the fourth derivative of $f(x) = x^{1.4}$ in the interval [1,2]. A calculation leads to

$$|f^{(4)}(x)| = |(-1.6)(-.6)(.4)(1.4)x^{-2.6}| \leq .5376$$

Thus the error is bounded by

$$\frac{1^5}{180(10)^4}(.5376) < 3 \times 10^{-8} \ ,$$

telling us that the calculation of example 10.10 is correct in all six decimal places.

Problems, Chapter 10.

1. Since $\tan(\pi/6) = 1/\sqrt{3}$,and therefore, $\pi = 6\arctan(1/\sqrt{3})$ we can use the Taylor series for the arc tangent to estimate π. Do this, using the first three nonzero terms.

2. Since $\sin(\pi/6) = 1/2$, we can also find π by solving the equation $\sin x = 1/2$. We can approximate the solution by replacing sin by an approximating Taylor polynomial, and then using Newton's method. Do this with the cubic Taylor polynomial for $\sin x$.

3. Find a solution, by Newton's method, of the equation

$$x^5 - x^4 + x^3 - x^2 = 4$$

correct to five decimal places.

4. Here is another way of estimating π. We know that

$$\pi/4 = \int_0^1 \frac{dx}{1+x^2} \ .$$

Estimate this integral by the trapezoid rule, using steps of size 1/10. How many steps should we take to be sure of an estimate correct to 4 decimal places?

5. Define

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)(n+1)!} \ .$$

Evaluate $J_0(1)$ correctly to 4 decimal places.

6. Find an estimate for

$$\int_0^2 \frac{\sin x}{x} dx$$

using Simpson' s rule with $N = 20$ subdivisions.

XI. Conics and Polar Coordinates

11.1 Quadratic Relations

A quadratic relation between the variables $x,\ y$ is an equation of the form

(11.1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey = F \quad \text{so long as one of} \quad A, B, C \quad \text{is not zero .}$$

If we substitute a number for x, we obtain a quadratic equation in y, which we can then solve by the quadratic formula. In this way (11.1) defines y as a function of x implicitly, although for some x there may be two solutions or no solutions. In any event, the set of points in the x, y-plane satisfying equation (11.1) is a curve. These curves are called *conics* or *conic sections*, for they represent, in suitably chosen coordinates, the curve on a cone in three dimensions cut out by a plane.

As a first example, consider the equation

(11.2) $$x^2 + y^2 = F\ .$$

This is the circle of radius $\sqrt{F}$ when $F > 0$, just the origin if $F = 0$, and has no points if $F < 0$.

We will see that in general, the curve defined by the quadratic relation (11.1) is one of these three curves: a) parabola, b) ellipse, c) hyperbola. As we have seen in (11.2), for special values of the coefficients, there may be no curve. There are other possibilities; for example, the equation $x^2 - y^2 = a$ describes a hyperbola if $a \neq 0$, but if $a = 0$, we get the two lines $x = \pm y$.

First we list the *standard forms* of the basic curves. These are standard in the sense that any other curve given by a quadratic equation is obtained from one of these by moving the curve in the plane by translating and/or rotating.

The Parabola. The standard form is one of these:

(11.3) $$y = ax^2\ , \qquad x = ay^2$$

The sign of a determines the orientation of the parabola. This gives us four possibilities, the graphs of which are shown in figures 11.1-11.4.

Figure 11.1

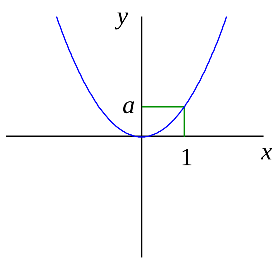


Figure 11.2

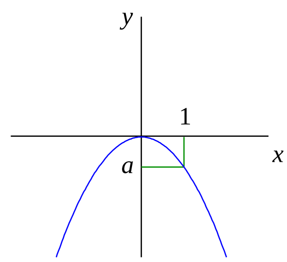


Figure 11.3

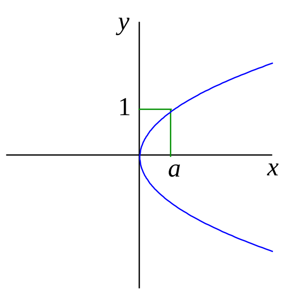
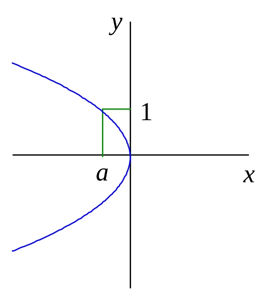


Figure 11.4



The magnitude of a determines the spread of the parabola: for $|a|$ very small, the curve is narrow, and as $|a|$ gets large, the parabola broadens. The origin is the *vertex* of the parabola. In the first two cases, the y-axis is the it axis of the parabola, in the second two cases it is the x-axis. The parabola is symmetric about its axis.

The Ellipse. The standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{11.4}$$

The values x can take lie between $-a$ and a and the values of y lie between $-b$ and b.

If $a > b$ (as shown in figure 11.5), the *major axis* of the ellipse is the x-axis, the *minor axis* is the y-axis and the points $(\pm a, 0)$ are its *vertices*. If $a < b$ (as shown in figure 11.6), the major axis of the ellipse is the y-axis, the Ax-axis is the minor axis, and the points $(0, \pm b)$ are its *vertices*.

Of course, if $a = b$, the curve is the circle of radius $|a|$, and there are no special vertices or axes.

Figure 11.5

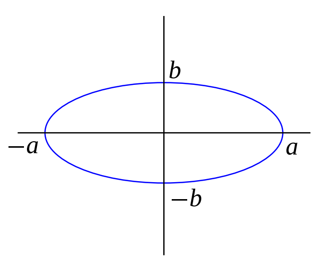
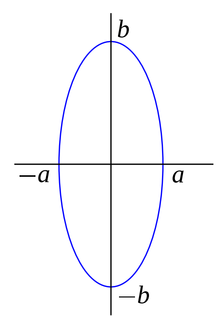


Figure 11.6



The Hyperbola. The standard form is one of these:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \ , \tag{11.5}$$

corresponding to the graphs (11.7),(11.8) respectively.

The x-axis is the *axis* of the hyperbola (11.7). The points $(\pm a, 0)$ are the *vertices* of the hyperbola; for x between these values, there corresponds no point on the curve. Similarly for the hyperbola (11.8), the y-axis is its *axis* and the points $(0, \pm b)$ are its *vertices*.

The lines

$$y = \pm \frac{b}{a} x$$

are the *asymptotes* of the hyperbola, in the sense that, as $x \to \infty$, the curve gets closer and closer to these lines. We see this by multiplying the defining equation by b^2/x^2, and consider what happens as $x \to \infty$. For example, using the first equation of (11.5), we get

$$\frac{b^2}{a^2} - \frac{y^2}{x^2} = \frac{b^2}{x^2} \quad \text{or}$$

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2} \ .$$

Thus, as $|x|$ gets large, the hyperbola approaches the graph of

$$\frac{y^2}{x^2} = \frac{b^2}{a^2}$$

which amounts to the two equations $y = \pm(b/a)x$. The dotted lines of figures 11.7 and 11.8 represent the asymptotes.

Figure 11.7

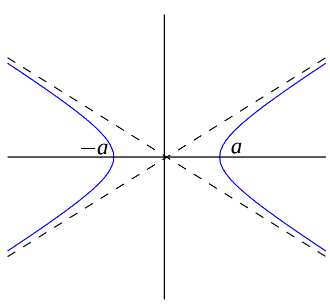
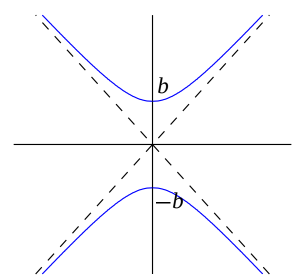


Figure 11.8



Now, let's return to the general quadratic relation (11.1), with $C = 0$:

$$Ax^2 + By^2 + Dx + Ey + F = 0 \ ,$$

and see how to relate the equation to the above standard forms. By completing the square in both x and y we are led to an equation which looks much like one of the standard forms, but with the center removed to a new point (x_0, y_0). If $C \neq 0$, the situation is more difficult: a rotation of the figure is also required to get it into standard form. We leave that for a later discussion, and here consider only the case $C = 0$. First, some examples:

Example 11.1. Let's graph the curve

$$3x^2 - 30x - y + 73 = 0$$

We have to complete the square in x. We get

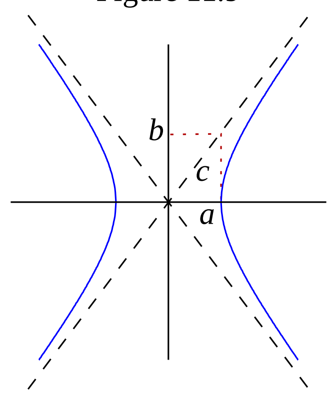
$$3(x^2 - 10x + 25) - y + 73 - 75 = 0$$

which can be rewritten in the standard form

$$y + 2 = 3(x - 5)^2 \ ,$$

where the vertex is at (5,-2) rather than the origin (see figure 11.9).

Figure 11.9



Example 11.2. Graph the curve

$$9x^2 + 4y^2 - 18x - 16y = 11$$

Completing the squares:

$$9(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 11 + 9 + 16 = 36 \quad \text{or}$$

$$9(x - 1)^2 + 4y - 2)^2 = 36$$

which can be rewritten in standard form (with the point (1,2) replacing the origin):

$$\frac{(x-1)^2}{2^2} + \frac{(y-2)^2}{3^2} = 1 \ .$$

See figure 11.10.

