NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 18TH, 2017

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1. Regular rings continued

Definition 1.1. A Noetherian ring is called *regular* if all of its localizations at prime ideals are regular local rings.

This is a bit confusing, since it is not clear yet whether a regular local ring satisfies this property.

Theorem 1.2 (Serre, Auslander-Buchsbaum)**.** *If R is a regular local ring and $Q \in \operatorname{Spec} R$ is a prime ideal, then R_Q is a regular local ring.*

The point is that a Noetherian local ring is regular if and only if it has finite global dimension (the projective dimension of every module is finite and in fact $\leq \dim$).

Exercise 1.1. Use the fact that a Noetherian local ring is regular if and only if it has finite global dimension to prove that if R is a regular local ring, then so is R_Q for any $Q \in \operatorname{Spec} R$.

Here are some facts about regular rings which we will use without proof (including the one just mentioned above). Recall first the following definitions.

Definition 1.3. An R-module M is called *flat* if the functor $\_\_ \otimes_R M$ is (left) exact. A module M is called *projective* if the functor $\operatorname{Hom}_R(M, \_\_)$ is (right) exact.

Theorem 1.4. *(a) A regular ring is normal*[1].

(b) If R is regular so is $R[X]$ and $R[\![x]\!]$.

(c) A regular local ring is a UFD.

(d) If $(A, \mathfrak{m}) \subseteq (B, \mathfrak{n})$ is a local[2] *extension of Noetherian local rings with A regular and B Cohen-Macaulay (for example, regular) and if we have $\dim B = \dim A + \dim(B/\mathfrak{m}B)$, then B is a flat A-module.*

(e) A local ring $(R, \mathfrak{m})$ is regular if and only if the global dimension of R is finite (in other words, the projective dimension of any module is finite, and in particular $\leq \dim R$. It is sufficient to find the global dimension of $k = R/\mathfrak{m}$.).

Proof. See for example [**?**, Theorem 19.2, Theorem 19.4, Theorem 19.5, Theorem 20.3, Theorem 23.1]. □

Let's give some examples which show that the conditions of (d) are sharp.

[1] This means that R is its own integral closure in its ring of fractions, recall $x \in K(R)$ is integral over R if it satisfies a monic equation $x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$ for $r_i \in R$.

[2] This just means that $\mathfrak{m} \subseteq \mathfrak{n}$, note $\mathbb{Z} \subseteq \mathbb{Q}$ is not local.

Example 1.5 (Non-flat local extension with the wrong dimensions)**.** Consider $R = k[x,y]_{\langle x,y\rangle} \subseteq k[x,y/x]_{\langle x,y/x\rangle} = S$. It is easy to check that this is a local extension since $\langle x, y/x\rangle_S \cap R$ contains both x and Y and so must equal $\langle x,y\rangle_R$. Now, $\dim R = 2$ (since we just localize $\mathbb{A}^2$ at the origin) and likewise $\dim S = 2$ since S is abstract isomorphic to R. Finally, let $\mathfrak{m} = \langle x,y\rangle_R$ and consider $\mathfrak{m}S = \langle x,y\rangle_S = \langle x\rangle_S = xS$. Thus $S/\mathfrak{m}S = S/xS = k[y/x]$ and so $\dim S/\mathfrak{m}S = 1$. Thus note that

$$2 = \dim S \neq \dim R + \dim(S/\mathfrak{m}S) = 2 + 1 = 3$$

and so Theorem 1.4(d) does not apply. Let's next verify that S is *not* a flat R-module. Consider the injection of R-modules,

$$k[y] = R/xR \hookrightarrow R/xR = k[y].$$

Example 1.6 (Non-flat local extensions where the base is not regular)**.** Consider $R = k[x^2, xy, y^2]_{\langle x^2,xy,y^2\rangle} \subseteq k[x,y]_{\langle x,y\rangle} = S$ and set $\mathfrak{m} = \langle x^2, xy, y^2\rangle_R$ and $\mathfrak{n} = \langle x,y\rangle_S$. We view S as an R-module.

First note that $2 = \dim S = \dim R + \dim(S/\mathfrak{m}S) = 2 + 0$. On the other hand R is not smooth since $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3 > 2$.

Now we show that S is not a free R-module (next lecture, we will see that flat and locally free are equivalent for finitely generated modules, so this is relevant). Indeed, consider $S/\mathfrak{m}S = S/\langle x^2, xy, y^2\rangle_S$, this is clearly a 3-dimensional k-vector space (with basis $\{\overline{1}, \overline{x}, \overline{y}\}$) and so S needs at least 3 generators to generate as an R-module. On the other hand when we work at the generic point, consider $k(x^2, xy, y^2) \subseteq k(x,y)$. Note that $k(x^2, xy, y^2) = k(x^2, xy)$ since $y^2 = (xy)^2/x^2$ and hence we only need consider $k(x^2, xy) \subseteq k(x,y)$. This is obviously a rank 2 field extension since we only need to adjoin x which satisfies the quadratic polynomial $X^2 - x^2$. In particular, we see that S is not a free R module since it requires 3 generators but has generic rank 2.

Our goal for the short term is to prove the following theorem of Kunz.

Theorem 1.7 (Kunz)**.** *If R is a Noetherian ring of characteristic $p > 0$, then R is regular if and only if F_*R is a flat R-module.*

2. Some notes on Kunz' theorem and the easy direction

Remember, we are trying to show that R is regular if and only if F_*R is a flat R-module. We will give two proofs of the easy direction (that regular implies flat). The first is quite easy but not very illuminating (especially since it relies on facts we haven't proven). Second, we will give essentially Kunz's original proof which I think yields quite a bit more intuition.

Proof #1. We suppose that R is regular and will show that $R^{1/p}$ is a flat R-module. Note that $(R^{1/p})_\mathfrak{m} = (R_\mathfrak{m})^{1/p}$ since inverting a pth power is the same as inverting the element. Thus, since flatness can be checked locally, we may localize R (and $R^{1/p}$) at a maximal ideal and from here on out assume that $(R, \mathfrak{m})$ is local.

Next consider the extension $R \subseteq R^{1/p}$. This is a local extension since $\mathfrak{m} \subseteq \mathfrak{m}^{1/p}$ and both R and $R^{1/p}$ are regular rings. It follows from Theorem 1.4(d) that $R^{1/p}$ is a flat R module and so we are done. □