EXERCISES FOR CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 3RD, 2017

DUE, FEBRUARY 13TH, 2017

(1) If we have a morphism of complexes, $A^{\bullet} \xrightarrow{\alpha} B^{\bullet}$, then we can always take the cone $C(\alpha)^{\bullet} = C^{\bullet} = A[1]^{\bullet} \oplus B^{\bullet}$ with differential

$$C^i = A^{i+1} \oplus B^i \xrightarrow{-d_A^{i+1}, \alpha^i + d_B^i} A^{i+2} \oplus B^{i+1}.$$

Verify that this really is a complex.

(2) Suppose that $0 \longrightarrow A^{\bullet} \xrightarrow{\alpha} B^{\bullet} \xrightarrow{\beta} D^{\bullet} \longrightarrow 0$ is a short exact sequence of complexes. Show that $D^{\bullet}$ is quasi-isomorphic to the cone $C(\alpha)^{\bullet}$.

(3) Consider the ring $R = \overline{\mathbb{F}}_p[x]$. Show that the perfection R^{∞} of R is $\overline{\mathbb{F}}_p[x, x^{1/p}, x^{1/p^2}, \ldots]$ and that the map $\operatorname{Spec} R^{\infty} \longrightarrow \operatorname{Spec} R$ is a bijection.

(4) More generally, suppose that R is a Noetherian ring and R^{∞} is its perfection (which as we've seen is usually not Noetherian). Is $\operatorname{Spec} R^{\infty} \longrightarrow \operatorname{Spec} R$ always a bijection?

(5) Suppose R is an F-finite Noetherian ring. Show that R is F-split if and only if $R_{\mathfrak{m}}$ is F-split for every maximal ideal $\mathfrak{m} \subseteq R$.

(6) Suppose we have an extension of rings $R \subseteq S$ and that M is an S-module. Suppose that $\operatorname{Hom}_R(S, R) \cong S$ as S-modules. We have a map

$$\operatorname{Hom}_S(M, S) \times \operatorname{Hom}_R(S, R) \longrightarrow \operatorname{Hom}_R(M, R)$$

by composition $(\alpha, \beta) \mapsto \beta \circ \alpha$. Show that this map is surjective.

(7) Suppose that we have inclusions $R \subseteq S \subseteq T$ of rings and that $\operatorname{Hom}_R(S, R) \cong_S S$ and that $\operatorname{Hom}_S(T, S) \cong_T T$. Choose $\Phi \in \operatorname{Hom}_R(S, R)$ which generates the Hom-module as an S-module and $\Psi \in \operatorname{Hom}_S(T, S)$ generating the Hom-module as a T-module. Prove that $\Phi \circ \Psi$ generates $\operatorname{Hom}_R(T, R)$ as a T-module.