NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 20TH, 2017

KARL SCHWEDE

1. Some notes on Kunz' theorem and the easy direction

Remember, we are trying to show that R is regular if and only if F_*R is a flat R-module. We will give two proofs of the easy direction (that regular implies flat). The first is quite easy but not very illuminating (especially since it relies on facts we haven't proven). Second, we will give essentially Kunz's original proof which I think yields quite a bit more intuition.

Proof #1. We suppose that R is regular and will show that $R^{1/p}$ is a flat R-module. Note that $(R^{1/p})_{\mathfrak{m}} = (R_{\mathfrak{m}})^{1/p}$ since inverting a pth power is the same as inverting the element. Thus, since flatness can be checked locally, we may localize R (and $R^{1/p}$) at a maximal ideal and from here on out assume that $(R, \mathfrak{m})$ is local.

Next consider the extension $R \subseteq R^{1/p}$. This is a local extension since $\mathfrak{m} \subseteq \mathfrak{m}^{1/p}$ and both R and $R^{1/p}$ are regular rings. It follows from **??**(d) that $R^{1/p}$ is a flat R module and so we are done. □

Let's consider in a bit more detail the difference between flat, projective and free modules. It is easy to see that free modules are both flat and projective (think about how Hom and $\otimes$ work).

Lemma 1.1. *If M is a projective module over a local ring $(R, \mathfrak{m})$, then M is free.*

Proof. We only prove the case when M is finitely generated, the general case is hard and due to Kaplansky.

Let $n = \dim_{R/\mathfrak{m}} M/\mathfrak{m}M$. Then, by Nakayama's lemma, we have a surjection $\kappa : R^{\oplus n} \twoheadrightarrow M$. Since M is projective, we have a map $\sigma : M \longrightarrow R^{\oplus n}$ so that $\kappa \circ \sigma : M \longrightarrow R^{\oplus n} \longrightarrow M$ is the identity (and in particular σ is injective). It follows that

$$M/\mathfrak{m}M \xrightarrow{\overline{\sigma}} (R/\mathfrak{m})^{\oplus n} \xrightarrow{\overline{\kappa}} M/\mathfrak{m}M$$

is also the identity. Thus $\overline{\sigma}$ must also be an isomorphism (since it is an injective map between vector spaces of the same dimension). Hence by Nakayama's lemma, $\sigma : M \longrightarrow R^{\oplus n}$ is also surjective. But thus σ is both injective and surjective and hence an isomorphism, which proves that M is free. □

Next let's verify that flatness is local (this was asserted without proof earlier).

Lemma 1.2. *If M is an R-module, and $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module, for every maximal ideal $\mathfrak{m} \subseteq R$, then M is flat.*

Proof. Suppose that $0 \longrightarrow A \longrightarrow B$ is an injection of R-modules. Consider $K = \ker(A \otimes M \longrightarrow B \otimes M)$, we will show that $K = 0$. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes M \longrightarrow B \otimes M,$$

since localization is exact, for every maximal ideal $\mathfrak{m} \subseteq R$, we have that

$$0 \longrightarrow K \otimes R_{\mathfrak{m}} \longrightarrow A \otimes M \otimes R_{\mathfrak{m}} \longrightarrow B \otimes M \otimes R_{\mathfrak{m}},$$

is exact. But this is the same as saying that

$$0 \longrightarrow K_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}} \otimes M_{\mathfrak{m}} \longrightarrow B_{\mathfrak{m}} \otimes M_{\mathfrak{m}}$$

is exact. Since $M_{\mathfrak{m}}$ is flat by hypothesis, this implies that $K_{\mathfrak{m}} = 0$, and this holds for every maximal ideal. Thus $K = 0$. □

Next let's show that finite flat modules are projective.

