NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA JANUARY 30TH, 2017

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1. Triangulated categories continued...

Fact 1.1. Given a triangle $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet$ in the derived category of R-modules, taking cohomology yields a long exact sequence

$$\ldots \longrightarrow h^{i-1}(C^\bullet) \longrightarrow h^i(A^\bullet) \longrightarrow h^i(B^\bullet) \longrightarrow h^i(C^\bullet) \longrightarrow h^{i+1}(A^\bullet) \longrightarrow \ldots$$

Exercise 1.1. Verify that fact.

Exercise 1.2. Suppose that

$$A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \xrightarrow{0} T(A^\bullet)$$

is a distinguished triangle in $D(R)$. Show that $B^\bullet \simeq_{\mathrm{qis}} A^\bullet \oplus B^\bullet$ compatibly so that α and β are identified with the canonical inclusion and projections.

In particular, show that there exist maps $p : B^\bullet \longrightarrow A^\bullet$ and $s : C^\bullet \longrightarrow B^\bullet$ so that $p \circ \alpha$ is the identity on $A^\bullet$ and that $\beta \circ s$ is the identity on $C^\bullet$.

2. Common functors on our derived categories

Suppose we are forming the derived category $D(R)$ of the category of R-modules for some ring R. We have lots of functors we like to apply to R-modules, notably Hom and $\otimes$ but also things like Γ_I (the submodule of things killed by a power of I). Associated to any of these functors we get derived functors, as follows.

Derived functors are functors between triangulated categories which preserve the triangulation structure (ie, send triangles to triangles and commute with the $T(\bullet)/[\bullet]$ operation) and which satisfy a certain universal property which we won't need too much (see for example [Wei94, Section 10.5] for details). The point for us is that derived functors exist for the functors we care about.

Lemma 2.1. [Wei94, Corollary 10.5.7] *Suppose $F : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ is an additive functor which takes R-modules to S-modules. Then the right derived functors $\mathbf{R}F : D^+(R) \longrightarrow D(S)$ are morphisms between triangulated categories and can be computed by $\mathbf{R}F(C^\bullet) = F(I^\bullet)$ where $I^\bullet$ is a complex of injectives quasi-isomorphic to $C^\bullet$. In particular, $h^i\mathbf{R}F(C^\bullet) = \mathbb{R}^iF(C^\bullet)$.*

Likewise, the left derived functors $\mathbf{L}F : D^-(R) \longrightarrow D(S)$ can be computed by $\mathbf{R}F(C^\bullet) = F(P^\bullet)$ where $P^\bullet$ is a complex of projectives quasi-isomorphic to $C^\bullet$.

If you are really Hom'ing or tensoring two complexes together, you typically need to actually compute this by forming the associated double complex and then taking the total complex, see for example page 8 of [Wei94]. For example if $M^\bullet$ and $N^\bullet$ are complexes made up of projectives (or at least one of them is), then the total complex of the double complex represents the object $M^\bullet \otimes^{\mathbf{L}}_R N^\bullet$.

Notably, we have

- $\mathbf{R}\operatorname{Hom}_R(A^{\bullet}, B^{\bullet})$ can be computed by taking a complex of projectives quasi-isomorphic to $A^{\bullet} \in D^-(R)$ or a complex of injectives quasi-isomorphic to $B^{\bullet} \in D^+(R)$. Note if A, B are modules, then $h^i\mathbf{R}\operatorname{Hom}(A, B) = \operatorname{Ext}^i(A, B)$
- $A^{\bullet} \otimes_R^{\mathbf{L}} B^{\bullet}$ can be computed by taking a complex of projectives quasi-isomorphic to either $A^{\bullet}$ or $B^{\bullet}$ in $D^-(R)$. Note if A, B are modules, then $h^{-i}(A \otimes_R^{\mathbf{L}} B) = \operatorname{Tor}_i^R(A, B)$.
- For any ideal $I \subseteq R$, recall that $\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \gg 0\}$. Then $\mathbf{R}\Gamma_I(A^{\bullet})$ is computed by finding a complex of injectives quasi-isomorphic to $A^{\bullet} \in D^+(R)$. Note that if A is a module, then $h^i\mathbf{R}\Gamma_I(A) = H_I^i(A)$ is just local cohomology.

