# NOTES ON CHARACTERISTIC $p$ COMMUTATIVE ALGEBRA JANUARY 11TH, 2017

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## 1. Rings of interest and Frobenius continued

**Example 1.1.** Consider $R = \mathbb{F}_p[a, b]/\langle a^3 - b^2\rangle = \mathbb{F}_p[x^2, x^3] \subseteq \mathbb{F}_p[x]$. Let's try to understand the structure of $R^{1/p}$ as an $R$-module at least for some specific $p$.

We begin in the case that $p = 2$. $R^{1/p} = \mathbb{F}_p[x, x^{3/2}]$. Let's try to write down a minimal set of monomial generators of $R^{1/p}$ over $R$. So a first computation suggests that we need $1, x, x^{3/2}, x^{5/2}$, in particular we need at least four elements and it is easy to see that these are enough. On the other hand, $R^{1/p}$ cannot be a free module of rank 4 since if $R^{1/p} = R^{\oplus d}$ then if $W = R \setminus \{0\}$,

$$W^{-1}R^{1/p} = ((W^p)^{-1}R)^{1/p} = \mathbb{F}_p(x)^{1/p}$$

since any fraction of $k(x)$ can be written as $f(x)/g(x) = f(x)g(x)^{p-1}/g(x)^p \in (W^p)^{-1}R$. But $\mathbb{F}_p(x)^{1/p}$ has rank 2 as a $\mathbb{F}_p(x)$-module. Thus it can't be free since if $R^{1/p}$ needs three generators, if free it must be $R \oplus R \oplus R$, so $W^{-1}R^{1/p}$ would be isomorphic to $k(x) \oplus k(x) \oplus k(x)$.

Ok, how do we really check that $R^{1/p}$ needs at least 4 generators in characteristic $p$ (maybe there is a more clever way to pick just two generators)? One option is to localize. If $M$ is a module which can be generated by $d$ elements, then for any multiplicative set $W$, $W^{-1}M$ can also be generated by $d$ elements (why?) So let's let $W$ be the elements of $R$ not contained in $\langle x^2, x^3\rangle$. Set $S = W^{-1}R$. Then it's enough to show that $S^{1/p}$ is not a free $S$-module. Note $S$ is local with maximal ideal $\mathfrak{m} = \langle x^2, x^3\rangle$. So consider $S^{1/p}/\mathfrak{m}S^{1/p}$, this is a $\mathbb{F}_p = S/\mathfrak{m}$-module of rank equal to the number of generators. We rewrite it as

$$S^{1/p}/\mathfrak{m}S^{1/p} = S^{1/p}/\langle x^2, x^3\rangle_{S^{1/p}}$$

and then obviously $1, x, x^{3/2}, x^{5/2}$ are nonzero in the quotient and they are linearly independent over $k$ since they are different degree, and so $R^{1/p}$ has at least 3 generators as an $R$-module.

**Exercise 1.1.** If $R = k[x^2, x^3]$, verify that $R^{1/p}$ is not a free $R$-module for any prime $p$.

Our work above leads us to a lemma.

**Lemma 1.2.** *Suppose $R$ is a ring and $W$ is a multiplicative set. In this case, $W^{-1}F^e_*R \cong F^e_*(W^{-1}R)$ where the second $F^e_*$ can be viewed as either as an $W^{-1}R$-module or as an $R$-module. This can be viewed as either an isomorphism of rings or of $F^e_*R$-modules.*

*Note this is the same $W^{-1}R^{1/p^e} \cong (W^{-1}R)^{1/p^e}$ if these terminologies make sense (ie, $R$ is a domain).*

*Proof.* There is an obvious map $W^{-1}F^e_*R \to F^e_*(W^{-1}R)$ which sends $1/g.F^e_*r \mapsto F^e_*(r/g^{p^e})$. It is certainly surjective by the argument above since

$$F_*(x/g) = F_*(xg^{p^e-1}/g^{p^e}) = 1/g.F^e_*xg^{p^e-1}$$

it is also easily verified to be linear in all relevant ways. Thus we simply need to check that it is injective. Hence suppose that $F^e_*(r/g^{p^e}) = 0$. This means that there exists $h \in W$ such that $hr = 0$. We want to show that $1.F_* r = 0$ as well in $W^{-1}F^e_* R$. But to show that it suffices to show that $h^{p^e} r = 0$ which obviously follows from $hr = 0$. □

Finally, we will frequently extend ideals via Frobenius, and so we need a notation for that.

**Notation 1.3.** Given an ideal $I = \langle f_1, \dots, f_n \rangle \subseteq R$, we write $I^{[p^e]} := \langle f_1^{p^e}, \dots, f_n^{p^e} \rangle$. Note that

$$I \cdot R^{1/p^e} = (I^{[p^e]} R)^{1/p^e} \text{ and } I \cdot F^e_* R = F^e_* I^{[p^e]}.$$

1.1. **Frobenius and Spec.** Remember, associated to any ring $R$ there is $\operatorname{Spec} R$, the set of prime ideals. It is given the Zariski topology (the set of primes containing any fixed ideal $I$ is closed). To any ring homomorphism $f : R \longrightarrow S$, note we get a (continuous) map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ which sends $Q \in \operatorname{Spec} S$ to $f^{-1}(Q) \in \operatorname{Spec} R$.

