NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 13TH, 2017

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1. Regular rings continued

Recall the following.

Definition 1.1. A local (implicitly Noetherian) ring $(R, \mathfrak{m}, k)$ is called *regular* if $\mathfrak{m}$ can be generated by $\dim R$ number of elements.

The dimension of the tangent space is just $\dim_k \mathfrak{m}/\mathfrak{m}^2$. Thus to call a ring regular is exactly the same as requiring the tangent space to have the same dimension as the ambient space (the Spec of the germ of functions).

Fortunately, for most rings there is a convenient way to check whether a local ring is regular (rather than messing about with derivations).

Proposition 1.2 (Essentially taken from [**?**]). *Suppose $k = \overline{k}$ and $R = k[x_1, \dots, x_n]/I = S/I$ is a domain with $I = \langle f_1, \dots, f_t \rangle$. If $P \subseteq R$ is a maximal ideal then R_P is regular if and only if the Jacobian matrix $||(\partial f_i/\partial x_j)(P)||$ has rank $n - r$ where $r = \dim R$.*

Proof. Let $\mathfrak{m} = \langle g_1 = x_1 - a_1, \dots, g_n = x_n - a_n \rangle$ be the ideal corresponding to P in the polynomial ring S. Consider the map $\rho : S \longrightarrow k^n$ defined by

$$\rho(f) = \Big(\frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P) \Big).$$

Note $\rho(g_i)$ form a basis for k^n and that $\rho(\mathfrak{m}^2) = 0$. Hence we get $\rho : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow k^n$. This is an isomorphism (by direct computation, we are still in the polynomial ring setting). Now, $\rho(I)$ has k-vector space dimension equal to the rank of the Jacobian matrix at that point. Thus the rank of the Jacobian matrix is the same as $\dim_k (I + \mathfrak{m}^2)/\mathfrak{m}^2$.

The dimension of the tangent space of $V(I)$ at P however is

$$\dim_k (\mathfrak{m}/I)/(\mathfrak{m}/I)^2 = \dim_k (\mathfrak{m}/(\mathfrak{m}^2 + I)).$$

It follows that

$$n = \dim_k (\mathfrak{m}/(\mathfrak{m}^2 + I)) + \dim_k (I + \mathfrak{m}^2)/\mathfrak{m}^2 = (\text{tangent space dimension}) + (\text{Jacobian rank}).$$

The result follows immediately. □

Remark 1.3 (Warning!). This only works over algebraically closed fields since we can write their maximal ideals in a very special way.

The above gets us a way to identify the locus where a ring (of finite type over an algebraically closed field) is regular.

Algorithm 1.4. Given $k = \overline{k}$ and a domain $R = k[x_1, \dots, x_n]/I$ of dimension r, compute the following algorithm to find a canonical ideal defining the locus where R is not regular (where it localizations are not regular).

(a) Let $M = ||\partial f_i / \partial x_j||$ denote the Jacobian matrix where $I = \langle f_1, \dots, f_t \rangle$.
(b) Let J the ideal defined by the determinants of all $(n-r) \times (n-r)$-minors of J.
(c) $J + I$ is the desired ideal.

By the Nullstellensatz, a maximal ideal contains $J + I$ if and only if it is in $V(I)$ and if $M(P)$, the evaluation of M at P, has rank $< n - r$.

Example 1.5. Consider $k[x, y, z]/\langle x^2 y - z^2 \rangle$, a ring of dimension 2. The Jacobian matrix has only three entries, $2xy, x^2, 2z$, and we want to form the ideal made up by the 1×1 determinants. $J + I = \langle 2xy, x^2, 2z, x^2 y - z^2 \rangle$. We have two cases.

If $\operatorname{char} k = 2$, $J + I = \langle x^2, z^2 \rangle$. If $\operatorname{char} k \neq 2$, then $J + I = \langle xy, x^2, z \rangle$. Notice that in both cases, the radical of the ideal $\sqrt{J + I} = \langle x, z \rangle$, so the singular *locus* is the same, even though the Jacobian ideals are somewhat different.

However, it can even happen that an equation can define a ring which is singular in some characteristics but nonsingular in others.

Example 1.6. Consider $k = \mathbb{F}_p(t)$ and $R = k[x]/\langle x^p - t \rangle$. Obvious R is regular since it is a field $\cong k(t^{1/p})$. The Jacobian matrix has a single entry, $\frac{\partial}{\partial x} x^p - t = 0$ and hence the rank of the Jacobian matrix is 0. On the other hand $n = 1$ since there is one variable and $r = 0$ since a field has dimension zero. In particular,

$$0 = (\text{Jacobian rank}) \neq 1 - 0 = 1.$$

Definition 1.7. A Noetherian ring is called *regular* if all of its localizations at prime ideals are regular local rings.

This is a bit confusing, since it is not clear yet whether a regular local ring satisfies this property.

Theorem 1.8 (Serre, Auslander-Buchsbaum)**.** *If R is a regular local ring and $Q \in \operatorname{Spec} R$ is a prime ideal, then R_Q is a regular local ring.*