MATH 6310 – MIDTERM REVIEW SHEET

1. Short answer questions. No explanation is necessary, but little or no partial credit will be given without a justification.

(a) Let k be a field. How many prime ideals does $k[x]/(x^3)$ have?

(b) Suppose $R = \mathbb{Z}$. Is every quotient-module of $\mathbb{Z}^{\oplus 3}$ a free module?

(c) If R is a ring, what is the definition of a *left ideal* of R?

(d) Give an example of split short exact sequence of nonzero modules.

(e) Is every PID automatically a unique factorization domain.

(f) Give an example of a unique factorization domain that is not a PID.

(g) Give an example of a maximal ideal that is not principal.

(h) Let $\mathbb{F}_3 = \mathbb{Z}/(3)$ and let $R = \mathbb{F}_3[x]$. How many R-module homomorphisms are there from $M = \mathbb{F}_3[x] \oplus \mathbb{F}_3[x]/(x) \oplus \mathbb{F}_3[x]/(x-1)$ to $N = \mathbb{F}_3[x]/(x^2)$.

1. Short answer questions continued.

(a) Consider $R = (\mathbb{Z}[x])[y]$ viewed as a polynomial ring in y with coefficients in $\mathbb{Z}[x]$. What is the content of the polynomial $f(x, y) = (2x^2 + 3x)y^2 + 6xy + 2x^2$?

(b) What is the length of the $\mathbb{Z}$-module $\mathbb{Z}/(12) \oplus \mathbb{Z}(25)$?

(c) Give an example of a short exact sequence of modules that is not split.

(d) Give an example of a ring with exactly 3 prime ideals.

(e) Consider $\mathbb{Z}[i] \subseteq \mathbb{C}$, the ring of Gaussian integers. Briefly justify why $\mathbb{Z}[i]$ is a PID.

(f) What is the number of distinct ring homomorphisms from $\mathbb{Z}[i]$ to $\mathbb{C}$?

(g) State a definition of what it means for a ring to be a Euclidean domain.

(h) State the snake lemma.

2. Find all the prime ideals in the ring $R = \mathbb{Q}[x, y]/(x^4 - x^2, y^2 + x)$.

3. Find all the maximal ideals in the ring $\mathbb{Z}[x, y]/(12, x^2 + 5, xy - 1)$.

4. Let $k = \mathbb{Z}/(2)$ and $R = k[x]$. Let M denote the cokernel of the map $R^3 \to R^3$ given by the matrix:

$$\begin{bmatrix} x & x+x^2 & x \\ 1+x & 0 & 1+x+x^2 \\ 1+x & x^2 & 1+x \end{bmatrix}$$

How many elements $m \in M$ satisfy $xm = 0$?

5. Let $R = \mathbb{Z}$ and consider the R-module with three generators x, y, z subject to the relations $2x+4y+6z = 0$ and $3x + 6y + 9z = 0$. How many elements are in $\mathrm{Hom}(\mathbb{Z}/(5), M)$?

6. Let $R = \mathbb{C}[x]$ and consider the R-module M which is the cokernel of the map $R^3 \to R^3$ defined by the matrix

$$\begin{bmatrix} x & x^2+2 & x^3 \\ x-1 & x & 1-x \\ 0 & x-1, & 0 \end{bmatrix}$$

Write down M as a direct sum of cyclic modules.

7. Let M be the $\mathbb{Q}[x]$-module defined by:

(a) Generators a, b, c and

(b) Relations $ax^2 = 0, a + bx = 0, a + b + cx = 0$.

Find the length of M.

8. Let R be a commutative ring, $I \subseteq R$ a finitely generated ideal, and M an R-module. Define

$$\Gamma_I(M) = \{m \in M \mid \exists n > 0 \text{ such that } \forall f \in I^n, f \cdot m = 0\}.$$

Here $I^n = I \cdot I \cdots I$ denotes the nth power of I.

(a) Prove that $\Gamma_I(M) \subseteq M$ is an R-submodule.

(b) Suppose that $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence. Show that there is an exact sequence

$$0 \to \Gamma_I(L) \xrightarrow{\alpha'} \Gamma_I(M) \xrightarrow{\beta'} \Gamma_I(N)$$

where the maps α' and β' are α and β restricted to $\Gamma_I(L)$ and $\Gamma_I(M)$ respectively.

9. Suppose $I \subseteq R$ is an ideal. Recall that if M is an R-module, then $IM = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in I, x_i \in M\}$.

Suppose R is a commutative ring and suppose $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence of R-modules. Show that there is an exact sequence

$$L/IL \xrightarrow{\overline{\alpha}} M/IM \xrightarrow{\overline{\beta}} N/IN \to 0$$

where $\overline{\alpha}$ and $\overline{\beta}$ are the maps induced by α and β respectively.