NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 27TH, 2017

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1. Triangulated categories continued...

We remind ourselves of the axioms of triangulated categories we discussed last time.

Definition 1.1 (Triangulated categories)**.** A *triangulated category* is an additive[1] category with a fixed automorphism T equipped with a distinguished set of triangles and satisfying a set of axioms (below). A *triangle* is an ordered triple of objects (A, B, C) and morphism $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$, $\gamma : C \longrightarrow T(A)$,

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A).$$

A *morphism of triangles* $(A, B, C, \alpha, \beta, \gamma) \longrightarrow (A', B', C', \alpha', \beta', \gamma')$ is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & T(A) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle T(f)} \\ A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \xrightarrow[\gamma']{} & T(A') \end{array}$$

We now list the required axioms to make a triangulated category.

(a) The triangle $A \longrightarrow A \xrightarrow{0} 0 \xrightarrow{0} T(A)$ is one of the distinguished triangles.
(b) A triangle isomorphic to one of the distinguished triangles is distinguished.
(c) Any morphism $A \longrightarrow B$ can be embedded into one of the distinguished triangles $A \longrightarrow B \longrightarrow C \longrightarrow T(A)$.
(d) Given any distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A)$, then both

$$B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A) \xrightarrow{-T(\alpha)} T(B)$$

and

$$T^{-1}C \xrightarrow{-T^{-1}(\gamma)} A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

are also distinguished.
(e) Given distinguished triangles with maps between them as pictured below, so that the left square commutes,

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & T(A) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle \exists h}\,\vdots & & \downarrow{\scriptstyle T(f)} \\ A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \xrightarrow[\gamma']{} & T(A') \end{array}$$

[1]Hom sets are Abelian groups and composition is bilinear.

then the dotted arrow also exists and we obtain a morphism of triangles.

(f) We finally come to the feared *octahedral axiom*. Given objects A, B, C, A', B', C' and three distinguished triangles:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{\partial} T(A)$$

$$B \xrightarrow{v} C \xrightarrow{x} A' \xrightarrow{i} T(B)$$

$$A \xrightarrow{v \circ u} C \xrightarrow{y} B' \xrightarrow{\delta} T(A)$$

then there exists a fourth triangle

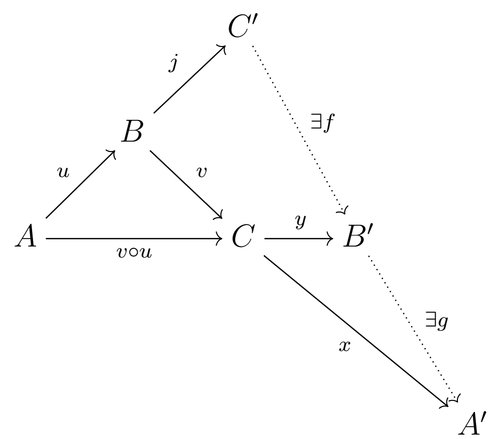
$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{(T(j)) \circ i} T(C')$$

so that we have

$$\partial = \delta \circ f, x = g \circ y, y \circ v = f \circ j, u \circ \delta = i \circ g.$$

These can be turned into a nice octagon (with these equalities being commuting faces) that I am too lazy to LaTeX.

Remark 1.2. It is much easier to remember the octahedral axiom (without the compatibilities at least) with the following diagram.



Any of the derived categories we have discussed are triangulated categories with $T(\bullet) = \bullet[1]$. The main point is if we have a morphism of complexes, $A^{\bullet} \xrightarrow{\alpha} B^{\bullet}$, then we can always take the cone $C(\alpha)^{\bullet} = A[1]^{\bullet} \oplus B^{\bullet}$ with differential

$$C^i = A^{i+1} \oplus B^i \xrightarrow{-d_A^{i+1}, \alpha^i + d_B^i} A^{i+2} \oplus B^{i+1}$$

Exercise 1.1. Verify that this really is a complex.

Exercise 1.2. Suppose that $0 \to A^{\bullet} \xrightarrow{\alpha} B^{\bullet} \xrightarrow{\beta} D^{\bullet} \to 0$ is an exact sequence of complexes. Show that $D^{\bullet}$ is quasi-isomorphic to $C(\alpha)^{\bullet}$.

Then we have $A^{\bullet} \xrightarrow{\alpha} B^{\bullet} \xrightarrow{\beta} C(\alpha)^{\bullet} \gamma A[1]^{\bullet}$ a distinguished triangle where β and γ are given by maps to and projecting from the direct summands that make up $C(\alpha)^{\bullet}$. Note that morphisms in the derived category are more complicated than maps between complexes (since we might have formally inverted some quasi-isomorphisms) but this still is enough for our purposes since the cone of a quasi-isomorphism is exact.

We now do a worksheet!

References