F-rational mod p implies rational in mixed characteristic and characteristic zero

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AMS Sectional Meeting, Denton Texas
September 2017

Definitions

- R ess. f.t. char 0. has *rational singularities* if for all/any res.

$$\pi : Y \to \operatorname{Spec} R$$

one has $\mathbf{R}\pi_* O_Y \cong R$. (WRITE)

- R has *pseudo-rational singularities* if CM & for any proper birat. map

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one has $H^d_{\mathfrak{m}}(R) \to \mathbb{H}^d_{\mathfrak{m}}(\mathbf{R}\pi_* O_Y)$ injects (WRITE)

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History

Theorem (Elkik, 1978)

$(R, \mathfrak{m})$ local char. 0. $f \in \mathfrak{m}$ reg. elt.

$$R/f \text{ is rational} \Rightarrow R \text{ is rational.}$$

Theorem (Smith, 1997)

In general, if R_p char $p > 0$.

$$R_p \text{ is } F\text{-rat.} \Rightarrow R_p \text{ is pseudo-rational.}$$

In particular, $R_{\mathbb{Q}}$ char. 0, $R_{\mathbb{Z}}$ spread out. $p > 0$ big enough.

$$R_{\mathbb{Z}}/p \text{ is } F\text{-rational} \Rightarrow R_{\mathbb{Q}} \text{ is rational}$$

Converse [Hara, 1998], [Mehta-Srinivas]

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Our results

Theorem (Ma-S.)

Suppose $(R, \mathfrak{m})$ is nice mixed char local domain.

$$R/p \text{ is } F\text{-rat.} \Rightarrow R \text{ is pseudo-rational} \Rightarrow R \otimes_{\mathbb{Z}} \mathbb{Q} \text{ is rational.}$$

Several steps in proof.

Deduce something in mixed char.

Rough idea: $R \subseteq B$ in bal. big CM-alg.

$$\begin{tikzcd}
H^{d-1}_{\mathfrak{m}}(R) = 0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/p) \arrow[r] \arrow[d] & H^{d}_{\mathfrak{m}}(R) \arrow[r, "\cdot p"] \arrow[d] & H^{d}_{\mathfrak{m}}(R) \arrow[d] \\
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Socle chase! So $H^d_{\mathfrak{m}}(R) \hookrightarrow H^d_{\mathfrak{m}}(B)$. $\square$

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Say $\pi : Y \to \operatorname{Spec} R$ blowup. Choose B big enough (related to Rees algebra of blowup). Then [Ma] says

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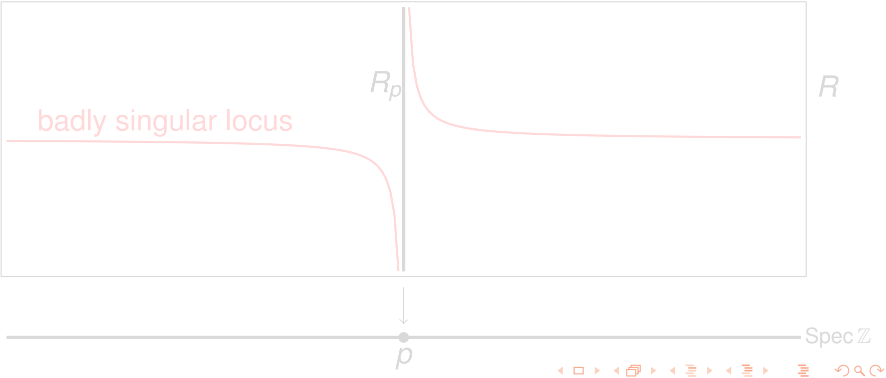
Warning

Our main result only works locally (or for proper varieties).

- It can happen that R finite type / $\mathbb{Z}$.
- R/p is F-rational for some $p > 0$.
- But R/p is not F-rational for other $p > 0$
- and $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is not rational.



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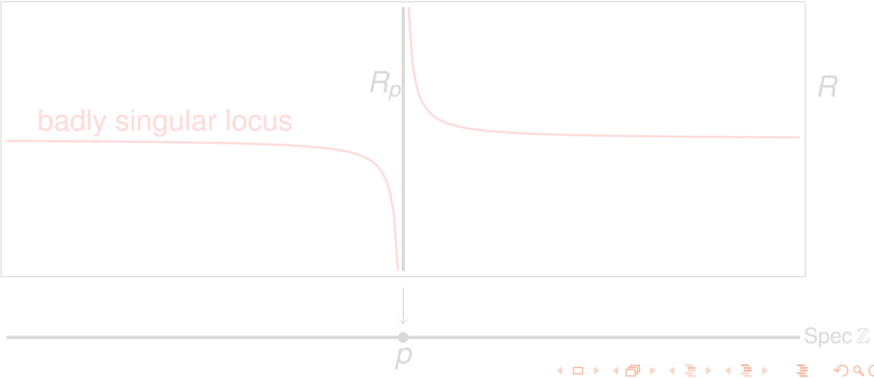

