EXERCISES FOR CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 3RD, 2017

DUE, FRIDAY APRIL 14TH, 2017

(1) Suppose R is an F-finite normal domain and that $0 \neq \phi \in \mathrm{Hom}_R(F^e_* R, R)$ and $0 \neq \psi \in \mathrm{Hom}_R(F^d_* R, R)$. Find a formula for $\Delta_{\phi \circ F^e_* \psi}$ in terms of Δ_ϕ and Δ_ψ.

(2) Suppose that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on a normal domain R. Consider the subset

$$S := \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R) \subseteq \mathrm{Hom}_R(F^e_* R, R)$$

induced by contravariance of Hom.

Show that S is equal to:

$$\begin{aligned} &= \{\phi \in \mathrm{Hom}_R(F^e_* R, R) \mid \phi \text{ factors through the inclusion } R \subseteq R(\lceil (p^e-1)\Delta \rceil)\} \\ &= \{\phi \in \mathrm{Hom}_R(F^e_* R, R) \mid \Delta_\phi \geq \Delta\} \end{aligned}$$

(3) Suppose that L/K is a finite extension of characteristic $p > 0$ fields and $x \in L \setminus K$ but $x^p \in K$. Show that if $\phi : K^{1/p^e} \longrightarrow K$ extends to $L^{1/p^e} \longrightarrow L$, then ϕ is the zero map on K.

(4) Using the previous problem, conclude in general that if L/K is finite and inseparable, then no nonzero $\phi : K^{1/p^e} \longrightarrow K$ can extend to $L^{1/p^e} \longrightarrow L$.

Hint: Suppose such a map did extend to ϕ_L. Let K' denote the separable closure of K in L. Show that $\phi_L(K'^{1/p^e}) \subseteq K'$ and the reduce to the previous exercise.

(5) Now fix a normal F-finite Noetherian domain R (with $\mathrm{Hom}_R(F_* R, \omega_R) \cong F_* \omega_R$ if you'd like) and a nonzero $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. In this case we define the *test ideal of* (R, ϕ) to be the smallest nonzero ideal J such that

$$\phi(F^e_* J) \subseteq J,$$

if it exists. The test ideal is denoted by $\tau(R, \phi)$. In the following steps, we will show that it always exists.

i. Suppose first that R is strongly F-regular and that ϕ generates $\mathrm{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module. Show that for every $0 \neq c \in R$, there is an $n > 0$ such that $\phi^n(F^{ne}_* cR) = R$. Conclude that $\tau(R, \phi) = R$.

Hint: We know there exists a $d > 0$ and $\psi \in \mathrm{Hom}_R(F^d_* R, R)$ such that $\psi(F^e_* c) = 1$. Show that you can choose d such that e divides d. In this case, ψ is a pre-multiple of some ϕ^n.

ii. Show there exists $b \in R$ such that $R_b = R[b^{-1}]$ is strongly F-regular and ϕ_b, the image of ϕ, generates $\mathrm{Hom}_{R_b}(F^e_* R_b, R_b)$ as an $F^e_* R_b$-module.

Hint: First invert some b' to make $R_{b'}$ regular (and hence strongly F-regular). Then invert b'' contained in all the ideals that make up D_ϕ. Let $b = b'b''$. Show that the image of ϕ in $\mathrm{Hom}_{R_b}(F^e_* R_b, R_b)$ corresponds to the zero divisor.

iii. Now fix b as in the previous problem. For any $0 \neq c \in R$, show that $b^{m_c} \in \phi^n(F^{ne}_* cR)$ for some $n > 0$ (m_c depends on c).

Hint: We know that $1 \in \phi^n_b(F^{ne}_* cR_b)$ for some n. Clear denominators.

iv. We can write $b^{m_1} \in \phi^{n_1}(F_*^{n_1 e} R)$ for some integers n_1, m_1. Show that $b^{2m_1} = (b^{m_1})^2 \in \phi^{l n_1}(F_*^{l n_1 e} R)$ for all integers l.

Hint: Start by showing that $b^{\lceil m_1/p^{n_1 e}\rceil} b^{m_1} \in \phi^{n_1}(F_*^{n_1 e} b^{m_1} R) \subseteq \phi^{2n_1}(F_*^{2n_1 e} R)$. Then do the general case.

v. Now fix $0 \neq c \in R$, and so $b^{m_c} \in \phi^{n_c}(F_*^{n_c e} cR)$ for some integers m_c and n_c. Conclude that $b^{3m_1} \in \phi^N(F_*^{Ne} cR)$ for some integer $N > 0$.

Hint: Apply ϕ^{n_1} until the result is obtained.

vi. Set $a = b^{3m_1}$. Suppose that J is any nonzero ideal such that $\phi(F_*^e J) \subseteq J$. Show that $a \in J$.

vii. Show that $\tau(R, \phi) = \sum_{n \geq 0} \phi^n(F_*^{ne} aR)$ and hence the test ideal exists.

Remark. Remember, ϕ corresponds to a divisor Δ_ϕ. One can define $\tau(R, \Delta_\phi) := \tau(R, \phi)$, note that any two ϕ defining the same divisor are unit multiples. Now, start with R of finite type over a field of characteristic zero with some divisor Δ such that $K_R + \Delta \sim_{\mathbb{Q}} 0$, then for all $p \gg 0$ we have that $(1 - p^e)(K_R + \Delta) \sim 0$ for some e. Hence if one reduces[1] R to R_p, a characteristic $p \gg 0$ ring, one can choose ϕ_p corresponding to Δ_p (the reduction mod p of Δ – done for the primes defining Δ). In this case, it is a theorem that if $\mathcal{J}(R, \Delta)$ is the multiplier ideal, then

$$\mathcal{J}(R, \Delta) \bmod p = \tau(R_p, \Delta_p)$$

for $p \gg 0$. This was proven by Takagi in this generality. Smith, Hara and Hara-Yoshida previously or simultaneously proved similar results.

[1] In the case that R is finite type over $\mathbb{Q}$, we write that $R = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ for some $R_{\mathbb{Z}}$ of finite type over $\mathbb{Q}$, then $R_p = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$.