EXTRA PROBLEMS #7

DUE: FRI NOVEMBER 9TH

We'll be doing some exercises related to vector functions. See sections 3-10 through 3-12 in the text for additional discussion.

A *vector function* is a function $\mathbf{F}$ that takes a real number in as input, and outputs a vector in the plane.

Exercise 0.1. Show that every vector function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ can be described by a rule

$$\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions of a single variable.

The following definition can be found on pages 187 and 188 of the text. We say that

$$\lim_{t\to c} \mathbf{F}(t) = \mathbf{v}$$

if for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $t \in (c-\delta, c+\delta)$, $t \neq c$, then $|\mathbf{F}(t)-\mathbf{v}| < \epsilon$.

This second condition can be rephrased as requiring that $\mathbf{F}(t)$ is inside the circle of radius ϵ centered at $\mathbf{v}$.

We can also define what it means for a vector function to be continuous. A vector function $\mathbf{F}(t)$ is said to be *continuous at* $t = c$ if

$$\lim_{t\to c} = \mathbf{F}(c).$$

If $\mathbf{F}$ is continuous at every point in its domain, we simply say that $\mathbf{F}$ is *continuous*.

Definition 0.2. We say that a subset $U \subseteq \mathbb{R}^2$ is *open* if for every point $P \in U$, there exists an $\epsilon > 0$ such that the interior of the disc of radius ϵ centered at P is completely contained in U.

In other words, if $P = (p_1, p_2)$, then all points $\mathbf{v} = (x_1, x_2)$ which satisfy the inequality

$$|\mathbf{v} - P| = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2} < \epsilon,$$

are contained inside U.

Exercise 0.3. Prove that $U \subset \mathbb{R}^2$ is open if and only if for every point $P \in U$, there exists a $e > 0$ such that the interior of the square with side length $2e$ and with central point P is completely contained in U.

See extra problems #4 for the definition of an open subset U of $\mathbb{R}$. Also see extra problems #4 for the definition of $f^{-1}(U)$ where f is a function and U is a subset of the codomain.

Exercise 0.4. Prove that a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is continuous if and only if for every open subset $U \subseteq \mathbb{R}^2$, the subset of $\mathbb{R}$, $\mathbf{F}^{-1}(U)$, is also open.

Hint: Try something similar to the proofs of 2.2 and 2.3 in extra problems #4.

Given a vector function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$, we say that $\mathbf{F}$ is *differentiable at* $t_0 \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{1}{h} \left(\mathbf{F}(x_0 + h) - \mathbf{F}(x_0) \right)$$

exists. If the limit exists, we define $\mathbf{F}'(t_0)$ to be the value of that limit, and we say $\mathbf{F}'(t_0)$ is the *derivative of* $\mathbf{F}(t)$ *at* t_0. See section 3-11 in the text.

Exercise 0.5. Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is differentiable at every point, prove that $\mathbf{F}$ is also continuous at every point.

Exercise 0.6. Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is a vector function that is differentiable at t_0. Suppose further that $g : \mathbb{R} \to \mathbb{R}$ is a function of a single variable that is also differentiable at t_0.

Consider a new function $\mathbf{H}(t) = g(t)\mathbf{F}(t)$ (here we are viewing $g(t)$ as a scalar). Prove that $\mathbf{H}'(t) = g'(t)\mathbf{F}(t) + g(t)\mathbf{F}'(t)$.

Now we define a new notion. Whether a subset of $\mathbb{R}^2$ (or even $\mathbb{R}$) is "connected".

Definition 0.7. We say that a subset $C \subseteq \mathbb{R}^2$ (or in $\mathbb{R}$) is *disconnected* if there are two open subsets, U and V, of $\mathbb{R}^2$ such that $U \cap V = \emptyset$, $C \subseteq U \cup V$ and $C \cap U \neq \emptyset$ and $C \cap V \neq \emptyset$.

In words, this means that there are two open subsets of $\mathbb{R}^2$ (or $\mathbb{R}$) such that C has points in common with each of them, every point of C is contained in one of them and the two open sets have no points in common.

Definition 0.8. We say that a subset $C \subseteq \mathbb{R}^2$ (or $\mathbb{R}$) is *connected* if it is not disconnected.

The following theorem is just a restatement of what we've already said so far.

Theorem 0.9. *A subset $C \subseteq \mathbb{R}^2$ (or in $\mathbb{R}$) is connected if whenever there are two open subsets U and V of $\mathbb{R}^2$ (or $\mathbb{R}$) such that $C \in U \cup V$ and $U \cap V = \emptyset$, then either*
(a) *$C \subseteq U$ or*
(b) *$C \subseteq V$.*

It is a fact that any open interval (a, b) in $\mathbb{R}$ is connected. It also true that $\mathbb{R}$ is a connected subset of itself. You don't need to prove either of these facts. We may assume them for what follows. (The way it is proven typically is by the least upper bound axiom). Intuitively, a subset of $\mathbb{R}^2$ (or $\mathbb{R}$) is connected, if you can get from one point to any other point without the need to "jump".

Exercise 0.10. Suppose that $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ is a continuous vector function. Prove that the range of $\mathbf{F}$ is a connected subset of $\mathbb{R}^2$.

Hint: Suppose not, then there exist two open subsets U and V of $\mathbb{R}^2$ satisfying various properties. Consider $\mathbf{F}^{-1}(U)$ and $\mathbf{F}^{-1}(V)$.

Exercise 0.11. Give a brief explanation about why the result of the previous exercise is closely related to the intermediate value theorem.