A FACT ABOUT REGULAR LOCAL RINGS

MATH 538 FALL 2011

1. Prime avoidance and regular local rings are domains

We will prove that regular rings are integral domains. Before continuing however, I need a stronger form of prime avoidance.

Lemma 1.1 (Prime avoidance #2)**.** *Suppose that R is a ring and that $P_2, \dots, P_t \subseteq$ are prime ideals and P_1 is any other ideal that isnot necessarily prime. Suppose that I is an ideal such that*

$$I \subseteq \left(\bigcup_i P_i \right)$$

Then I is contained in at least one of the P_i.

Proof. The proof is by induction. If $t = 1$, there is nothing to prove. We now assume it for $t-1$ and so we may assume that I is not contained in the union of any proper subcollection of the P_i. In particular, we may pick $x_i \in I \setminus \left(\bigcup_{j \neq i} P_j \right)$ for each $i = 1, \dots, t$. Consider $y = (x_1 \cdot \dots \cdot x_{n-1}) + x_n$ noting that P_n is prime. This element is clearly in I, so it must be in one of the P_i. There are two cased.

- $y \in P_n$**:** Since $x_n \in I$ and $x_n \notin P_j$ for $j \neq n$, we have that $x_n \in P_n$. Then $y - x_n = x_1 \cdot \dots \cdot x_{n-1} \in P_n$ as well so at least one of the $x_1, \dots, x_{n-1} \in P_n$. But that is a contradiction.
- $y \in P_j, j \neq n$**:** Now, $x_j \in I$ and so $x_j \in P_j$ and thus $x_1 \cdot \dots \cdot x_{n-1} \in P_j$ as well. So $x_n = (x_1 \cdot \dots \cdot x_{n-1}) - y \in P_j$ which is also a contradiction.

We have proven our result. □

Remark 1.1. It is actually possible to prove the same result while assuming both P_1 and P_2 are *NOT* necessarily prime.

Theorem 1.2. *Suppose that $(R, \mathfrak{m})$ is a regular local ring. Then R is an integral domain.*

Proof. We proceed by induction on the dimension of R – the case of dimension zero being obvious and the case of dimension 1 being clear from our previous work on DVRs. We thus assume that $\dim R \geq 1$. Indeed, set $P_1, \dots, P_t$ to be the set of minimal primes of R. We have $\mathfrak{m} \supsetneq P_i$ by the minimality of the P_i and the fact that R is not zero dimensional. By Nakayama's lemma, we know that $\mathfrak{m} \supsetneq \mathfrak{m}^2$.

Suppose now that $\mathfrak{m} \subseteq (\mathfrak{m}^2) \cup (\bigcup_i P_i)$, but then the previous lemma provides a contradiction and so we may choose $x \in \mathfrak{m}$, $x \notin P_i$ for any i and $x \notin \mathfrak{m}^2$.

We have two claims:

(i) $\dim R/\langle x \rangle = (\dim R) - 1$. We prove this claim. Choose

$$P_i = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d = \mathfrak{m}.$$

to be a chain of primes of maximal length (in other words, $\dim R = d$). This chain of primes is also maximal length in R/P_i. Indeed, by the above, x is a regular element on R/P_i and so $\dim R/(P_i + \langle x \rangle) = \dim(R/P_i) - 1 = \dim R - 1$. In particular,

$\dim R/\langle x\rangle \geq \dim R - 1$. On the other hand, if $\dim R/\langle x\rangle = \dim R$, then we may choose a chain of primes $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d \subseteq R$ which remains a proper sequence of primes after passing to $R/\langle x\rangle$. But again, we can assume that $Q_0 = P_i$ for some i (since Q_0 must be some minimal prime). This sequence must stay a proper sequence of primes after passing to $(R/\langle x\rangle)/\overline{Q_0} = (R/\langle x\rangle)/\overline{P_i} \cong (R/P_i)/\langle \overline{x}\rangle$. But this is impossible by the above argument. Thus we have proved (i).

(ii) $R/\langle x\rangle$ is regular. This is easy $\dim R/\langle x\rangle = (\dim R) - 1$ and $\mathfrak{m}/\langle x\rangle$ has $(\dim R) - 1$ generators. This proves (ii)

Our inductive hypothesis then implies $R/\langle x\rangle$ is an integral domain and in particular that $\langle x\rangle$ is prime. In particular, $P_i \subseteq \langle x\rangle$ for some i. Choose $y \in P_i$ and write $y = rx$ for some $r \in R$. Since $x \notin P_i$ and $rx = y \in P_i$, we see that $r \in P_i$. It follows that $x \cdot P_i = P_i$ and so $\mathfrak{m} \cdot P_i = P_i$. This contradicts Nakayama's Lemma and completes the proof. $\square$