HW #3 – MATH 6320
SPRING 2015

DUE: TUESDAY FEBRUARY 17TH

(1) Exhibit an *explicit* isomorphism between the splitting fields of $x^3 - x + 1$ and $x^3 - x - 1$ over $\mathbb{F}_3$.

(2) Explicitly show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is a simple extension of $\mathbb{Q}$ by finding a generator.

(3) Show that $\mathbb{F}_p(x, y)/\mathbb{F}_p(x^p, y^p)$ is *not* a simple extension.

(4) Is $\mathbb{Q}(2^{1/3})$ a subfield of a cyclotomic extension of $\mathbb{Q}$?

(5) Let K/F be any finite extension, let L be a Galois extension of F containing K and let $H \leq \mathrm{Gal}(L/F)$ be the subgroup corresponding to K. For any $\alpha \in K$ define the *norm* of α to be

$$\mathrm{N}_{K/F}(\alpha) = \prod_{\overline{\sigma} \in G/H} \overline{\sigma}(\alpha).$$

Note here that G/H is just the set of cosets of H. Likewise we define the *trace* of α to be

$$\mathrm{Tr}_{K/F}(\alpha) = \sum_{\overline{\sigma} \in G/H} \overline{\sigma}(\alpha).$$

In other words, the norm is a product of Galois conjugates and the trace is a sum of Galois conjugates (if K/F is Galois, it is a product/sum over all Galois conjugates). Of course,

(a) Show that $\mathrm{N}_{K/F}(\alpha) \in F$ and $\mathrm{Tr}_{K/F}(\alpha) \in F$.

(b) Prove that the norm is multiplicative ($\mathrm{N}_{K/F}(\alpha\beta) = \mathrm{N}_{K/F}(\alpha)\mathrm{N}_{K/F}(\beta)$) and that the trace is additive ($\mathrm{Tr}_{K/F}(\alpha\beta) = \mathrm{Tr}_{K/F}(\alpha) + \mathrm{Tr}_{K/F}(\beta)$)

(c) If $K = L = F(\sqrt{c})$ is a degree two extension, find formulas for $\mathrm{N}(a+b\sqrt{c})$ and $\mathrm{Tr}(a+b\sqrt{c})$.

(6) Let K/F be a finite separable extension. Then each element $\alpha \in K$ gives us an F-linear map $(\times\alpha) : K \longrightarrow K$. We can of course pick a basis for K/F and so consider $(\times\alpha)$ as a matrix m_α. Then consider maps $t : K \longrightarrow F$ and $n : K \longrightarrow F$ defined by

$$t(\alpha) = \mathrm{trace}(m_\alpha), n(\alpha) = \det(m_\alpha)$$

where here trace just means sum up the diagonal. Is it true that $t = \mathrm{Tr}_{K/F}$ and $n = \mathrm{N}_{K/F}$?

(7) Fix notation as in the last two problems. If $\alpha \in F$, show that $\mathrm{N}(\alpha) = \alpha^n$ and $\mathrm{Tr}(\alpha) = n\alpha$ where $n = [K : F]$.

(8) Fix notation as in the last three problems. Let $F = \mathbb{Q}, K = \mathbb{Q}(2^{1/3})$ and $\alpha = 2^{1/3}$. Compute $\mathrm{N}_{K/F}(2^{1/3})$ and $\mathrm{Tr}_{K/F}(2^{1/3})$.