F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 10/26-2010

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1. Characteristic $p > 0$ analogs of LC-centers and subadjunction

We recall the following definition (for now, we work in characteristic zero). Most of the results found here (including more details of various proofs) can be found in [Sch09].

Definition 1.1. Suppose (X, Δ) is a pair and $W \subseteq X$ is an irreducible subvariety set η to be the generic point of W. We say that W is a *non-KLT-center* if there exists a divisor E_i on some birational model $\pi : \widetilde{X} \to X$ of X such that $W = \pi(E_i)$ where the discrepancy $a_i \leq -1$ (as usual, $\sum a_i E_i = K_{\widetilde{X}} - \pi^*(K_X + \Delta)$. We say that W is an *LC-center* if W is a non-KLT-center and furthermore, $(X, \Delta)_\eta$ is log canonical.

Lemma 1.2. *Given a pair (X, Δ) as above further assume that X is affine. Then $W \subseteq X$ is a non-KLT-center if and only if for every $d \in \mathcal{O}_X$ such that $\eta \in V(d)$, we have that for every $\varepsilon > 0$ that the pair $(X, \Delta + \varepsilon \operatorname{div}_X(d))_\eta$ is not log canonical.*

Proof. If W is a non-KLT-center, then the conclusion of the lemma is obvious. Suppose conversely that W satisfies the condition of the lemma but is not an LC-center.

If η is a codimension 1 point, then the result is also clear (no birational models are needed). On the other hand, if $(X, \Delta)_\eta$ is not log canonical, we are already done, so we may assume that $(X, \Delta)_\eta$ is log canonical. Choose a log resolution $\pi : \widetilde{X} \to X$ of (X, Δ) such that $I_W \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$ is also a SNC divisor (and by hypothesis, all the discrepancies of E_i such that $\pi(E_i) = W$ satisfy $a_i > -1$). We choose a general element d of $\mathcal{O}_X$ vanishing at η so that π is a log resolution of $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ for every $\varepsilon > 0$. Because d is general, for $0 < \varepsilon \ll 1$ $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ is log canonical on $X \setminus W$. Of course, for E_i such that $\pi(E_i) = W$, the associated a_i for $(X, \Delta + \varepsilon \operatorname{div}_X(d))$ is still > -1 for $\varepsilon > 0$ small enough. But this implies that $(X, \Delta + \varepsilon \operatorname{div}_X(d))_\eta$ is log canonical. $\square$

In analogy with the previous lemma, we make the following definition.

Definition 1.3. Suppose that $X = \operatorname{Spec} R$ is an F-finite normal scheme of characteristic $p > 0$ and that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(1 - p^e)(K_X + \Delta)$ is Cartier. For an element $Q \in \operatorname{Spec} R$, we say that $V(Q) = W \subseteq X$ is an *non-F-regular-center* if for every element $d \in Q$ and every $\varepsilon > 0$, we have that $(X, \Delta + \varepsilon \operatorname{div}(d))_Q$ is not F-pure. It is an *F-pure center* if we additionally require that $(X, \Delta)_Q$ is F-pure. These definitions generalize to the non-affine setting by requiring them on affine charts.

Lemma 1.4. *With notation as above, suppose that ϕ to be a map $F^e_* \mathscr{L} \to \mathcal{O}_X$ a map corresponding to Δ. Then W is a non-F-regular center if and only if $\phi(F^e_* Q \mathscr{L}) \subseteq Q$.*

Proof. Without loss of generality, we may assume that R is a local ring and thus that $\mathscr{L} = \mathcal{O}_X$. Furthermore, we can localize at Q and assume that Q is the maximal ideal of

R. First we claim that $\phi(F^e_* Q) \subseteq Q$ if and only if $\phi^n(F^{ne}_* Q) \subseteq Q$ for some $n > 0$. The ($\Rightarrow$) direction is clear, for the reverse, if $\phi(F^e_* Q) \not\subseteq Q$, then $\phi(F^e_* Q) = R$, but then it follows easily that $\phi^n(F^{ne}_* Q) = R$ for all $n > 0$. From this, it follows that $\phi(F^e_* Q) \subseteq Q$ if and only if for every $\psi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ such that $\Delta_\psi \geq \Delta_\phi$ we have that $\psi(F^e_* Q) \subseteq Q$.

Of course, we may assume that the $\varepsilon > 0$ we consider is of the form $\frac{1}{p^{ne}-1}$. Now, $(X, \Delta + \frac{1}{p^{ne}-1} \operatorname{div}(d))$ is not F-pure if and only if $\phi^n(F^{ne}_* dR) \subseteq Q$. But if we require this for all $d \in Q$, this just says that $\phi^n(F^{ne}_* Q) \subseteq Q$. □

Corollary 1.5. *(X, Δ) is strongly F-regular if and only if it has no non-F-regular centers.*

Theorem 1.6. *Non-klt centers in characteristic zero reduce to non-F-regular centers in characteristic $p > 0$.*

Proof. It follows easily from the fact that we have the map $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$, which induces for each effective divisor G,

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G).$$

The theorem then follows once one observes that any non-klt center can be written as $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + G)$ for some appropriate G. □

Here is a characteristic $p > 0$ version of Kawamata's subadjunction theorem.

