

MATH 185

DUE WEDNESDAY, NOVEMBER 11TH

We'll now consider functions which take vectors as input and output other vectors. We'll consider vectors in the plane (denoted $\mathbb{R}^2$) and in 3-space (denoted $\mathbb{R}^3$).

Exercise 0.1. Write down a precise definition of a function that inputs a vector in $\mathbb{R}^2$ and outputs a vector in $\mathbb{R}^2$ (based on the definition of a function from Spivak). Such a function will often be denoted by $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Also give a precise definition of a function that inputs a vector from $\mathbb{R}^2$ and outputs a vector in $\mathbb{R}^3$. Such a function will often be denoted by $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Definition 0.2. A *function* $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a collection of pairs of planar vectors $(\mathbf{u}, \mathbf{v})$ such that whenever $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{w})$ are both in the collection, then $\mathbf{v} = \mathbf{w}$.

Definition 0.3. A *function* $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a collection of pairs of vectors $(\mathbf{u}, \mathbf{v})$ where $\mathbf{u}$ is planar and $\mathbf{v}$ is in 3-space, such that whenever $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}, \mathbf{w})$ are both in the collection, then $\mathbf{v} = \mathbf{w}$.

Consider the following functions of vectors.

- (i) The function h which takes any vector and $\mathbf{v}$ and halves its magnitude (but leaves its direction alone). That is, $h(\mathbf{v}) = \frac{1}{2}\mathbf{v}$.
- (ii) Given any fixed vector $\mathbf{w}$, consider the function $g_{\mathbf{w}}$ which takes an input vector $\mathbf{v}$ and outputs $\mathbf{w} + \mathbf{v}$, that is $g_{\mathbf{w}}(\mathbf{v}) = \mathbf{w} + \mathbf{v}$.
- (iii) The function f which takes a vector $\mathbf{v}$ and reverses its direction (but leaves the magnitude alone).
- (iv) Given any fixed non-zero vector $\mathbf{w}$, consider the function $L_{\mathbf{w}}$ which takes an input vector $\mathbf{v}$ and outputs a vector with the same direction as $\mathbf{w}$ and the same length as $\mathbf{v}$.
- (v) Fix a basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$. Consider the function P which takes a vector $\mathbf{t} \in \mathbb{R}^3$, writes it as

$$\mathbf{t} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$$

and then outputs $a\mathbf{u} + b\mathbf{v}$. This can also be viewed as a function from $\mathbb{R}^3$ to $\mathbb{R}^2$ because $\mathbf{u}$ and $\mathbf{v}$ span a plane.

- (vi) Fix a basis $\mathbf{u}$ and $\mathbf{v}$ for $\mathbb{R}^2$. Consider the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which sends any $\mathbf{t} = a\mathbf{u} + b\mathbf{v}$ to $b\mathbf{u} - a\mathbf{v}$.

Exercise 0.4. Come up with a 7th function that takes an input vector, and then outputs a different vector.

Fix any non-zero vector $\mathbf{w} \neq \mathbf{0}$. Define $T(\mathbf{u}) = \mathbf{w}$ to be the constant function.

Recall that a function is called *one-to-one* if, whenever you have two different input vectors $\mathbf{v}$ and $\mathbf{v}'$, the associated output vectors are also different.

Exercise 0.5. Which of the above functions (including your function) are one-to-one? Justify your answer.

- (i) One-to-one. For if $h(\mathbf{u}) = h(\mathbf{v})$ then $\frac{1}{2}\mathbf{u} = \frac{1}{2}\mathbf{v}$ and thus $\mathbf{u} = \mathbf{v}$.
- (ii) One-to-one. For if $g_{\mathbf{w}}(\mathbf{u}) = g_{\mathbf{w}}(\mathbf{v})$ then $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ so that $\mathbf{u} = \mathbf{v}$.
- (iii) One-to-one. Note that $f(\mathbf{u}) = -\mathbf{u}$ so if $f(\mathbf{u}) = f(\mathbf{v})$ then $-\mathbf{u} = -\mathbf{v}$ and so $\mathbf{u} = \mathbf{v}$.
- (iv) Not one-to-one. Let $\mathbf{u}$ and $\mathbf{v}$ be two different vectors with the same length. Then $L_{\mathbf{w}}(\mathbf{u}) = L_{\mathbf{w}}(\mathbf{v})$.
- (v) Not one-to-one. Notice that $P(\mathbf{u} + \mathbf{w}) = \mathbf{u} = P(\mathbf{u})$ but $\mathbf{u} \neq \mathbf{u} + \mathbf{w}$.
- (vi) One-to-one. Write $\mathbf{t} = a\mathbf{u} + b\mathbf{v}$ and $\mathbf{s} = c\mathbf{u} + d\mathbf{v}$. Then if $T(\mathbf{t}) = T(\mathbf{s})$ we have that $b\mathbf{u} - a\mathbf{v} = d\mathbf{u} - b\mathbf{v}$ so that $b = d$ and that $a = b$ (since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent). But then $\mathbf{t} = \mathbf{s}$ as desired.
- (vii) Non one-to-one. The function I made up is not one-to-one because for any two different vectors $\mathbf{u}, \mathbf{v}$, $T(\mathbf{u}) = \mathbf{w} = T(\mathbf{v})$.

