NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 23RD, 2017

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1. Some notes on Kunz' theorem and the easy direction continued

What the Cohen-Structure theorem lets us reduce the problem of flatness of F_*R over R to flatness of $F_*\widehat{R}$ over $\widehat{R}$, and for regular rings, we have just reduced to the power series case (which is essentially the same as the

Theorem 1.1 (Kunz)**.** *If R is Noetherian and regular then F_*R is a flat R-module.*

Proof. Since flatness can be checked locally, as we showed above, we may assume that R is local. We know that $\widehat{R} \cong k[\![x_1, \dots, x_n]\!]$. There is an induced map

$$\widehat{R} \longrightarrow \lim_{\leftarrow}(F_*R)/\mathfrak{m}^n(F_*R) = \lim_{\leftarrow}(F_*R)/(F_*(\mathfrak{m}^n)^{[p]}R) = F_* \lim_{\leftarrow}(R/(\mathfrak{m}^n)^{[p]}) \cong F_*\widehat{R}$$

where the final equality is due to the fact that $(\mathfrak{m}^n)^{[p]}$ defines the same topology of $\mathfrak{m}^n$ (the powers are cofinal with each other). Note that this map is the Frobenius map. We have the following diagram:

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{F} & F_*\widehat{R} \\ \uparrow & & \uparrow \\ R & \xrightarrow[F]{} & F_*R \end{array}$$

The top horizontal arrow is flat by direct computation that we now do. Write $R = k[\![x_1, \dots, x_n]\!]$. Then notice that $F_*R = R^{1/p} = k^{1/p}[\![x_1^{1/p}, \dots, x_n^{1/p}]\!]$ and so we can factor $R \subseteq R^{1/p}$ as

$$k[\![x_1, \dots, x_n]\!] \subseteq k[\![x_1^{1/p}, \dots, x_n^{1/p}]\!] \subseteq k^{1/p}[\![x_1^{1/p}, \dots, x_n^{1/p}]\!]$$

The first extension is flat because it is free (using the same basis you are writing down in the homework). The second extension is flat because it is just a residue field extension (technically, tensor up with $\otimes_k k^{1/p}$ and then complete again, remember completion of Noetherian rings yields flat extensions). The vertical arrows are flat since completion is always flat (note the right vertical arrow is just F_* of the left arrow). It follows that $F_*\widehat{R}$ is flat over R. We need to show that the bottom horizontal arrow is flat, this is a basic commutative algebra fact but lets prove it.

Suppose that $M' \hookrightarrow M$ injects but $M' \otimes_R F_*R \longrightarrow M \otimes_R F_*R$ does not and so let K be the nonzero kernel so that we have an exact sequence $0 \longrightarrow K \longrightarrow M' \otimes_R F_*R \longrightarrow M \otimes_R F_*R$

of F_*R-modules. We tensor this with $\_\_ \otimes_{F_*R} F_*\widehat{R}$ to obtain

$$\begin{array}{ccccccc} 0 \longrightarrow & K \otimes_{F_*R} F_*\widehat{R} & \longrightarrow & M' \otimes_R F_*R \otimes_{F_*R} F_*\widehat{R} & \longrightarrow & M \otimes_R F_*R \otimes_{F_*R} F_*\widehat{R} \\ & \updownarrow \sim & & \updownarrow \sim & & \updownarrow \sim \\ 0 \longrightarrow & K \otimes_{F_*R} F_*\widehat{R} & \longrightarrow & M' \otimes_R F_*\widehat{R} & \longrightarrow & M \otimes_R F_*\widehat{R} \end{array}$$

Since completion is *faithfully* flat, $K \otimes_{F_*R} F_*\widehat{R} \neq 0$ hence $M' \otimes_R F_*\widehat{R} \to M \otimes_R F_*\widehat{R}$ is not injective. But the contradicts the flatness of $F_*\widehat{R}$ over R. $\square$

2. Macaulay2

We head downstairs...

References