NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 25TH, 2017

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1. A crash course in using derived categories

Before doing the other direction of the proof, it will be very helpful if we learn a little bit about the derived category.

In a nutshell, taking Hom and Ext is very useful, but dealing with individual cohomology groups can be a hassle.

Solution: Deal with the complexes instead!

Definition 1.1. A *complex* of R-modules (or $\mathscr{O}_X$-modules if you prefer) is a collection of $\{C^n\}_{n\in\mathbb{Z}}$ of R-modules plus maps $d^n : C^n \longrightarrow C^{n+1}$ such that $d^{i+1} \circ d^i = 0$.

$$\ldots \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \ldots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \ldots$$

A complex is *bounded below* if $C^i = 0$ for $i \ll 0$, it is *bounded above* if $C^i = 0$ for $i \gg 0$, and it is *bounded* if $C^i = 0$ for $|i| \gg 0$.

Remark 1.2. In a *chain complex*, the differentials take C_i to C_{i-1}, we will deal exclusively with complexes however.

There are some problems. The category of complexes isn't quite right, so we fix it. We only consider morphisms of complexes up to homotopy equivalence (two maps of complexes are homotopic if their difference is null homotopic), and we *declare* two complexes to be isomorphic if there is a map between them which gives us an isomorphism on cohomology (formally add an inverse map to our category).

Examples 1.3. Here are some examples you hopefully all are familiar with.

(a) Given a module M, we view it as a complex by considering it in degree zero (all the differentials of the complex are zero) and all the other terms in the complex.

(b) Given any complex $C^\bullet$, (like a modules viewed as a complex as above), we can form another complex by shifting the first complex $C[n]^\bullet$. This is the complex where $(C[n])^i = C^{i+n}$ and where the differentials are shifted likewise and multiplied by $(-1)^n$. Note this shifts the complex n spots to *the left*.

(c) Given a module M, and a projective resolution

$$\ldots \longrightarrow P^{-n} \longrightarrow \ldots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

it is easy to see that there is a map $P^\bullet \longrightarrow M$ and this is a map of complexes in the sense that M is a complex via (a).

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & P^{-n} & \longrightarrow & \ldots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & 0 & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \ldots \end{array}$$

This map is a quasi-isomorphism (an isomorphism in the derived category).

(d) Given a module M and an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

we get a map of complexes $M \to I^{\bullet}$

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \dots \end{array}$$

which is also a quasi-isomorphism (again viewing M as a complex via (a)).

(e) Given two modules M and N, we form $\mathbf{R}\operatorname{Hom}_R(M, N)$. This is the complex whose cohomologies are the $\operatorname{Ext}^i(M, N)$. It is computed by either taking a projective resolution of M or an injective resolution of N. Note that while you get different complexes in either of those cases, it turns out the resulting objects in the derived category are isomorphic (in the derived category).

(f) Given two modules M and N, we form $M \otimes^L_R N$, the cohomologies of this complex are the $\operatorname{Tor}^i_R(M, N)$. It is obtained by taking a projective resolution of M or N.

(g) Note that not every quasi-isomorphism between complexes is invertible. Indeed, consider

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

This obviously induces a quasi-isomorphism since the top row is a projective resolution of the bottom, but the map of complexes is not invertible. In the *derived* category, we formally adjoin an inverse morphism.

(h) Not every pair of complexes with isomorphic cohomologies are quasi-isomorphic, indeed consider the complexes

$$\dots \to 0 \to \mathbb{C}[x, y] \xrightarrow{0} \mathbb{C} \to 0 \to \dots$$

and

$$\dots \to 0 \to \mathbb{C}[x, y]^{\oplus} \xrightarrow{[x,y]} \mathbb{C}[x, y] \to 0 \to \dots.$$

It is easy to see that they have isomorphic cohomologies. Some discussion of the fact that these complexes are not quasi-isomorphic can be found in the responses to this question on `math.stackexchange`, [hf].

Remark 1.4. There are different ways to enumerate things, but complexes have maps that go left to right. Thus a projective resolution of a module M has entries only in *negative* degrees. Thus when I write $\operatorname{Tor}^i_R(M, N)$ above, the i that can have interesting cohomology are the $i \leq 0$.

Definition 1.5. The *derived category of R-modules* denoted $D(R)$ is the category of complexes with morphisms defined up to homotopy and with quasi-isomorphisms formally inverted.

If we look at the full subcategory of complexes bounded above, and then construct the derived category as above, the result is denoted by $D^-(R)$. From bounded below complexes, we construct $D^+(R)$. Finally, if the complexes are bounded on both sides the result is denoted by $D^b(R)$.

2. Triangulated categories

Derived categories are *not* an Abelian category, short exact sequences don't exist, but we have something almost as good, exact triangles. In particular, the derived category is a triangulated category.

Remark 2.1. The notation from the following axioms is taken from [Wei94] (you can find different notation on for instance Wikipedia).

Definition 2.2 (Triangulated categories)**.** A *triangulated category* is an additive[1] category with a fixed automorphism T equipped with a distinguished set of triangles and satisfying a set of axioms (below). A *triangle* is an ordered triple of objects (A, B, C) and morphism $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$, $\gamma : C \longrightarrow T(A)$,

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A).$$

A *morphism of triangles* $(A, B, C, \alpha, \beta, \gamma) \longrightarrow (A', B', C', \alpha', \beta', \gamma')$ is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \xrightarrow[\gamma']{} & T(A') \end{array}$$

We now list the required axioms to make a triangulated category.

(a) The triangle $A \longrightarrow A \xrightarrow{0} 0 \xrightarrow{0} T(A)$ is one of the distinguished triangles.
(b) A triangle isomorphic to one of the distinguished triangles is distinguished.

The rest of the axioms will be discussed *next time...*

References

[hf] B. F. (http://math.stackexchange.com/users/56960/brian fitzpatrick): *Proving two complexes are not quasi-isomorphic*, URL:http://math.stackexchange.com/q/373085 (version: 2013-04-26).

[Wei94] C. A. Weibel: *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)

[1]Hom sets are Abelian groups and composition is bilinear.