Definition 0.6. Given a function T (which takes vectors as input, and outputs vectors), we say that T is a *linear transformation* if the following two properties hold.

- (#1) For any two vectors $\mathbf{v}$ and $\mathbf{v}'$, we always have $T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$. In other words, adding the vectors and then using the function gives the same result as using the function, then adding the two outputs.
- (#2) For any $\mathbf{v}$ and any real number c , we have $T(c\mathbf{v}) = cT(\mathbf{v})$. In other words, if you scale the vector and then apply the function, you get the same result as if you'd applied the function and then scaled the output.

Exercise 0.7. Which of the functions above (including the one you created) are linear transformations? Give justification for each answer.

- (i) Linear transformation. For property #1, notice that $h(\mathbf{u} + \mathbf{v}) = (\frac{1}{2})(\mathbf{u} + \mathbf{v}) = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = h(\mathbf{u}) + h(\mathbf{v})$. For property #2, notice that $h(c\mathbf{u}) = \frac{1}{2}(c\mathbf{u}) = c\frac{1}{2}\mathbf{u} = ch(\mathbf{u})$.
- (ii) Not a linear transformation unless $\mathbf{w} = \mathbf{0}$. First suppose that $\mathbf{w} \neq \mathbf{0}$, then notice that $g_{\mathbf{w}}(2\mathbf{u}) = \mathbf{w} + 2\mathbf{u}$ but $2g_{\mathbf{w}}(\mathbf{u}) = 2\mathbf{w} + 2\mathbf{u}$ and if $\mathbf{w} \neq \mathbf{0}$, then $\mathbf{w} \neq 2\mathbf{w}$, thus $g_{\mathbf{w}}$ is not a linear transformation. On the other hand if $\mathbf{w} = \mathbf{0}$ then $g_{\mathbf{w}}(\mathbf{u} + \mathbf{v}) = \mathbf{u} + \mathbf{v} = g_{\mathbf{w}}(\mathbf{u}) + g_{\mathbf{w}}(\mathbf{v})$ and likewise $g_{\mathbf{w}}(c\mathbf{u}) = c\mathbf{u} = cg_{\mathbf{w}}(\mathbf{u})$ and so $g_{\mathbf{w}}$ is a linear transformation.
- (iii) Linear transformation. Note that $f(\mathbf{u}) = -\mathbf{u}$ so that $f(\mathbf{u} + \mathbf{v}) = -(\mathbf{u} + \mathbf{v}) = -\mathbf{u} - \mathbf{v} = f(\mathbf{u}) + f(\mathbf{v})$ and likewise that $f(c\mathbf{u}) = -c\mathbf{u} = c(-\mathbf{u}) = cf(\mathbf{u})$ as desired.
- (iv) Not a linear transformation. Let $\mathbf{u}$ be a non-zero vector with the same length as $\mathbf{w}$ and set $\mathbf{v} = -\mathbf{u}$ (same length as $\mathbf{w}$ also, but opposite direction from $\mathbf{u}$). Then $\mathbf{0} = L_{\mathbf{w}}(\mathbf{0}) = L_{\mathbf{w}}(\mathbf{u} + \mathbf{v})$. But $L_{\mathbf{w}}(\mathbf{u}) + L_{\mathbf{w}}(\mathbf{v}) = \mathbf{w} + \mathbf{w} = 2\mathbf{w} \neq \mathbf{0}$.
- (v) Linear transformation. Write $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ and write $\mathbf{y} = d\mathbf{u} + e\mathbf{v} + f\mathbf{w}$. Then