The rest of the chapter is devoted to how these functors play with each other.

Theorem 2.2 (Composition of derived functors, left-exact case)**.** *Given left exact functors $G : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ and $F : \mathscr{M}\mathrm{od}(S) \longrightarrow \mathscr{M}\mathrm{od}(T)$ (or suitable Abelian categories with enough injectives), and suppose that G sends injective objects to F-acyclic objects, then $\mathbf{R}F \circ \mathbf{R}G \cong \mathbf{R}(F \circ G)$ as functors from $D^+(R) \longrightarrow D^+(T)$.*

For a more general statement, see [Wei94, Theorem 10.8.2]. Things that imply the above, make a lot of the formulas we already know relating Hom and $\otimes$ and other functors hold in the derived category as well.

We list some of them here without proof, see for example [Wei94, Section 10.8] or [Har66, II, Section 5].

Proposition 2.3. *The following hold:*

(a) Let $f : R \longrightarrow S$ be a map of rings with functors $f^* : \mathscr{M}\mathrm{od}(R) \longrightarrow \mathscr{M}\mathrm{od}(S)$ defined by $f^*(M) = M \otimes_R S$ and $f_* : \mathscr{M}\mathrm{od}(S) \longrightarrow \mathscr{M}\mathrm{od}(R)$ defined by f_*N is N viewed as an R-module via restriction of scalars. Then for $A^{\bullet} \in D^-(R), B^{\bullet} \in D^-(S)$ we have

$$\mathbf{L}f^*(A^{\bullet}) \otimes_S^{\mathbf{L}} B^{\bullet} \cong A^{\bullet} \otimes_R^{\mathbf{L}} f_*B^{\bullet}.$$

This is a special case (the affine case) of the derived projection formula you might have seen in your algebraic geometry class.

(b) For $A^{\bullet}, B^{\bullet} \in D^-(R)$ and $C^{\bullet} \in D^+(R)$, we have

$$\mathbf{R}\operatorname{Hom}_R(A^{\bullet}, \mathbf{R}\operatorname{Hom}_R(B^{\bullet}, C^{\bullet})) \cong \mathbf{R}\operatorname{Hom}_R(A^{\bullet} \otimes_R^{\mathbf{L}} B^{\bullet}, C^{\bullet})$$

in $D^+(R)$. This is just derived Hom, $\otimes$ *adjointness.*

(c) For $A^{\bullet} \in D^-(R)$ and $B^{\bullet} \in D^+(R)$ and $C^{\bullet} \in D^b(R)$ of bounded Tor*-dimension (for example, bounded projective dimension), ie the projective resolution of anything in a regular ring, then*

$$\mathbf{R}\operatorname{Hom}_R(A^{\bullet}, B^{\bullet}) \otimes_R^{\mathbf{L}} C^{\bullet} \cong \mathbf{R}\operatorname{Hom}_R(A^{\bullet}, B^{\bullet} \otimes_R^{\mathbf{L}} C^{\bullet}).$$

(d) Consider two ideals $I, J \subseteq R$ in a Noetherian ring. Then $\Gamma_I \circ \Gamma_J = \Gamma_{I+J} = \Gamma_{\sqrt{I+J}}$ as is easily checked. Next suppose that M is an injective module, we want to show that $\Gamma_J(M)$ is Γ_I-acyclic. This is normally done by showing that $\Gamma_J(M)$ is flasque and I won't reproduce it here. Thus we have that

$$\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J = \mathbf{R}\Gamma_{I+J}$$

In the case case that $I \supseteq J$ we see that

$$\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J = \mathbf{R}\Gamma_I.$$

References

[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR0222093 (36 #5145)

[Wei94] C. A. Weibel: *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)