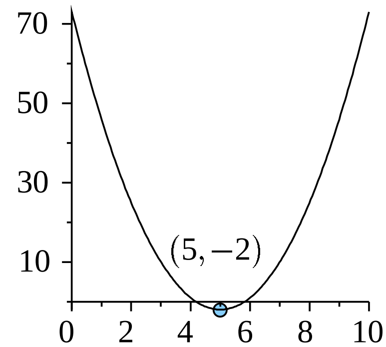


Figure 11.10

Example 11.3. Graph the curve

$$-5x^2 + y^2 + 30x + 4y - 46 = 0$$

Completing the squares:

$$-5(x^2 - 6x + 9) + (y^2 + 4y + 4) = 46 - 45 + 4 = 5 \ ,$$

$$\frac{(y+2)^2}{(\sqrt{5})^2} - (x-3)^2 = 1 \ .$$

See figure (11.11).

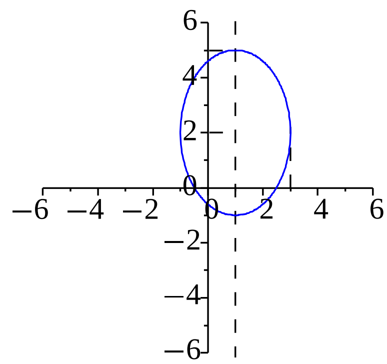


Figure 11.11

Proposition 11.1. The equation

$$Ax^2 + By^2 + Dx + Ey = F$$

can be put into one of the following forms by completing the square:

$a)$ (parabola) : $\quad y - y_0 = A(x - x_0)^2, \quad$ if $B = 0$.

The vertex of the parabola is at (x_0, y_0), and the axis is the line $x = x_0$.

b) (parabola) : $$x - x_0 = C(y - y_0)^2 \qquad \text{if } A = 0\ .$$

The vertex of the parabola is at (x_0, y_0), and the axis is the line $y = y_0$.

c) (ellipse) : $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \qquad \text{if } A \text{ and } B \text{ are of the same sign}\ .$$

The center of the ellipse is at (x_0, y_0), and its axes are the lines $x = x_0,\ y = y_0$.

d) (hyperbola) : $$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-y_0)^2}{b^2} - \frac{(x-x_0)^2}{a^2} = 1$$

if A and B are of different signs. The center of the hyperbola is (x_0, y_0), and its axes are the lines $x = x_0,\ y = y_0$.

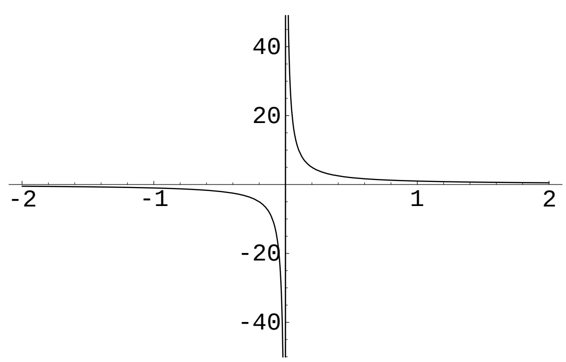
e) If both A and B are zero, the curve is a line. The following degenerate cases may also result:

$$A(x-x_0)^2 + B(y-y_0)^2 \le 0\ : \quad \text{no graph or just the point } (x_0, y_0).$$

$$A(x-x_0)^2 - B(y-y_0)^2 = 0\ : \quad \text{two lines crossing at } (x_0, y_0).$$

Example 11.4. Finally, just to illustrate the situation of a quadratic whose coefficient of xy is nonzero, we consider the curve $xy = 1$. This curve is symmetric about the lines $y = \pm x$, and has the asymptotes $x = 0,\ y = 0$. This appears to be a hyperbola with major axis the line $x = y$. In fact, if we make the linear change of variables $x = u + v,\ y = u - v$, this becomes the curve $u^2 - v^2 = 1$ in the new variables. (This change of variables represents a rotation by 45°, with a slight change of scale.) See figure (11.12).

Figure 11.12



Problems 11.1

In problems 1-4 put the conic in standard form, and find the center and vertices. In the next section you will learn about foci of the conic sections. After reading that section, return to these problems and find the foci.

1. $$y - 8x^2 + 32x - 29 = 0$$

2. $$9x^2 + 4y^2 - 36x + 8y + 4 = 0$$

3. $$4x^2 - y^2 + 2y = 5$$

4. $$x^2 - 5y^2 - 4x + 10y = 1$$

5. Find the equation of the parabola with vertex (2,5), axis $y = 5$ that goes through the point (4,2).

6. Find the equation of the ellipse with vertices (0,0), (0,10) that goes through the point (4,6).

7. Find the equation of the hyperbola with vertices at (3,0), (1,0) and asymptotes of slope ± 5.

In problems 8-10 your are given the equation of a curve C and a point P on the curve. Find the point of intersection of the tangent line to C at P with the x-axis.

8. $$x^2 + 5y = 0 \ , \quad (10, -20) \ .$$

9. $$x^2 + 4y^2 = 16 \ , \quad (2\sqrt{3}, 1) \ .$$

10. $$4x^2 - y^2 = 1 \ , \quad (\sqrt{2}/2, 1) \ .$$

11.2 Eccentricity and Foci

Curves described by quadratic equations are called the *conic sections* because they can be visualized as the intersection of a cone with a plane. We shall now consider another definition, dating from the ancient Greeks, which leads to important properties of the conics.

Fix a point F and a line L in the plane such that L does not go through F.

Pick a positive number e. We consider the locus C of all points X in the plane such that

$$|XF| = e|XL| \tag{11.6}$$

where $|XY|$ means the distance from X to Y. e is the *eccentricity* of C; F the *focus* and L the *directrix.* Note that the curve C is symmetric about the line through the focus and perpendicular to the directrix. This is the *axis* of the curve. There is one point between F and L on C which is on this axis; this point is the *vertex* of C. See figure 11.13.

Figure 11.13

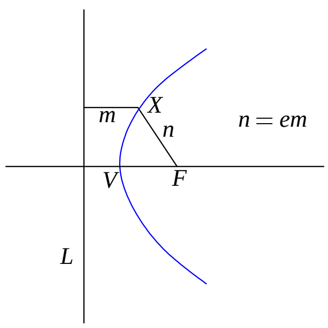
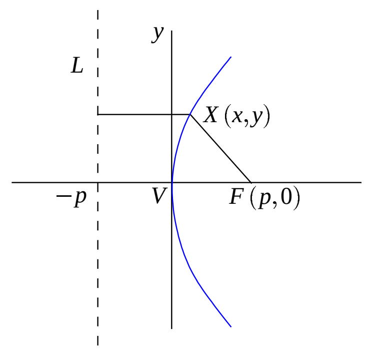


Figure 11.14



We now show that if $e = 1$, C is a parabola, if $e < 1$, C is an ellipse and if $e > 1$, C is a hyperbola. Let's take the axis of C to be the x axis, and place the vertex at the origin, O. Then the focus is some point $(p, 0)$; we take $p > 0$. Since $|OF| = p$, from (11.6) we find that the directrix is the line $x = -p/e$ (see figure 11.14).

Now, for a point $X = (x, y)$ on the curve, we have

$$|XL| = x + p/e \quad \text{and} \quad |XF| = \sqrt{(x-p)^2 + y^2}$$

and so equation (11.6) in coordinates is given by

$$\sqrt{(x-p)^2 + y^2} = e(x + p/e) = ex + p \ . \tag{11.7}$$

Case $e = 1$. Squaring both sides we get

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2 \quad \text{simplifying to} \quad y^2 = 4px \ .$$

This of course is the standard form of a parabola. It also locates the focus (at $(p, 0)$) and the directrix (the line $x = -p$) of the parabola.

Proposition 11.2 . The focus of the parabola $y^2 = ax$ is $a/4$ units on one side of the vertex of the parabola along the axis, and and the directrix intersects the axis $a/4$ units on the other side.

Example 11.5. Find the vertex, focus and directrix of the parabola given by the equation

$$2x^2 + 12x - y + 20 = 0$$

We put the equation in standard form. First we move the terms not involving x to the other side:

$$2x^2 + 12x = y - 20 \ ,$$

and then, completing the square, we have

$$2(x^2 + 6x + 9) = y - 20 + 18 \ , \quad \text{or} \quad (x+3)^2 = \frac{1}{2}(y-4) \ .$$

Thus the vertex is at $(-3, 4)$, axis of the parabola is the line $y = 4$ and the parabola opens up (since $4p = 1/2 > 0$). We have $p = 1/8$, so the focus is 1/8=.125 units above the vertex at (-3,4.125) and the directrix is the line $y = 3.875$.

Example 11.6. Find the equation of the parabola whose vertex is at $(4, 2)$ and whose directrix is the line $x = -1$. Find the focus of this parabola.

Since the directrix is a vertical line, the axis is horizontal, so the equation has the form

$$(y - 2)^2 = 4p(x - 4) \ ,$$

since the vertex is at $(4, 2)$. Now p is the distance between the vertex and the directrix, so $p = 4 - (-1) = 5$. Thus the equation of the parabola is

$$(y - 2)^2 = 20(x - 4) \ .$$

The focus is 5 units to the right of the vertex, so is at $(9, 2)$.

Example 11.7. Find the equation of the parabola whose focus is the origin and whose vertex is at the point $(a, 0)$ with $a > 0$.

Since both the focus and vertex are on the x=axis, that is the axis of the parabola. Since the vertex is to the right of the focus, the parabola opens to the left. Thus the equation has the form

$$y^2 = -4p(x - a) \ ,$$

where p is the distance between focus and vertex. But that is a, so the equation is

$$y^2 = -4a(x - a) \ .$$

Case $e \neq 1$. Squaring both sides of (11.7) gives us

$$x^2 - 2px + p^2 + y^2 = e^2(x^2 + 2px + p^2)$$

which simplifies to

$$(1 - e^2)x^2 + y^2 - 2p(1 + e)x = 0 \qquad (11.8)$$

Thus, if $e < 1$, this is an ellipse, and if $e > 1$ this is a hyperbola. Notice, because of symmetry in the minor axis, ellipses and hyperbolas have two foci; one on each side of the minor axis.