**Proposition 1.4.** *The Frobenius morphism $F : R \longrightarrow R$ induces the identity map on $\operatorname{Spec} R$.*

*Proof.* Choose $Q \in \operatorname{Spec} R$. Since $Q$ is an ideal $F(Q) \subseteq Q$. Thus $Q \subseteq F^{-1}(Q)$. Conversely, if $x \in F^{-1}(Q)$, then $x^p \in Q$ and so since $Q$ is prime and hence radical, $x \in Q$. □

The Frobenius is a morphism that acts as the identity on points of Spec but acts by taking powers/roots on functions.

## 2. Regular rings

Commutative algebra can be viewed as "local" algebraic geometry. If you are studying manifolds locally, you are going to be pretty bored, but in algebraic geometry, not everything is a manifold, so we have to study a lot of "singularities". To talk about singularities, we need to have some variant of tangent spaces / some variant of dimension.

**Definition 2.1.** The *Krull dimension* of a ring $R$ is defined to be the maximal length $n$ of a chain of prime ideals

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \subsetneq R.$$

More generally, given any prime ideal $Q$, the height $m$ of $Q$ is the maximal length of a chain of prime ideals

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_m = Q.$$

**Example 2.2.** Here are some examples of dimension.

(a) The dimension of a field $k$ is zero, as it has only one prime ideal $\langle 0 \rangle$. Notice that $\operatorname{Spec} k$ is a single point.
(b) The dimension of a PID (such as $k[x]$) is one since nonzero prime ideals are incomparable. Notice that if $k = \overline{k}$ then $\operatorname{Spec} k$ is a copy of $k$ plus the zero ideal (the generic point).
(c) The dimension of $k[x, y]$ is 2 assuming $k$ is a field. A maximal chain of prime ideals is $0 \subseteq \langle x \rangle \subseteq \langle x, y \rangle$.

We recall some facts about dimension, most of which are easy to prove.

**Lemma 2.3.** *Suppose $R$ is a ring.*

*(a) For any ideal $I$, $\dim R \geq \dim R/I$.*
*(b) For any multiplicative set $W$, $\dim R \geq \dim W^{-1}R$.*
*(c) If $(R, \mathfrak{m})$ is a Noetherian local ring, then $\dim R$ is finite and*

$$\dim R = \textit{ least number of generators of an ideal } I \textit{ with } \sqrt{I} = \mathfrak{m}.$$

*(d) If $R$ is a Noetherian local ring and $x$ is not a zero divisor, then $\dim R - 1 \geq \dim(R/\langle x \rangle)$.*
*(e) If $R$ is a domain of finite type over a field and $Q \in \operatorname{Spec} A$, then*

$$\dim A = \textit{ height} Q + \dim R/Q.$$

*((this condition is close to something called being* catenary*, not all rings satisfy it?!?!?))*

**Definition 2.4.** A local (implicitly Noetherian) ring $(R, \mathfrak{m}, k)$ is called *regular* if $\mathfrak{m}$ can be generated by $\dim R$ number of elements.

Note the minimal number of generators of $\mathfrak{m} = \dim_k \mathfrak{m}/\mathfrak{m}^2$ by Nakayama's lemma. Hence it seems reasonable to study $\mathfrak{m}/\mathfrak{m}^2$.

**Proposition 2.5.** *Suppose that $k = \overline{k}$, $R = k[x_1, \dots, x_n]/I$ is a domain and that $A = R_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m} \subseteq R$. Then $\mathfrak{m}/\mathfrak{m}^2$ is canonically identified with the dual of the tangent space of $V(I) \subseteq k^n$ at the point corresponding to $\mathfrak{m}$.*

*Proof.* Before proving this, let's fix our definition of the tangent space of $V(I)$ at $P = V(\mathfrak{m})$ to be the $k$-vector space of derivations at $P$. Remember, the set of derivations at $P$ is the set of $k$-linear functions $A \longrightarrow k$ which satisfy a Leibniz rule (note, $A$ is basically germs of functions at $P$). Note if $f, g \in \mathfrak{m}$ and $T$ is a derivation, then $T(fg) = f(P)T(g) + g(P)T(f)$ and so $T(fg) = 0$ since $f(P) = g(P) = 0$. It follows from the same argument that $T(\mathfrak{m}^2 A) = 0$.

On the other hand, consider derivations acting on constants $T(1) = T(1 \cdot 1) = 1T(1) + 1T(1) = 2T(1)$. Hence $T(1) = 0$. Thus a derivation is completely determined by its action on $\mathfrak{m}/\mathfrak{m}^2$. In particular, derivations are $k$-linear maps $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow k$ satisfying a Leibniz rule. But all $k$-linear maps $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow k$ satisfy a Leibniz rule, and so the set of derivations is just the $k$-vector space dual of $\mathfrak{m}/\mathfrak{m}^2$. □

In other words, the dimension of the tangent space is just $\dim_k \mathfrak{m}/\mathfrak{m}^2$. Thus to call a ring regular is exactly the same as requiring the tangent space to have the same dimension as the ambient space (the Spec of the germ of functions).