Lemma 1.3. *If R is Noetherian and M is a finite R-module that is flat, then M is projective.*

Proof. Since localization commutes with Hom from finitely presented modules, and exactness may be checked locally, we may assume that $(R, \mathfrak{m})$ is a local ring. In this case, we will show that M is free. Choose a minimal generating set for M and the corresponding surjection $R^{\oplus t} \longrightarrow M \longrightarrow 0$ with kernel K (which is finitely generated). We tensor with $R/\mathfrak{m}$ to obtain

$$\operatorname{Tor}_1(M, R/\mathfrak{m}) \longrightarrow K \otimes R/\mathfrak{m} \longrightarrow (R/\mathfrak{m})^{\oplus t} \xrightarrow{\alpha} M \otimes R/\mathfrak{m} \longrightarrow 0$$

Since we picked a minimal generating set for M, α is bijective. Since M is flat, $\operatorname{Tor}_1(M, \mathbb{R}/\mathfrak{m}) = 0$ and hence $K \otimes R/\mathfrak{m} = K/\mathfrak{m}K = 0$ and so $K = \mathfrak{m}K$. By Nakayama's lemma this implies that K is zero. □

Note we didn't really need that R was Noetherian above, we just needed M to be finitely presented.

We now give another proof of the "easy" direction of Kunz's theorem, that regular implies Frobenius is flat. We have already checked this for polynomial rings over perfect fields, and now we want to essentially reduce to that case.

To do that, we need *completion.* Suppose that R is a ring and I is an ideal. Then

$$\widehat{R} := \lim_{\leftarrow} R/I^n$$

is called the completion of R with respect to the I-adic topology (the powers of I form a neighborhood basis of 0). Most often, $(R, \mathfrak{m})$ is a local ring and we are completing with respect to the maximal ideal $\mathfrak{m}$. In this case, $R/\mathfrak{m}$ is the residue field, $R/\mathfrak{m}^2$ records first order tangent information, $R/\mathfrak{m}^3$ records second order tangent information, etc. Thus $\widehat{R}$ somehow knows all the tangent information around R.

Definition 1.4. A local ring is called *complete* if it is complete (for example, equal to its own completion) with respect to the maximal ideal.

Example 1.5. It is easy to verify that if $R = k[x_1, \ldots, x_n]$ is a polynomial ring and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ is the ideal defining the origin. Then $\widehat{R} = k[\![x_1, \ldots, x_n]\!]$, formal power series in the x_i. Note this ring is still Noetherian.

Example 1.6. Consider $R = k[x, y]/\langle y^2 - x^3 + x \rangle$ and complete with respect to $\mathfrak{m} = \langle x, y \rangle$. Now, any polynomial in x, y can be rewritten, say up through some fixed degree n, as a power series only in y (for example, replace all the xs with $x^3 - y^2$, and repeat, until the x degree is too high). It follows that the completion is isomorphic to $k[\![y]\!]$ (a picture will explain why this is reasonable).

In particular, even though the completion of R is a powerseries ring in one variable, the localization $R_{\langle x,y\rangle}$ is not.

Completion should be thought of as one analog of a local analytic neighborhood (localization still remembers too much about the global geometry for some applications). In particular, the follow theorem makes this precise.

Theorem 1.7. (Cohen Structure Theorem, [Mat89, Section 29]) *Suppose that* $(R, \mathfrak{m}, k)$ *is a complete local Noetherian ring containing a field* F*. Then* $R \cong k[\![x_1, \ldots, x_n]\!]/I$*. Furthermore, if* R *is regular then* $R \cong k[\![x_1, \ldots, x_n]\!]$*.*

We are not going to prove this, but we will take it on faith. Note that hard part is to show that R actually contains a copy of k (this copy is not unique in characteristic $p > 0$). Also note that the x_is are a set of generators of the maximal ideal $\mathfrak{m}$.

We need one other lemma.

Lemma 1.8. [Mat89, Theorem 8.8] *If* $(R, \mathfrak{m})$ *is a local ring with completion* $\widehat{R}$*, then* $\widehat{R}$ *is a faithfully flat* R*-module.*

References

[Mat89] H. Matsumura: *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR1011461 (90i:13001)