We now want show how to locate the foci of an ellipse ($e < 1$) given in standard form. Thus we start by putting (11.8) in standard form, and then compare it to the formula of Proposition 11.1c). Dividing equation (11.8) by the coefficient of x^2 gives us

$$x^2 - \frac{2p}{1-e}x + \frac{y^2}{1-e^2} = 0$$

Now completing the square, we come to

$$(x - \frac{p}{1-e})^2 + \frac{y^2}{1-e^2} = \frac{p^2}{(1-e)^2} \tag{11.9}$$

Comparing this to

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{11.10}$$

we see that the center of the ellipse is at $(p/(1-e), 0)$ and $a^2 = p^2/(1-e)^2,\ b^2 = a^2(1-e^2)$. Let c be the distance of the center from the focus. Since the focus is at $(p, 0)$,

$$c = \frac{p}{1-e} - p = p(\frac{1}{1-e} - 1) = e\frac{p}{1-e} = ea$$

and $c^2 = e^2a^2 = a^2 - b^2$. Summarizing

Proposition 11.3. If an ellipse is in standard form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1\ , \tag{11.10}$$

with $a > b$, then the foci of the ellipse are on the major axis, c units away from the center where

$$c^2 = a^2 - b^2$$

The eccentricity of the ellipse is given by the equations

$$b^2 = (1-e^2)a^2 \qquad \text{or} \qquad e = c/a$$

A similar argument for the case $e > 1$, the hyperbola, leads to

Proposition 11.4 . If a hyperbola is in standard form

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \tag{11.11}$$

then the foci of the hyperbola are on the major axis, c units away from the center where

$$c^2 = a^2 + b^2$$

The eccentricity of the hyperbola is given by the equations

$$b^2 = (e^2 - 1)a^2 \qquad \text{or} \qquad e = c/a$$

Example 11.8. Find the foci of the conic given by the equation

$$x^2 + 4y^2 - 2x = 8 \ .$$

First, we complete the square to get the equation in standard form:

$$\frac{(x-1)^2}{3^2} + \frac{y^2}{(3/2)^2} = 1 \ .$$

This conic is an ellipse centered at (1,0), with major axis the line $x = 1$, and $a^2 = 9,\ b^2 = 9/4$. Thus $c^2 = a^2 - b^2 = 9(3/4)$, so $c = (3/2)\sqrt{3}$. This is the distance of the foci from the center (along the major axis), so the foci are at $(1 \pm (3/2)\sqrt{3}, 0)$.

Example 11.9. Find the foci of the conic given by the equation

$$y^2 - x^2 + 4x = 13 \ .$$

Complete the squares, and get the standard form

$$\frac{y^2}{3^2} - \frac{(x-2)^2}{3^2} = 1 \ .$$

This is a hyperbola with center at $(2, 0)$, and major axis the line $x = 2$. We have $c^2 = a^2 + b^2 = 18$, so $c = 3\sqrt{2}$ is the distance of the foci from the center along the line $x = 2$. Thus the foci are at $(2, \pm 3\sqrt{2}$. The vertices are at $(2, \pm 3)$.

Example 11.10. Find the equation of the ellipse centered at the origin, with a focus at $(2, 0)$ and a vertex at $(3, 0)$.

The equation of an ellipse centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we are given $a = 3,\ c = 2$. Thus $b^2 = a^2 - c^2 = 5$, and the equation is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Problems 11.2.

1. Find the equation of the parabola whose vertex is at (0,2) and focus is the origin.

2. Find the vertex of the parabola with focus at (0,7) and passes through the points $(\pm 2, 7)$ and $(\pm 1, 5)$.

3. Find the equation of the parabola with focus at (2,3) and directrix the line $y = -1$.

4. Find the foci and eccentricity of the ellipse given by the equation

$$\frac{(x-1)^2}{4} + \frac{y^2}{16} = 1 \ .$$

5. A hyperbola has vertices at $(\pm 3, 0)$ and foci at $(\pm 4, 0)$. What are the equations of its asymptotes?

6. Find the foci and eccentricity of the hyperbola

$$\frac{(y-2)^2}{25} - \frac{(x+1)^2}{16} = 1 \ .$$

7. A hyperbola has asymptotes $y - 1 = \pm 0.8(x + 2)$. What is its eccentricity?

In each of problems 8 thorugh 10, the curve described depends upon a parameter. Identify the parameter, and find the equation of the curve in terms of the parameter.

8. A parabola with axis the x-axis and focus at the origin.

9. A hyperbola with foci at (-1,0), (1,0).

10. An ellipse with foci at (-1,0), (1,0).

11. Show that the hyperbola and the ellipse of problems 9 and 10 intersect orthogonally; that is, at a point of intersection their tangent lines are orthogonal.

11.3 String and Optical Properties of the Conics

We have seen that the parabola can be defined as the locus of points X equidistant from a given point F and a given line L. The ellipse and the hyperbola have similar definitions.

Proposition 11.5. Given two points F_1 and F_2 and a number a greater than half the distance between F_1 and F_2, the locus of points X such that

$$|XF_1| + |XF_2| = 2a \tag{11.13}$$

is an ellipse with foci at F_1 and F_2 and major axis of length $2a$.

Choose coordinates so that the points F_1 and F_2 lie on the x-axis, equidistant from the origin. Then F_1 has coordinates $(-c, 0)$, and F_2 has coordinates $(c, 0)$ for some $c < a$. Let X have the coordinates (x, y). Then (11.13) becomes

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Eliminate the radicals to verify that we end up with a quadratic equation which is that of an ellipse.

We should point out that every ellipse has property (11.13). For there is only one ellipse with given foci and length $2a$ of the major axis. So, if we start with a given ellipse, and then construct the curve satisfying (11.13) with the foci and major axis of the given ellipse, since that curve is an ellipse, it is the given ellipse.

We have a similar description of the hyperbola:

Figure 11.15

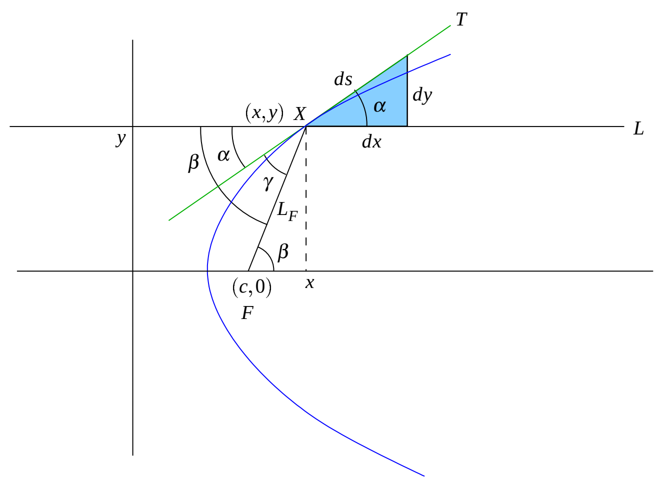


Figure 11.16

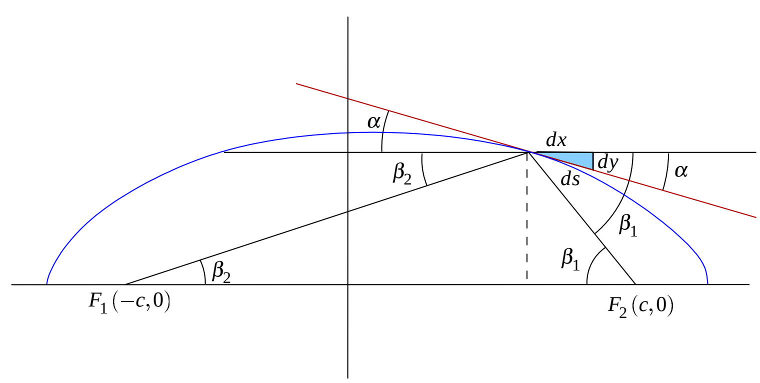
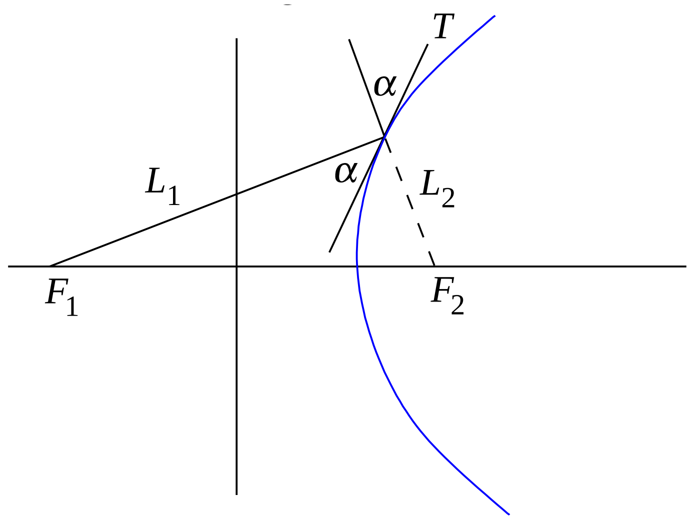


Figure 11.17



Proposition 11.6. Given two points F_1 and F_2 and a positive number a, the locus of points X such that

$$|XF_1| - |XF_2| = 2a \tag{11.13}$$

is a hyperbola with foci at F_1 and F_2.

Actually, this is just the branch of the hyperbola which wraps around the focus F_2; the other branch is given by the equation

$$|XF_2| - |XF_1| = 2a$$

The optical properties of the conics follow from these string characterizations. Let's start with the parabola. Suppose that the parabola is coated with a light-reflecting material. The rays of a beam of light originating far away along the axis of the parabola will approach the parabola along lines parallel to its axis. According to the physics of the situation, the angle of reflection off the parabola is equal to the angle of incidence. The optical property of the parabola is that these reflected rays all meet at the focus. See figure 11.15.

Proposition 11.7. Let X be a point on the parabola, and T the tangent line to the parabola at X. Let L_F be the line from the focus to X, and L the line through X parallel to the axis of the parabola. Then the angle between T and L_F is equal to the angle between T and L.

What we want to show, referring to figure 11.15, is that $\gamma = \alpha$. From the figure we see that $\gamma = \beta - \alpha$, so this amounts to showing that $\beta - \alpha = \alpha$.