$$\begin{aligned} P(\mathbf{x} + \mathbf{y}) &= P((a + d)\mathbf{u} + (b + e)\mathbf{v} + (c + f)\mathbf{w}) = (a + d)\mathbf{u} + (b + e)\mathbf{v} \\ &= (a\mathbf{u} + b\mathbf{v}) + (d\mathbf{u} + e\mathbf{v}) = P(\mathbf{x}) + P(\mathbf{y}), \end{aligned}$$

which proves property #1. For property #2 simply notice that

$$P(n\mathbf{x}) = P(na\mathbf{u} + nb\mathbf{v} + nc\mathbf{w}) = na\mathbf{u} + nb\mathbf{v} + nc\mathbf{w} = n(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = nP(\mathbf{x}).$$

- (vi) Linear transformation. Write $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ and $\mathbf{y} = c\mathbf{u} + d\mathbf{v}$. Then

$$\begin{aligned} T(\mathbf{x} + \mathbf{y}) &= T((a + c)\mathbf{u} + (b + d)\mathbf{v}) = (b + d)\mathbf{u} - (a + c)\mathbf{v} \\ &= (b\mathbf{u} - a\mathbf{v}) + (d\mathbf{u} - c\mathbf{v}) = T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

proving property #1. For property #2 we have

$$T(n\mathbf{x}) = T(na\mathbf{u} + nb\mathbf{v}) = nb\mathbf{u} - na\mathbf{v} = n(b\mathbf{u} - a\mathbf{v}) = nT(\mathbf{x}).$$

- (vii) Not a linear transformation. Notice that $T(\mathbf{u}) = \mathbf{w} = T(2\mathbf{u})$. But $2\mathbf{w} \neq \mathbf{w}$ since $\mathbf{w} \neq \mathbf{0}$.

Exercise 0.8. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are a basis for $\mathbb{R}^3$ and that $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation. Then show that the function T is uniquely determined (among linear transformations) by the values of $T(\mathbf{u}), T(\mathbf{v})$, and $T(\mathbf{w})$.

Suppose that T and S are two linear transformations such that

$$T(\mathbf{u}) = S(\mathbf{u})$$

$$T(\mathbf{v}) = S(\mathbf{v})$$

$$T(\mathbf{w}) = S(\mathbf{w})$$

We will show that $T = S$. So choose any vector $\mathbf{x}$ and using the fact that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a basis write $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

$$\begin{aligned} T(\mathbf{x}) &= T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) \\ &= aS(\mathbf{u}) + bS(\mathbf{v}) + cS(\mathbf{w}) = S(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = S(\mathbf{x}). \end{aligned}$$

Therefore $S = T$ and we are done.

Exercise 0.9. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are a basis for $\mathbb{R}^3$ and that $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation. Prove that T is injective if and only if $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ also form a basis for $\mathbb{R}^3$.

First we prove a lemma.

Lemma 0.10. Suppose that T is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Proof. Note $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$ since T is a linear transformation. But then cancelling $T(\mathbf{0})$ from both sides gives us $T(\mathbf{0}) = \mathbf{0}$ as desired. $\square$

Now we return to the problem at hand. First suppose that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ also forms a basis. We will show that T is injective. So suppose that $T(\mathbf{x}) = T(\mathbf{y})$. Write $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ and write $\mathbf{y} = d\mathbf{u} + e\mathbf{v} + f\mathbf{w}$ (we can do this since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a basis). Thus

$$\begin{aligned} aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) &= T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = T(\mathbf{x}) \\ &= T(\mathbf{y}) = T(d\mathbf{u} + e\mathbf{v} + f\mathbf{w}) = dT(\mathbf{u}) + eT(\mathbf{v}) + fT(\mathbf{w}) \end{aligned}$$

Therefore $a = d$, $b = e$ and $c = f$ because $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ is a linearly independent set. But then $\mathbf{x} = \mathbf{y}$ as desired.

For the converse direction, suppose that T is injective. Then suppose that $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$. We will show that necessarily $a = 0, b = 0, c = 0$ proving that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ is linearly independent and thus a basis. Now

$$T(\mathbf{0}) = \mathbf{0} = aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w})$$

by the Lemma. Then since T is injective $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$. Therefore $a = 0, b = 0$ and $c = 0$ as desired.