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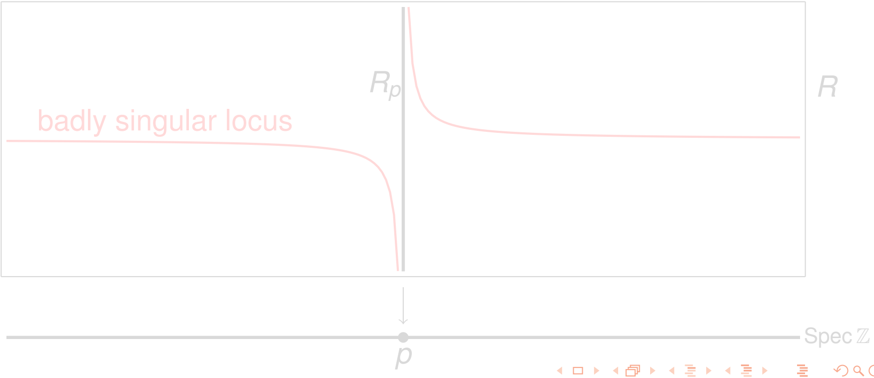

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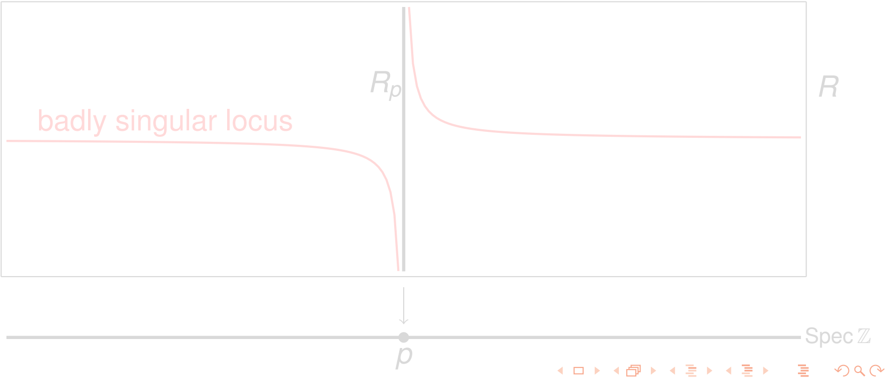

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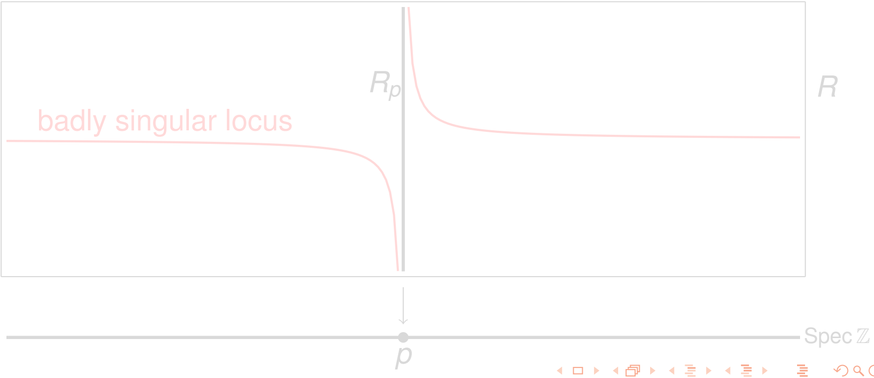

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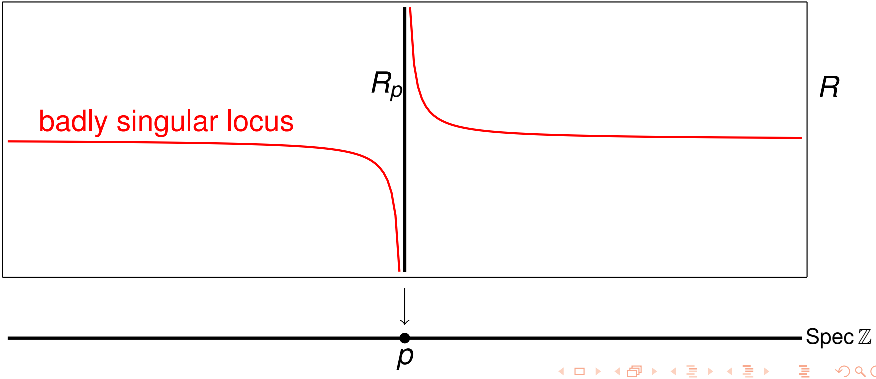

Say more

We we can say a lot more.

- $(R_{\mathbb{Q}}, \mathfrak{m}_{\mathbb{Q}})$ char. 0. Then

$$\begin{array}{rl} & R_p \text{ is } F\text{-rational for one } p \\ \Rightarrow & R_{\mathbb{Q}} \text{ is rational} \\ \Rightarrow & R_p \text{ is } F\text{-rational for almost all } p > 0. \end{array}$$

- We believe we can prove a restriction theorem for something like param. test modules.

$$\mathrm{Image}(\tau(\omega_R)/p \to \omega_{R/p}) \supseteq \tau(\omega_{R/p}).$$

- Then we believe we can show via cyclic covering,

R is $\mathbb{Z}_{(p)}$-Gor. & R/p is strongly F-regular $\Rightarrow$ R is pseudo-KLT.

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Computational application

We can prove rings have rational singularities using a computer now!

- Start out with $(R, \mathfrak{m})$ char. 0, say over $\mathbb{Q}$.

$$(\mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n))_{\mathfrak{m}_0}$$

  $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$.
- Check if

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Use the new `TestIdeals` package (in build tree).

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Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralC
               LLLBases, PrimaryDecomposition, ReesAlgeb
i1 : S = ZZ/3[a,b,c,d,t]; m = 4; n = 3;
i4 : M = matrix{ {a^2 + t^m, b, d}, {c, a^2, b^n-d} };
i5 : I = minors(2, M);
o5 : Ideal of S
i6 : loadPackage "TestIdeals";
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This is from Singh's example of non-F-regular deformation.

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