Referring to figure 11.15, the focus-directrix definition of the parabola tells us that L_F (the distance from the point X to the focus F) is equal to $x + c$ (the distance from X to the directrix). Thus

$$L_F = x + c \ . \tag{11.15}$$

Squaring this equation gives us

$$(x - c)^2 + y^2 = (x + c)^2 \ .$$

Differentiate this equation with respect to arc length and divide by 2 to get

$$(x-c)\frac{dx}{ds} + y\frac{dy}{ds} = (x+c)\frac{dx}{ds} \ .$$

Now, divide by $x + c = L_F$ to get

$$\frac{x-c}{L_F}\frac{dx}{ds} + \frac{y}{L_F}\frac{dy}{ds} = \frac{dx}{ds} \ . \tag{11.16}$$

From figure 11.15, we have

$$\frac{dx}{ds} = \cos\alpha \ , \ \frac{dy}{ds} = \sin\alpha \ , \ \frac{x-c}{L_F} = \sin\beta \ , \ \frac{y}{L_F} = \cos\beta \ ,$$

so (11.16) becomes

$$\sin\beta\cos\alpha + \cos\beta\sin\alpha = \cos\alpha \ , \quad \text{or} \quad \cos(\beta-\alpha) = \cos\alpha \ ,$$

from which we conclude that $\beta - \alpha = \alpha$, as desired.

The optical property of the ellipse is that a ray of light emanating from one focus reflects off the ellipse so as to pass through the other focus (see figure 11.16).

Proposition 11.8. Let X be a point on the ellipse, and T the tangent line to the ellipse at X. Let L_1 be the line from the focus F_1 to X, and L_2 the line from the other focus F_2 to X. Then the angle between T and L_1 is equal to the angle between T and L_2.

What we want to show, referring to figure 11.16, is that $\beta_2 + \alpha = \beta_1 - \alpha$. We start with the string property, written in the coordinates as shown in the figure:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Differentiate with respect to arc length to arrive at

$$\frac{x+c}{\sqrt{(x+c)^2+y^2}}\frac{dx}{ds} + \frac{y}{\sqrt{(x+c)^2+y^2}}\frac{dy}{ds} + \frac{x-c}{\sqrt{(x-c)^2+y^2}}\frac{dx}{ds} + \frac{y}{\sqrt{(x-c)^2+y^2}}\frac{dy}{ds} = 0$$

Now make substitutions of the trigonometric functions, using the figure. We have to be careful: in our picture dy and $x - c$ are negative, so since the sine and cosine are ratios of *lengths*, we have

$$\cos\beta_1 = -\frac{x-c}{\sqrt{(x-c)^2+y^2}} \qquad \sin\alpha = \left|\frac{dy}{ds}\right| = -\frac{dy}{ds} \ .$$

Thus our equation becomes

$$\cos\beta_2\cos\alpha + \sin\beta_2(-\sin\alpha) + (-\cos\beta_1)\cos\alpha + \sin\beta_1(-\sin\alpha) = 0$$

or

$$(\cos\beta_2\cos\alpha - \sin\beta_2\sin\alpha) - (\cos\beta_1\cos\alpha + \sin\beta_1\sin\alpha) = 0$$

which is $\cos(\beta_2 + \alpha) - \cos(\beta_1 - \alpha) = 0$, so $\beta_2 + \alpha = \beta_1 - \alpha$ as desired.

The optical property of the hyperbola is that a ray of light emanating from one focus reflects off the opposite branch of the hyperbola so as to appear to have come from the other focus (see figure 11.17).

Proposition 11.9. Let X be a point on the hyperbola, and T the tangent line to the ellipse at X. Let L_1 be the line from the focus F_1 to X, and L_2 the line from the other focus F_2 to X. Then the exterior angles between T and L_1 and between T and L_2 are equal.

The verification is a computation completely analogous to the one for the ellipse: differentiate the string property, and then make the correct trigonometric substitutions

Problems 11.3

1. Find the point (x, y) on the parabola $y^2 = 12x$ for which the line from the focus meets the tangent line at an angle of 45°.

2. Give a proof, using the optical properties of the conics that confocal hyperbolae and ellipses intersect orthogonally. That is, suppose that H and E are respectively a hyperbola and an ellipse, and suppose that the have the same foci. Show that, at a point P of intersection of H and E that the curves have tangent lines which are perpendicular.

11.4 Polar Coordinates

Often a problem can be seen as that of understanding the motion of a particle in the plane relative to a fixed point. In such a situation it is desirable to be able to describe a position in terms of the length and the direction of the line between the two points. These are the **polar coordinates** of the point. We consider the fixed point as the origin of these coordinates, and take the positive x-axis as the "zero" direction. Then any other direction is described by the angle between it and the positive x axis, which we denote as θ. The distance of a point on this line from the origin is denoted r. These equations relate the cartesian coordinates (x, y) with the polar coordinates r, θ:

$$x = r\cos\theta\ , \quad y = r\sin\theta\ , \qquad r = \sqrt{x^2+y^2}\ , \quad \theta = \arctan\frac{y}{x} \tag{11.17}$$

See figure 11.18 to justify these formulas. Polar coordinates have two pecularities that need to be pointed out. Every value of (r, θ) determines a point in the plane. However, if $r = 0$, the point is the origin, and θ doesn't make sense. Secondly, the values (r, θ) and $(r, \theta + 2\pi)$, and in fact, $(r, \theta + 2n\pi)$ for any n give the same point. This ambiguity is sometimes of value: for example, when discussing the motion of a particle, n tells us how many times the particle has wound around the origin in the counterclockwise sense. Finally, it is also of convenience to let r take negative values, meaning a distance of $|r|$ in the opposite direction of the ray θ. Thus (r, θ) and $(-r, \theta + \pi)$ determine the same point. We now consider the graphs of equations in polar coordinates.

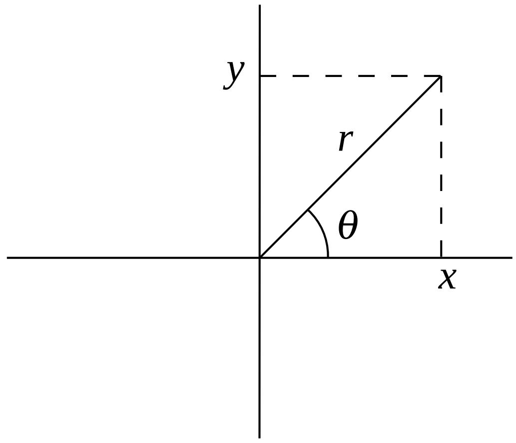


Figure 11.18

Example 11.11. The equation $r = a$, for $a > 0$ is satisfied by all points of distance a from the origin, so is polar equation of the circle of radius a centered at the origin.

Example 11.12. The equation $\theta = \theta_0$ is the line which makes an angle of θ_0 with the x-axis.

Example 11.13. $r = a\theta$ describes the motion of a point which rotates around the origin at angular velocity 1 while moving out along the ray at velocity a. This is the **Archimedean spiral**; see figure 11.19.

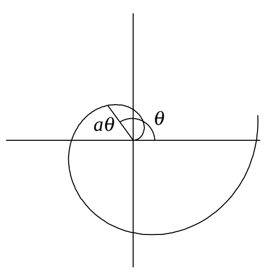


Figure 11.19

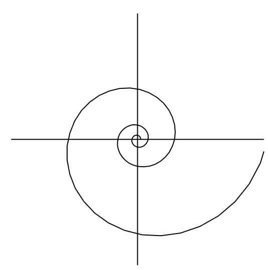


Figure 11.20

Example 11.14. $r = e^{a\theta}$ is another spiral, however, the point moves out along the ray at a rate exponential in the rate of rotation. This is the **logarithmic spiral**, depicted in figure 11.20.

Example 11.15. The equation $r = a\cos\theta$ is a the circle of diameter a with center on the x-axis which goes through the origin. For, if we multiply by r we get $r^2 = ar\cos\theta$, which can now be written in cartesian coordinates (using (11.17)) as

$$x^2 + y^2 = ax \qquad \text{or} \qquad (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4} .$$

Given an equation of the form $r = r(\theta)$, we can often trace out the graph by just studying the behavior of the function $r(\theta)$. Let's redo example 11.15 this way. We have this table

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
r	a	$\frac{a\sqrt{2}}{2}$	0	$-\frac{a\sqrt{2}}{2}$	$-a$	0	a

It is useful to follow the point on the curve of figure 11.21 as θ ranges from 0 to 2π. Between 0 and $\pi/2$ the point is in the first quadrant, and as the angle increases it moves toward the origin, reaching there at $\theta = \pi/2$. Then for θ between $\pi/2$ and π, the point is in the fourth quadrant (because $r < 0$), steadily moving away from the origin until we reach the point we've started with. This looks like a circle, and the argument above (in example 11.15) shows that it is. Note that as θ moves from π to 2π the circle is retraced.

Example 11.16. Similarly, the equation $r = a\cos(\theta - \theta_0)$ is the circle through the origin of radius a with center on the ray of angle θ_0. This amounts to the assertion that any equation of the form

$$r = a\cos\theta + b\sin\theta$$

is a circle with the origin the endpoint of one of its diameters (see problem 1 of this section).

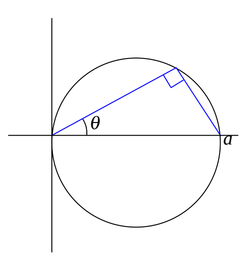


Figure 11.21

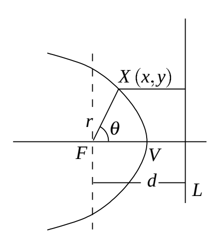


Figure 11.22

Example 11.17. If we are given the equation of a curve in cartesian coordinates, we can find its equation in polar coordinates through the substitution $x = r\cos\theta,\ y = r\sin\theta$. For example

(11.18) Equation of a line : $$r = \frac{c}{a\cos\theta + b\sin\theta}\ .$$

For, the general equation of a line is $ax + by = c$. After substitution this becomes

$$ar\cos\theta + br\sin\theta = c,$$

which gives us (11.18) when we solve for r.