Theorem 1.7. *Suppose that (X, Δ) is a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$. Suppose that $W \subseteq X$ is a normal F-pure center. Then, there exists a canonically determined divisor Δ_W on W such that $K_W + \Delta_W \sim_{\mathbb{Q}} (K_X + \Delta)|_W$ and such that:*

- *(W, Δ_W) is F-pure if and only if (X, Δ) is F-pure near W.*
- *(W, Δ_W) is strongly F-regular if and only if W is minimal with respect to inclusion of F-pure centers with respect to containment.*
- *The set of F-pure centers of (W, Δ_W) is the same as the set of F-pure centers of (X, Δ) which properly contain W.*

Proof. Given a map $\phi : F^e_* \mathscr{L} \to \mathcal{O}_X$ corresponding to Δ, suppose that Q is an ideal sheaf such that $V(Q) = W$. We immediately obtain a map $\phi|_W : F^e_* \mathscr{L}|_W \to \mathcal{O}_W$ obtained by modding out by Q. This map $\phi|_W$ corresponds to divisor Δ_W. The first statement follows from the fact that in a local ring, $\phi : F^e_* R \to R$ surjects if and only if the induced map $\phi : F^e_* R/Q \to R/Q$ surjects. The third statement follows the fact that $P \supseteq Q$ is ϕ-compatible if and only if P/Q is ϕ/Q-compatible, and the third statement implies the second. □

Remark 1.8. If W is not normal, one can always induce a divisor Δ_{W^N} on the normalization of W. Nice properties of (X, Δ) still induce nice properties of (W^N, Δ_{W^N}) but the converse statements don't necessarily hold (this seems to be due to inseparability and wild ramification in the normalization map $\eta : W^N \to W$).

If one knew that log canonical implied F-pure type, one could prove a number of interesting things about log canonical centers via reduction to characteristic $p > 0$.

This is very different from the behavior in characteristic zero. In particular, Δ_W is canonically determined which is not the case in characteristic zero. Consider the following example.

Example 1.9 (Speyer, –, Xu)**.** Suppose that $X \to \operatorname{Spec} k[t] = \mathbb{A}^1$ is the family of cones over elliptic curves defined by $zy^2 - x^3 + txz^2$ with a section $\sigma : \mathbb{A}^1 \to X$ mapping to the cone points. Further assume that there is a log resolution $\pi : \widetilde{X} \to X$ which is obtained by blowing up the image of σ (which we now call Z). Finally note that X is F-pure at the generic point of Z.

It then follows that Z is an F-pure center. Note that X is $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$, so we can set $\Delta = 0$. We can construct Δ_W as above. In this context, Δ_W has support exactly at those points such that the associated elliptic curve is not F-split (ie, supersingular).

Note that $(X, \operatorname{div}_X(t-\lambda))$ has a log canonical center at $W = (x, y, z, t-\lambda)$. Furthermore, by blowing up $\widetilde{X}$ at the inverse image of that point, one obtains a log resolution with two exceptional divisors, the one dominating Z and the one dominating W. Both of these exceptional divisors have discrepancy -1. It then follows that if $(X, \operatorname{div}_X(t-\lambda))$ is F-pure, the exceptional divisor is F-split. This implies that the associated elliptic curve is also F-split. But $(X, \operatorname{div}_X(t-\lambda))$ is F-pure if and only if $(W, \Delta_W + \operatorname{div}_W(t-\lambda))$ is F-pure. The latter is F-pure at W if and only if Δ_W does not have $\operatorname{div}_W(t-\lambda)$ as a component. This implies that if λ corresponds to a supersingular elliptic curve, then Δ_W must have $\operatorname{div}_W(t-\lambda)$ among its components.

Conversely, suppose that λ corresponds to an ordinary elliptic curve E_λ. The generating map on the associated elliptic curve $\psi : F_* \mathcal{O}_{E_\lambda} \to \mathcal{O}_{E_\lambda}$ is always the map induced by the pair $(X, \operatorname{div}_X(t-\lambda))$ on $\widetilde{X}$ as above. On the elliptic curve, the map ψ sends units to units, thus on $\widetilde{X}$, the map associated to $(X, \operatorname{div}_X(t-\lambda))$ has to send units to non-zero elements which restrict to units on E_λ. Thus back on X, units must be sent to elements that are units near W and the proof is complete.

We give one more application of these ideas. I do not know of an analog of this result in characteristic zero.

Theorem 1.10. *Suppose that S is a regular local ring and that $R = S/I$ is any reduced normal ring and Δ_R is a divisor on* $\operatorname{Spec} R$ *such that $K_R + \Delta_R$ is F-pure with index not divisible by $p > 0$. Then there exists a divisor Δ_S on S such that $K_S + \Delta_S$ is* $\operatorname{Spec} R \subseteq \operatorname{Spec} S$ *is an F-pure center of (S, Δ_S) and furthermore, that Δ_S and Δ_R are related as in Theorem 1.7. In particular, (R, Δ_R) is F-pure, then we may choose (S, Δ_S) also to be F-pure.*

References

[Sch09] K. Schwede: *F-adjunction*, Algebra Number Theory **3** (2009), no. 8, 907–950.