EXTRA PROBLEMS #6

DUE: FRI NOVEMBER 2ND

In this assignment, we give a proof for the chain rule. We gave an incomplete proof in class. In particular, we assumed that the inner-function, didn't actually achieve the value of its limit in a little interval around that point.

Consider the following function.

$$f(t) = \begin{cases} 0, & t = 0 \\ t^2 \sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

We've talked about why this function is continuous in class before. Even more can be said however, this function is actually differentiable everywhere.

FACTS: We've done the following things in class (or done things close enough to them). You may use them without proving them.

(i) The functions

$$s(t) = \begin{cases} 0, & t = 0 \\ \sin(\frac{1}{t}), & t \neq 0 \end{cases} \quad \text{and } c(t) = \begin{cases} 0, & t = 0 \\ \cos(\frac{1}{t}), & t \neq 0 \end{cases}$$

are *NOT* continuous at $t = 0$. Even more, the functions $s(t)$ and $c(t)$ even have undefined limits at $t = 0$.

(ii) The functions $ts(t)$ and $tc(t)$ *ARE* continuous at $t = 0$. Here $ts(t)$ is just the product of the functions $i(t) = t$ with the function $s(t)$. We can also view the function $ts(t)$ as the multipart function

$$ts(t) = \begin{cases} 0, & t = 0 \\ t\sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

Exercise 0.1. Prove that

$$f'(t) = \begin{cases} 0, & t = 0 \\ 2t\sin(\frac{1}{t}) - \cos(\frac{1}{t}), & t \neq 0 \end{cases}$$

Hint: The case where $t \neq 0$ should be easy, simply apply the chain and product rules. For the case of $t = 0$, you'll have to use the limit-definition of the derivative.

Exercise 0.2. Prove that the related function

$$k(t) = \begin{cases} 0, & t = 0 \\ 2t\sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

is continuous but not differentiable.

Hint: You'll need to use the limit definition of the derivative to prove the function is not differentiable.

Exercise 0.3. Prove that $f'(t)$ is not continuous.

Hint: Show that $k(t)$ can be added to a non-continuous function, to get $f'(t)$.

Now we get into the real work.

Theorem 0.4. *Let $f\colon (a,b) \to (c,d)$ and $g\colon (c,d) \to \mathbb{R}$ be differentiable functions. Then*

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

for every $x \in (a,b)$.

We only proved this in class in the case that f did not do exactly what the f function above did (equal its limit many times around the limiting value). We begin with a warm-up exercise that gives us a new perspective to the derivative.

Exercise 0.5. Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ and set $L = f'(x_0)$. Consider a new function $E\colon (-\delta,\delta) \setminus \{0\} \to \mathbb{R}$ defined by the formula

$$E(h) = \frac{f(x_0+h) - f(x_0)}{h} - L.$$

Here $\delta > 0$ is assumed to be chosen in such a way that E can be defined.

Show that

a. for every $h \in (-\delta,\delta) \setminus \{0\}$ we have
$$f(x_0+h) = f(x_0) + Lh + hE(h).$$

b. $\lim_{h\to 0} E(h) = 0$.

The function E defined in the exercise can be called as an *error term*. This error term actually characterizes the derivative as the following exercise shows.

Exercise 0.6. Let $f\colon (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. Suppose that there exists a function $E\colon (-\delta,\delta) \setminus \{0\} \to \mathbb{R}$ so that $\lim_{h\to 0} E(h) = 0$ and

$$f(x_0+h) = f(x_0) + Lh + hE(h)$$

where L is a number. Show that f is differentiable at x_0 and $f'(x_0) = L$.

These two exercises can be combined as a *lemma* which we will use later.

Lemma 0.7. *A function $f\colon (a,b) \to \mathbb{R}$ has derivative L at x_0 if and only if there exist $\delta > 0$ and $E_f\colon (-\delta,\delta) \setminus \{0\} \to \mathbb{R}$ so that*

(1) $\lim_{h\to 0} E_f(h) = 0$, *and*
(2) $f(x_0+h) = f(x_0) + Lh + hE_f(h)$

for every $0 < |h| < \delta$.

Exercise 0.8. Let $f\colon (a,b) \to (c,d)$ and $g\colon (c,d) \to \mathbb{R}$ be functions and $x_0 \in (a,b)$. Suppose that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Let $E_f\colon (-\delta_0,\delta_0) \to \mathbb{R}$ be an error term for f (here we define $E_f(0) = 0$) and let $E_g\colon (-\delta_1,\delta_1) \to \mathbb{R}$ be an error term for g (we also define $E_g(0) = 0$). Let also $\ell_f\colon (-\delta_0,\delta_0) \to \mathbb{R}$ be the function $\ell_f(h) = f'(x_0)h + hE_f(h)$.

a. Show that there exists $\delta > 0$ so that for every $h \in (-\delta,\delta)$ we have
$$|\ell_f(h)| < \delta_1.$$
(*Hint:* Use the limit $\lim_{h\to 0} \ell_f(h)$)

b. Show that with this δ we have
$$\begin{aligned} g(f(x_0+h)) &= g\left(f(x_0) + \ell_f(h)\right) \\ &= g(f(x_0)) + g'(f(x_0))\ell_f(h) + \ell_f(h)E_g\left(\ell_f(h)\right) \end{aligned}$$
for every $0 < |h| < \delta$.

c. Show that there exists a function $E\colon (-\delta, \delta) \setminus \{0\} \to \mathbb{R}$ so that

$$g'(f(x_0))\ell_f(h) + \ell_f(h)E_g\left(\ell_f(h)\right) = g'(f(x_0))f'(x_0)h + hE(h)$$

for every $0 < |h| < \delta$ and $\lim_{h\to 0} E(h) = 0$.

d. Prove Theorem 0.4.

The formula

$$f(x_0 + h) = f(x_0) + f'(x_0)h + hE(h) \tag{1}$$

can also be used to approximate the values of function f.

Exercise 0.9. Argue how (1) could be used to give a decimal approximation for $\sqrt{25.012}$ if you are able to assume that the error term in (1) is very small. Calculate an approximation using (1) and compare it to a result given by a calculator.