Example 11.18. The polar equation of a conic of eccentricity e, focus at the origin and directrix the line $x = d$ is

(11.19) Equation of a Conic : $$r = \frac{ed}{1 + e\cos\theta}\ .$$

To show (11.19), we start with the defining relation $|XF| = e|XL|$, referring to figure 11.22.

In polar coordinates this gives us

$$r = e(d - x) = e(d - r\cos\theta)$$

Solving for r brings us to (11.19). If the figure is rotated by θ_0, we just replace θ with $\theta - \theta_0$

Example 11.19. $r = a\cos 2\theta$. We first construct the table:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	a	0	$-a$	0	a

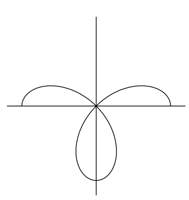


Figure 11.23

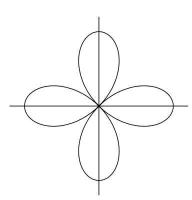


Figure 11.24

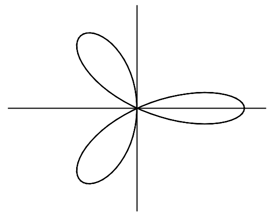


Figure 11.25

Follow this discussion along the graph in figure 11.23. This time the curve starts (at $\theta = 0$) at $r = a$, and decreases to zero by $\theta = \pi/4$. Between $\pi/4$ and $\pi/2$, r is negative, so the curve is in the third quadrant, and as θ rotates counterclockwise, r moves away from the origin finally to $r = -a$ for $\theta = \pi/2$. As θ increases from $\pi/2$ the point continues to move toward the origin (in the fourth quadrant), arriving there at $\theta = 3\pi/2$. Moving on, r becomes positive, so we enter the second quadrant with the distance from the origin steadily increasing until, at $\theta = \pi$ we are at $r = a$. Since $\cos\theta$ is an even function, as we move from π to 2π (or what is the same, from $-\pi$ to 0), we just get the same curve, reflected in the x-axis. The result is the **four-petalled rose** shown here in figure 11.24.

Example 11.20 $r = a\cos 3\theta$ is a three-petalled rose. Construct the table of important values between 0 and π and argue as in example 11.19. The table is

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	a	0	$-a$	0	a	0	$-a$

That completes the rose; of figure 11.25; as we proceed from π to 2π we traverse the rose again.

We conclude

Proposition 11.10. The graph of the equation $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ is a $2n$-petalled rose if n is even, and an n petalled rose if n is odd (traversed twice).

Limaçons. These are the curves defined by the equation $r = a + b\cos\theta$.

First, we consider the case: $a > b$. We have the table

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$a+b$	a	$a-b$	a	$a+b$

leading us to figure 11.26.

As b gets closer and closer to a, the value of r for $\theta = \pi$ goes to zero. Thus when $a = b$, we get the graph of figure 11.27, called the **cardioid**.

Then as b goes beyond a, r becomes negative as θ gets near π, and there is an inner loop of the limaçon.

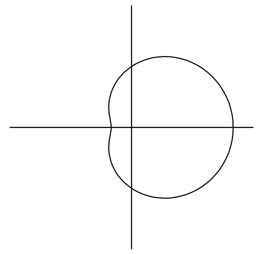


Figure 11.26

Figure 11.27

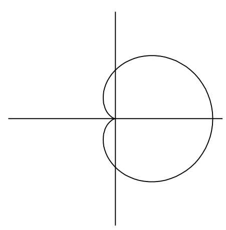
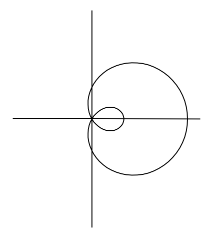


Figure 11.28



Example 11.21. $r = 2 + 4\cos\theta$. Our table is this:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	6	2	-2	2	6

When $\cos\theta = -1/2$, that is, for $\theta = \pm 2\pi/3$, the value of r is zero, and between these two values r is negative.Following these values, we arrive at the graph in figure 11.28.

We have drawn the curve so that it is tangent to the ray $\theta = \pm\pi/3$as the moving point comes to the origin. As we shall see in the next section, this is correct.

Finally, it is important to note that if the function $\cos\theta$ is replaced by $-\cos\theta$ the curve is reflected in the y-axis, and if it is replaced by $\pm\sin\theta$, it is rotated by a right angle.

Problems 11.4

1. Show that the graph of the polar equation $r = a\cos\theta + b\sin\theta$ is a circle of radius $\sqrt{a^2 + b^2}$ going through the origin. Where is its center?

2. Graph $r = 3(\cos\theta + \sqrt{3}\sin\theta)$.

3. What is the polar equation of an ellipse, with one focus at the origin, corresponding directrix the line $x = -3$ and corresponding vertex at the point (-1,0)?

4. Identify the curve: $r = 2\sin(5\theta)$.

5. Graph $r^2 = \cos(2\theta)$. This is called a *lemniscate*.

11.5 Calculus in polar coordinates

Arc length

Consider the curve given in polar coordinates by the equation $r = r(\theta)$. We can calculate the differential ds of arc length by the differential triangle in polar coordinates using the diagram in figure 11.29.

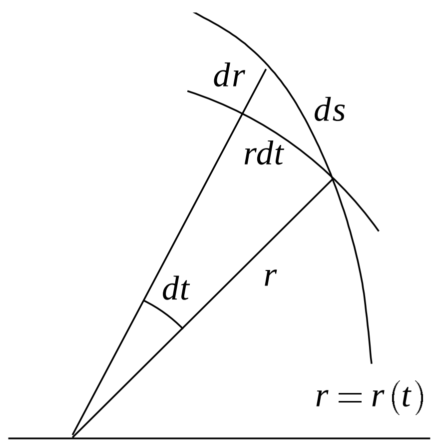


Figure 11.29

The length of the arc of the circle of radius r subtended by the angle $d\theta$ is $rd\theta$. The differential triangle is thus a right triangle with side lengths dr and $rd\theta$. By the pythagorean theorem

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{11.20}$$

Example 11.22. Find the length of the curve $r = \theta^2$ from 0 to 2π .

This curve is a spiral whose distance from the origin increases as the square of the angle. We have $dr = 2\theta d\theta$, so

$$ds^2 = dr^2 + r^2 d\theta^2 = 4\theta^2 d\theta^2 + \theta^4 d\theta^2 = \theta^2(4+\theta^2)d\theta^2$$

and thus the length is

$$\int_0^{2\pi} ds = \int_0^{2\pi} \theta\sqrt{4+\theta^2}d\theta = \frac{1}{3}(4+\theta^2)^{3/2})\Big|_0^{2\pi} = \frac{1}{3}(4+4\pi^2)^{3/2} - 4^{3/2})$$

Area

To caculate the area enclosed by a curve given, in polar coordinates, by $r = r(\theta)$, we calculate the differential of area, using figure 11.30

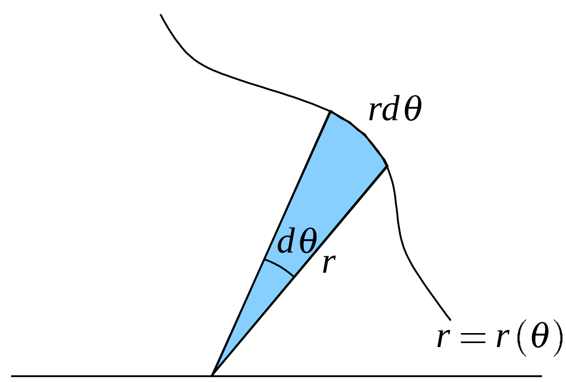


Figure 11.30

The area of the wedge given by the increment $d\theta$ is $(1/2)r^2d\theta$. To see this, we start with the area of the circle of radius r : $A = \pi r^2$. Now an angle α subtends a segment of the circle which is the $(\alpha/2\pi)$th part of the full circle, thus the area of that segment is $(1/2)r^2\alpha$. Thus, for $\alpha = d\theta$, we get

$$dA = \frac{1}{2}r^2 d\theta \tag{11.21}$$

Example 11.23. Find the area enclosed by the cardioid $r = 3(1 + \sin\theta)$.

The area is

$$Area = \frac{1}{2}\int_0^{2\pi}[3(1+\sin\theta)^2]d\theta = \frac{9}{2}\int_0^{2\pi}(1+2\sin\theta+\sin^2\theta)d\theta\ .$$

Now, we know that the integral of $\sin\theta$ over an entire period is zero, so we can neglect the middle term. We now use the double angle formula for the last term, and drop the integral of $\cos(2\theta)$ for the same reason:

$$Area = \frac{9}{2}\int_0^{2\pi}(1+\frac{1-\cos(2\theta)}{2})d\theta = \frac{9}{2}\int_0^{2\pi}\frac{3}{2}d\theta = \frac{27}{2}\pi\ .$$

Example 11.24. Find the area inside one petal of the rose $r = \sin 3\theta$.

At $\theta = 0$ we have $r = 0$, but then as the angle rotates, r increases to its maximum at $3\theta = \pi/2$, and then decreases back to zero for $3\theta = \pi$. Thus one petal is spanned as θ ranges from 0 to $\pi/3$. We now calculate;

$$Area = \frac{1}{2}\int_0^{\pi/3}\sin^2(3\theta)d\theta = \frac{1}{2}\int_0^{\pi/3}(\frac{1-\cos(6\theta)}{2})d\theta = \frac{1}{2}(\frac{\theta}{2}-\frac{\cos(6\theta)}{12}|_0^{\pi/3} = \frac{\pi}{12}\ .$$

Tangents

Given the polar equation $r = r(\theta)$ of a curve, we can find the tangent at any point as follows. First of all, the cartesian coordinates are given by $x = r(\theta)\cos\theta,\ y = r(\theta)\sin\theta$. If m is the slope of the tangent line, we have, by the chain rule

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}} \tag{11.22}$$

Notice that, as $r \to 0$, the right hand side approaches $\tan\theta$. Thus, if θ_0 is a value for which $r = 0$, then the curve approaches the origin along the ray $\theta = \theta_0$.

Example 11.25. What is the slope of the tangent to the inner loop of the limaçon

$$r = 2 + 5\cos\theta$$

at the origin?

First, we find the values of θ for which $r = 0$:

$$2 + 5\cos\theta = 0 \quad \text{or} \quad \cos\theta = -\frac{2}{5}$$

so that $\theta = \pm 0.63\pi$ radians or $113.6°$.

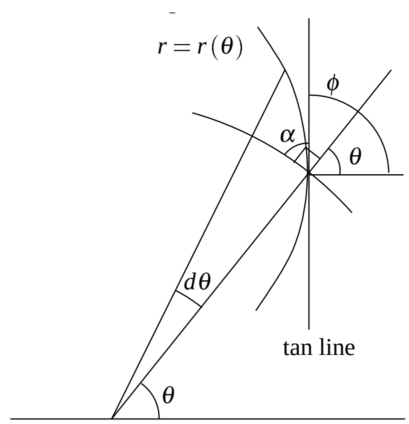


Figure 11.31

Problems 11.5

1. Find the length of the spiral $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

2. Find the length of the spiral $r = e^{-\theta}$ for $\theta \geq 0$.

3. Find the area inside the limaçon $r = 3 + \sin\theta$.

4. Find the area inside the cardioid $r = 1 - \sin\theta$ and above the x-axis.

5. What is the slope of the spiral $r = \theta$ at the points $\theta = 2\pi n$ for n a positive integer? What about the spiral $r = e^{\theta}$ at the same points?

6. Find the tangents to the curve $r = 2 + 3\sin\theta$ at the origin.

XII. Second Order Linear Differential Equations

12.1 Homogeneous Equations

A **differential equation** is an equation involving variables $x, y, y', y'', \ldots$. A **solution** is a function $f(x)$ defined in an interval I such that the substitution $y = f(x),\ y' = f'(x),\ y'' = f''(x),\ \ldots$ becomes an identity for x in the interval I. The differential equation is said to be **linear** if it is linear in the variables $y, y', y'', \ldots$. We first encountered solutions of differential equations in section 4 of Chapter 3, in the context of equations of the form $f(y)dy = g(x)dx$). In Chapter 6, section 3 we discussed the solution of first order linear equations (those in which only the variables x, y, y' appear). in this chapter we turn to second order linear equations with constant coefficients (those allowing also a second derivative). The general form of such an equation is

$$y'' + ay' + by = g(x) \ , \tag{12.1}$$

where a and b are constants, and $g(x)$ is a differentiable function of x. In section 6.4, we saw that a first order equation has a one-parameter family of solutions, and that the specification of an initial condition $y(x_0) = y_0$ uniquely determines a solution. In the case of second order equations, the basic theorem is this.

Theorem 12.1. Given x_0 in the domain of the differentiable function g, and numbers $y_0,\ y_0'$, there is a unique function $f(x)$ which solves the differential equation (12.1) and satisfies the initial conditions $f(x_0) = y_0,\ f'(x_0) = y_0'$.

In this section we shall see how to completely solve equation (12.1) when the function on the right hand side is zero:

$$y'' + ay' + by = 0 \ . \tag{12.2}$$

This is called the **homogeneous equation**. An important first step is to notice that if $f(x)$ and $g(x)$ are two solutions, then so is the sum; in fact, so is any linear combination $Af(x) + Bg(x)$. Thus, once we know two solutions (they must be *independent* in the sense that one isn't a constant multiple of the other) we can solve the initial value problem in theorem 12.1 by solving for A and B.

Example 12.1. Solve. $\quad y'' + y = 0 \ , \quad y(0) = 4 \ , \quad y'(0) = -1 \ .$

Now, we know that $\cos x$ and $\sin x$ are solutions of the equation, so we try a solution of the form $y(x) = A\cos x + B\sin x$. Evaluating at $x = 0$, we find that $A = 4$. Differentiate, getting $y'(x) = -A\sin x + B\cos x$, and evaluating at $x = 0$, we find $B = -1$. Thus the solution is $y(x) = 4\cos x - \sin x$.

The reason the answer worked out so easily is that $y_1 = \cos x$ is the solution with the particular initial values $y_1(0) = 1,\ y_1'(0) = 0$ and $y_1 = \sin x$ is the solution with $y_1(0) = 0.\ y_1'(0) = 1$. Then the solution with initial values $y(0)$ and $y'(0)$ is

$$y(x) = y(0)\cos x + y'(0)\sin x \tag{12.3}$$

Example 12.2. Solve $\quad y'' - y = 0$, with given initial values $\ y(0)\ ,\ y'(0)$.

Now e^x and e^{-x} are solutions of this differential equation, so the general solution is a linear combination of these. But we won't have as easy a time finding a solution like (11.3), since these functions do not have the initial values $1, 0; 0, 1$ respectively. However if we introduce the functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

these do have the right initial values:

$$\cosh 0 = 1 \ , \quad \sinh 0 = 0$$

$$\frac{d}{dx}(\cosh x) = \sinh x \ , \quad \frac{d}{dx}(\sinh x) = \cosh x \tag{12.4}$$

so $(\cosh)'(0) = 0, \ (\sinh)'(0) = 1$. Thus, the solution to our problem is

$$y(x) = y(0)\cosh x + y'(0)\sinh x \ .$$

This particular differential equation comes up so often that it is important to remember these functions, $\cosh x, \ \sinh x$, called the **hyperbolic functions** and their basic properties: equation (12.4) and

$$\cosh^2 x - \sinh^2 x = 1 \ . \tag{12.5}$$

Because of (12.5) these functions parametrize the standard hyperbola (and it is for this reason that they are called hyperbolic functions).

Using these examples as a guide, we return to the general second order equation.

Proposition 12.2. Let r be a root of the equation

$$r^2 + ar + b = 0 \ . \tag{12.6}$$

Then e^{rx} is a solution to the homogeneous equation:

$$y'' + ay' + by = 0 \ . \tag{12.7}$$

Equation (12.6) is called the **auxiliary equation** of the differential equation (12.7). To verify the proposition, let $y = e^{rx}$ so that $y' = re^{rx}, \ y'' = r^2 e^{rx}$. Substituting into equation (12.7):

$$r^2 e^{rx} + are^{rx} + be^{rx} = e^{rx}(r^2 + ar + b) \ ,$$

and this is zero if and only if r is a root of the auxiliary equation.

Now unfortunately a quadratic equation does not necessarily always have two real roots, so we have to examine the cases separately.

Case of two real roots. If the discriminant $a^2 - 4b > 0$, then there are two real roots, and it is straightforward to find the solution of the corresponding initial value problem.

Example 12.3. Solve : $\quad y'' + 6y' + 5y = 0 \ , \quad y(0) = 4 \ , \quad y'(0) = -1 \ .$

The auxiliary equation, $r^2 + 6r + 5 = 0$ has the roots $r = -1, -5$, so e^{-x} and e^{-5x} are solutions. The general solution is

$$y = Ae^{-x} + Be^{-5x} \qquad \text{with derivative} \qquad y' = -Ae^{-x} - 5Be^{-5x} \ .$$

Evaluating at $x = 0$, we have $4 = A + B,\ -1 = -A - 5B$. Solving this pair of equations, we get $A = 19/4$ and $B = -3/4$, so our solution is

$$y = \frac{19}{4}e^{-x} - \frac{3}{4}e^{-5x}$$

Example 12.4. A function $x = x(t)$ satisfies the differential equation

$$x'' - 2x' - 15x = 0 \ .$$

Under what conditions on the values of x at $t = 0$ will this function decay to 0 as $t \to \infty$?

The auxiliary equation $r^2 - 2r - 15$ has the roots $r = -3,\ 5$. Thus the general solution is $x(t) = Ae^{-3t} + Be^{5t}$. This will decay at infinity only if $B = 0$. Now, evaluating x and x' at 0 gives us the equations

$$x(0) = A + B \ , \qquad x'(0) = -3A - 5B \ .$$

Setting $B = 0$, the condition becomes $x'(0) + 3x(0) = 0$.

Case of complex roots. If the discriminant $a^2 - 4b < 0$, then the roots are two complex conjugate numbers $\alpha + i\beta,\ \alpha - i\beta$.

Let's look again at the case $y'' + y = 0$. Then the roots of $r^2 + 1 = 0$ are $\pm i$, and we'd like to say that the solutions are the functions $e^{ix},\ e^{-ix}$. This does work, and all the algebra in the case of real roots works just as well in this case, once we have given these expressions meaning. First of all, remember equation (12.3): the general solution of $y'' + y = 0$ is

(12.8) $$y(x) = y(0)\cos x + y'(0)\sin x$$

If $y(x) = e^{ix}$ is to represent a solution of this differential equation, we have $y(0) = e^0 = 1$, and $y'(0) = ie^0 = i$, so we must have

(12.9) $$e^{ix} = \cos x + i\sin x$$

Notice that if we differentiate this expression, we get

$$-\sin x + i\cos x = i(\cos x + i\sin x) \ ,$$

so this expression is consistent with the differentiation rule for the exponential:

$$\frac{d}{dx}e^{ix} = ie^{ix} \ .$$

In fact, defining the complex exponential by (12.9) is consistent with all the rules of exponentials. In particular (recall problem 10 of section 9.5), if we substitute the Maclaurin series for all the functions in (12.9) we get an identity:

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \ .$$

Proposition 12.3. For a complex number $\alpha + i\beta$ if we define the exponential function as

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))\ ,$$

then all the usual laws of exponents carry through.

Now, of course, we are interested only in real-valued functions. What we have shown is that if $\alpha \pm i\beta$ are the roots of the equation $r^2 + ar + b = 0$, then the functions $e^{(\alpha\pm i\beta)x}$ solve the differential equation $y'' + ay' + b = 0$. But then the real and imaginary parts of this function satisfy the equation as well, which gives us the desired two real-valued solutions.

Proposition 12.4 . If the auxiliary equation for the differential equation

$$y'' + ay' + b = 0$$

has the complex roots $\alpha \pm i\beta$, then every solution of the differential equation is of the form

$$Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \tag{12.10}$$

In solving initial value problems, we can work with the complex solutions or solutions of the form (12.10); usually the latter is more convenient.

Example 12.5. Find the general solution $x = x(t)$ of

$$x'' + a^2x = 0\ .$$

Since the roots of the auxiliary equation $r^2 = -a^2 = 0$ are $\pm ia$, the general solution is

$$x(t) = A\cos at + B\sin at \tag{12.11}$$

It is easy to see what this function looks like by defining

$$C = \sqrt{A^2 + B^2},\ \ \gamma = \arctan(B/A)$$

Then (12.11) becomes

$$x(t) = C(\cos\gamma\cos at + \sin\gamma\sin at = C\cos(at - \gamma) = C\cos a(t - \gamma/a))\ .$$

Thus the graph of $x = x(t)$ is a simple cosine curve of *amplitude* C, and *period* $2\pi/a$, shifted to the right by the *phase* γ. (See figure 12.1).

Example 12.6. Find the solution $y = y(x)$ of $y'' + 2y' + 5y = 0$, with initial values $y(0) = 2,\ y'(0) = -1$.

The auxiliary equation $r^2 + 2r + 5 = 0$ has the solutions $r = -1 \pm 2i$. Thus the general solution is $y = e^{-x}(A\cos(2x) + B\sin(2x))$. To solve for A and B using the initial values we must first differentiate y:

$$y' = -e^{-x}(A\cos(2x) + B\sin(2x)) + e^{-x}(-2A\sin(2x) + 2B\cos(2x))\ .$$

Substituting the initial values gives the equations $A = 2,\ -A + 2B = -1$, which has the solutions $A = 2, B = 1/2$. The answer thus is

$$y = e^{-x}(2\cos(2x) + \frac{1}{2}\sin(2x)) \ .$$

Case of a double root. If the discriminant $a^2 - 4b = 0$, then the auxiliary equation has one root r, which gives us only one solution e^{rx} of the differential equation. We find another solution by the technique of variation of parameters. We try $y = ue^{rx}$, where u is a new unknown function. Now, since the auxiliary equation has only the single root r, we have $a = -2r$ and $b = r^2$, so the differential equation can be written as:

$$y'' - 2ry' + r^2y = 0 \ .$$

Substituting $y = ue^{rx}$ in the left hand side, we get

$$y'' - 2ry' + r^2y = e^{rx}[(u'' + 2u'r + ur^2) - 2r(u' + ur) + r^2u] = e^{rx}u'' \ .$$

For this to be zero, we must have $u" = 0$, so that $u = Ax + B$.

Proposition 12.5 . If the auxiliary equation for the differential equation

$$y'' + ay' + b = 0$$

has only the root r, then every solution is of the form

$$(Ax + B)e^{rx}$$

Example 12.7. Find the solution of $y'' - 4y' + 4y = 0$ with initial values $y(0) = 2,\ y'(0) = -1$.

The auxiliary equation has just the root $r = 2$. The general solution is $y = (Ax + B)e^{2x}$, with derivative $y' = 2(Ax + B)e^{2x} + Ae2x$. Substituting the initial conditions gives the equations

$$2 = B \qquad -1 = 2B + A \ .$$

Thus $A = -5,\ B = 2$ and the answer is

$$y = (-5x + 2)e^{2x} \ .$$

Problems 12.1

1. Solve $y'' - 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

2. Solve $y'' + 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

3. Solve $y'' - 5y' + 6y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

4. Solve $y'' + 4y' + 5y = 0$ with the initial values $y(0) = 1,\ y'(0) = -1$.

5. Solve $y'' - y' = 0$ with the initial values $y(2) = 1,\ y'(2) = 2$.

6. Solve $y'' + 2y' + y = 0$ with the initial values $y(-1) = 1,\ y(1) = 1$.

12.2 Behavior of the Solutions

When these differential equations come up in applications, one usually wants to have some idea of the long term behavior of the solution. Since the roots of the auxiliary equation determine the solutions, they also determine their behavior. Here we summarize the results.

Both roots positive. Except for the identically zero solution, all solutions grow exponentially. See figure 12.2.

Both roots negative. Except for the identically zero solution, all solutions decay exponentially. See figure 12.3.

Figure 12.2

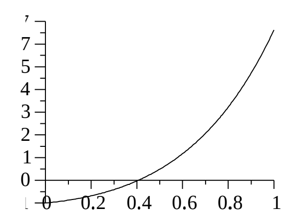
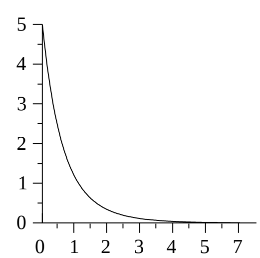


Figure 12.3



A negative and a positive root. All solutions grow exponentially, except for the multiples of the exponential with the negative root.

Both roots imaginary. In this case the equation is

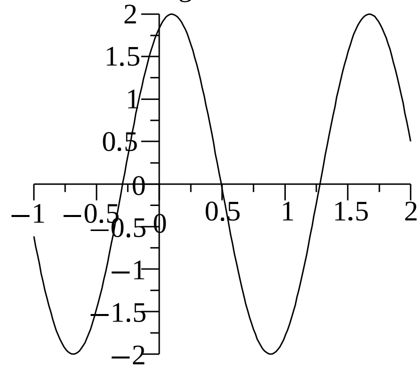
$$y'' + \omega^2 y = 0 \ ,$$

As we saw in example 12.5, the general solution can be written as

$$y(x) = A\cos\omega x) + B\sin(\omega x) \quad \text{or} \quad y(x) = C\cos(\omega x - \gamma) \ ,$$

an oscillation of period $2\pi/\omega$, amplitude C and phase γ (see figure 12.1).

Figure 12.1



Complex roots. In this case the roots are of the form $\alpha \pm i\beta$ and the general solution can be written in the form (following the previous discussion)

$$y(x) = Ce^{\alpha x}\cos(\omega x - \gamma)\ .$$

Thus if $\alpha > 0$, this gives an oscillation with exponentially increasing amplitude (figure 12.4), and if $\alpha < 0$, this gives an exponentially damped oscillation (figure 12.5).

Figure 12.4

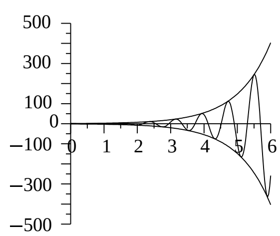
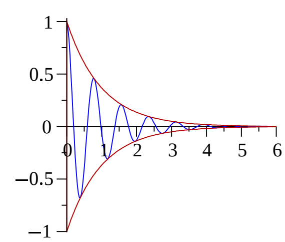


Figure 12.5



Problems 12.2

For each of the following differential equations, find the general solution in terms of the initial values $y(0),\ y'(0)$. Give conditions, if any, on the initial conditions for the solution to be bounded.

1. $$y'' - 6y' + 9y = 0\ .$$

2. $$'' - 6y' + 5y = 0\ .$$

3. $$y'' - y' + 6y = 0\ .$$

4. $$y'' + 4y' + 13y = 0\ .$$

5. $$y'' - 4y' + 13y = 0\ .$$

6. $$y'' + 9y = 0\ .$$

7. $$y'' - 9y = 0\ .$$

8. Find the solution of problem 3 with initial values $y(0) = 3,\ y'(0) = -2$.

9. Find the solution of problem 4 with initial values $y(0) = 0,\ y'(0) = 1$.

12.3 Applications

Springs

Suppose we place a mass m on the end of a vertically hanging spring, and then set it in motion; how can we describe the subsequent motion? As we have seen in chapter 5, section 4, the spring is subject to a restoring force proportional to its displacement from equilibrium. By Newton's second law of motion, this force is ma, where a is the acceleration of the mass m. Letting x represent the downward displacement from equilibrium, we have $a = x''$, and if the spring constant is k, this gives us the equation

$$mx'' = -kx\ , \quad \text{or} \quad x'' + \frac{k}{m}x = 0 \tag{12.12}$$

Letting $\omega = \sqrt{k/m}$, this has the solution (see example 12.5)

$$x(t) = C\cos(\omega t - \gamma)$$

where C and γ are to be determined by the initial data.

We have to be a little careful about units. In the metric system, when m is measured in kilograms and x in meters, then force is measured in newtons, and the units for the spring constant k are newtons/meter. On a smaller scale, m is in grams, x in centimeters, force in dynes, and k in dynes/meter. However, in the British system it is customary to refer to the *weight* w of the object (in pounds), rather than its mass. Then $m = w/g$, where $g = 32$ ft/sec^2 is the acceleration due to gravity. Finally, in the British system, the spring constant is given in lbs/foot.

Example 12.8. Suppose a mass of 10 g hangs from a spring with spring constant $k = .4$ dynes/cm. If the spring is extended an additional 8 cm. and then released, give the equation of subsequent motion.

The initial conditions are that when $t = 0$, $x(0) = 8$, $x'(0) = 0$. We also have $\omega = \sqrt{.4/10} = \sqrt{.04} = .2$. Thus the solution has the form

$$x(t) = C\cos(.2t + \gamma)$$

We get, from the initial conditions

$$8 = C\cos(-\gamma)\ , \quad 0 = -.2C\sin(-\gamma)$$

so $\gamma = 0$ and $C = 8$, and the equation of motion is

$$x(t) = 8\cos(.2t)$$

We could have concluded this more quickly, by observing that the initial conditions tell us that $x = 8$ when the velocity is 0, so the maximum extension (the amplitude) has to be 8.

Example 12.9. Suppose that we come upon the above configuration already in motion, and when we make our observation (at time $t = 0$), the mass is displaced 12 cm downward and is traveling downward at a velocity of 1 cm/sec. Find the equation of motion.

Again, the equation has the general form

$$x(t) = C\cos(.2t - \gamma)$$

and the initial conditions give

$$12 = C\cos(-\gamma)\ , \quad 1 = -.2C\sin(-\gamma)\ .$$

We solve for C and γ as follows. The equations are

$$C\cos(-\gamma) = 12\ . \quad C\sin(-\gamma) = -5\ .$$

Adding the squares of both equations gives us $C^2 = 12^2 + 5^2 = 169$, so $C = 13$, and dividing one equation by the other gives

$$\tan(-\gamma) = -5/12`, \quad \text{so that} \quad \gamma = .126\pi$$

and the equation of motion is

$$x(t) = 13\cos(.2t - .126\pi)\ .$$

Example 12.10. If a 16 lb. object is hung from a spring with spring constant k=9 lbs/foot and then is given an initial velocity of 24 ft/sec, what is the maximum extent of the spring?

Here $m = 16/32$ and $k = 9$, so we have the spring equation

$$\frac{1}{2}x'' + 9x = 0$$

so $x = A\cos(3\sqrt{2}t) + B\sin(3\sqrt{2}t)$. The initial conditions $x(0) = 0$, $x'(0) = 24$ lead to $A = 0$, $B = 8\sqrt{2}$. The solution thus is

$$x(t) = 8\sqrt{2}\sin(2\sqrt{2}t)$$

whose maximum value is $8\sqrt{2}$ feet.

Now, let us return to our spring with spring constant k and mass m, and suppose that it is inserted in a viscous fluid which imparts a retarding force proportional to the velocity of the mass. Letting $q > 0$ be the constant in this proportion, we see that equation (12.12) is replaced by

$$mx'' = -kx - qx' \ , \quad \text{or} \quad x'' + \nu x' + \omega^2 x = 0 \tag{12.13}$$

where $\omega = \sqrt{k/m}$ and $\nu = q/m$. The roots of this equation are

$$r = \frac{-\nu \pm \sqrt{\nu^2 - 4\omega^2}}{2} \ .$$

If $\nu > 2\omega$, then the roots are both real and negative, and there is exponential decay with no oscillation. if $\nu < 2\omega$, then we have complex roots, so there are oscillations. But the real part of the roots is $-\nu/2 < 0$, so the oscillations are exponentially damped. Thus, if we want to have a good damping effect (as for example in a shock absorber) we should be sure that ν is sufficiently large; that is, that the fluid is very viscous.

Example 12.11. A system consisting of a spring in a viscous fluid is installed so as to absorb the shock on a 100 kg mass. The spring constant is k=2500 and the constant of viscosity is q=600. Suppose a shock is sustained when the system is in equilibrium imparting an instantaneous velocity of 100 cm/sec. How long will it take for the amplitude of the oscillation to be reduced to 1 cm?

The basic differential equation is $100x'' + 600x' + 2500x = 0$. The roots of the auxiliary equation are $r = -3 \pm 4i$, so the general solution is

$$x(t) = e^{-3t}(A\cos 4t + B\sin 4t)$$

Now at $t = 0$, $x = 0$, $x' = 100$. Solving for A and B with those initial conditions, we find $x(t) = 25e^{-3t}\sin 4t$. Now the maximum amplitudes of this damped vibration occur at the values $t = k\pi/8$, for k an odd integer. Here is a table of the first few values:

t	.3927	1.1781	1.9634
A	7.697	.729	.069

Thus the first maximum amplitude occurs at .3927 seconds and is 7.697 cm, but by 1.1781 the maximum amplitude is less than 1 cm.

Problems 12.3.

1. A man drops out of a plane at 25,000 feet of altitude and immediately opens his parachute. For this man and parachute the deceleration due to air resistance is proportional to $4v$ where v is his velocity. How far has he fallen in one econd? How long does it take for him to hit the ground, and at what velocity does he hit the ground?

2. a) Let a mass m hang from a spring of spring constant k. Suppose that it is set in motion. Show that, throughout the motion, $mv^2 + kx^2$ is constant, where x represents displacement from equilibrium, and v is velocity.

b) Suppose that $k = 4$ dynes/cm and $m = 10$ g, and the spring is already in motion At a particular instant the srping is located 10 cm. from equilibrium and traveling at velocity 8 cm/sec. For this motion, what are the maximum velocity and maximum displacement of the mass?

3. The above configuration is put in a viscous fluid which exerts a retardant force proportional to the velocity, with constant of proportionality $q = 12$. Find the equation of motion of the mass, given that at time $t = 0$ it is at $x = 0$ and its velocity is 4.8 cm/sec. What is the maximum displacement of the mass?

4. A 10 g mass is hung on a spring with spring constant 0.8 dynes/cm. The mass is extended 40 cm beyond equilibrium and then released. What is the maximum speed the mass attains?

5. The system described in problem 4 is now inserted in a fluit with constant of viscosity $q = 4$. The mass is extended 10 cm and released. Find the equation of motion.

12.4 The Inhomogeneous Equation

We return now to the general second order equation with a nonzero right hand side;

$$y'' + ay' + by = g(x) \ , \tag{12.14}$$

Proposition 12.6 . Suppose that $y = y_p(x)$ is a particular solution of the equation (12.14). Then every solution is of the form $y = y_p + y_h$ where y_h is a solution of the homogeneous equation.

Example 12.12. Find the solution of the initial value problem:

$$y'' + y = x + 2 \ , \quad y(0) = 4 \ , \quad y'(0) = 2 \ .$$

It is easy to see that $y_p(x) = x+2$ is a particular solution of this equation. Since the homogeneous equation is $y'' + y = 0$, the general solution is of the form

$$y(x) = x + 2 + A\cos x + B\sin x$$

To find A and B we use the initial conditions (at $x = 0$):

$$4 = 0 + 2 + A\cos(0) \ , \qquad 2 = 1 - A\sin(0) + B\cos(0)$$

giving us $4 = 2 + A,\ 2 = 1 + B$, so $A = 2,\ B = 1$, and the solution is $y = x + 2 + 2\cos x + \sin x$.

Example 12.13. Knowing that $y = -(1/3)\cos(2x)$ solves the differential equation

$$y'' + y = \cos(2x)$$

find the solution with initial values $y(\pi/2) = 1,\ y'(\pi/2) = 3$.

We know the solution has the form $y = -(1/3)\cos(2x) + A\cos x + B\sin x$. Putting in the initial values gives us $1 = 1/3 + B,\ 3 = -A$, so the solution is $y = -(1/3)\cos(2x) - 3\cos x + (2/3)\sin x$.

In general, we may neither be given a particular solution, nor can we see one by inspection. It is usually very difficult to find that first particular solution. However, here is one technique (basically trial and error) which leads to a particular solution when the inhomogeneous function is an elementary function.

Undetermined coefficients. To solve $y'' + ay' + by = g(x)$, try a function of the same form as $g(x)$. More precisely:

If g is a polynomial of degree n, try the general polynomial of degree n.

If g is an exponential times a polynomial of degree n, try the general exponential times a polynomial of degree n.

If g is a cosine or a sine times a polynomial of degree n, try the general cosine and sine times a polynomial of degree n.

The reason this works is that successive differentiation keeps us in the same form, so we end up equating coefficients and solving for them. There is one caution: if g is a solution of the homogeneous equation, this will fail In this case we replace the phrase "polynomial of degree n" by "polynomial of degree n or more". Let us illustrate

Example 12.14. Find a particular solution of $y'' + 5y' - y = x^2 - 3x + 4$.

We try $y = ax^2 + bx + c$. First we calculate the first and second derivative: $y' = 2ax + b,\ y'' = 2a$. Substituting these in the given equation we obtain

$$2a + 5(2ax + b) - (ax^2 + bx + c) = x^2 - 3x + 4$$

This simplifies to

$$-ax^2 + (10a - b)x + 2a + 5b - c = x^2 - 3x + 4$$

We now equate coefficients:

$$-a = 1\ , \qquad 10a - b = -3\ , \qquad 2a + 5b - c = 4$$

giving the solutions $a = -1,\ b = -7,\ c = -41$, so the answer is

$$y = -x^2 - 7x - 41\ .$$

Example 12.15. Find a particular solution of $y'' + y' - y = xe^x$.

Try $y = (ax + b)e^x$. Differentiating, $y' = (ax + a + b)e^x,\ y'' = (ax + 2a + b)e^x$. Substituting in the given equation leads to

$$(ax + 2a + b)e^x + (ax + a + b)e^x - (ax + b)e^x = xe^x\ ,$$

reducing to the equations $a = 1,\ 3a + b = 0$. Thus the answer is

$$y = (x - 3)e^x$$

Example 12.16. Find a particular solution of $y'' + y = \cos x$.

Since $\cos x$ satisfies the homogeneous equation, we must try $y = ax\sin x + bx\cos x$. Then $y'' = 2a\cos x - 2b\sin x - ax\sin x - bx\cos x$, and we obtain the equation

$$2a\cos x - 2b\sin x = \cos x$$

so $a = 1/2$, and the solution is $y = (1/2)x\sin x$.

Problems 12.4

1. Solve $y'' + 2y' + y = x$ with the initial values $y(0) = 0,\ y'(0) = 0$.

2. Find the general solution of $y'' + 2y' + y = \sin x$.

3. Find the general solution of $y'' - 4y = \sin(2x)$.

4. Find the general solution of $y'' + 4y = \sin(2x)$.

5. A crystal glass consists of cells in a crystalline shape which oscillate at a natural frequency, so the motion is governed by a differential equation $x'' + \omega_0^2 x = 0$ where $2\pi/\omega_0$ is the frequency. If the ambient air is vibrating at a frequency of $\omega/2\pi$ (due to a monotonal sound, perhaps), then the motion of a crystal is modified by the force of the air in motion so as to be governed by the inhomogeneous equation

$$x'' + \omega_0^2 x = A\cos\omega t$$

Find a particular solution of this differential equation. What happens as ω approaches ω_0? (This phenomenon is called *resonance*.)