NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 1ST, 2017

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We started class off with a student presentation of the octahedral axiom problem from the recent worksheet.

1. The other direction of Kunz's theorem

Recall Kunz's theorem

Theorem 1.1. *If R is a Noetherian ring of characteristic $p > 0$, then R is regular if and only if $F_* R$ is a flat R-module.*

Earlier we proved that "regular $\Rightarrow F_* R$ is flat", and now we want to prove the converse.

Definition 1.2. Suppose R is a domain (or simply is reduced) of characteristic $p > 0$ and let $R^\infty = R^{1/p^\infty} = \bigcup_{e \geq 0} R^{1/p^e}$. This is called the *perfection* of R. If R is not reduced, we can still define

$$R^\infty = \lim_{\rightarrow} R = \lim_{\rightarrow} F^e_* R$$

where the transition maps are Frobenius.

Remark 1.3. Note that even if R is not reduced, the Frobenius map on R^∞ is injective (since if something is killed by Frobenius, it is also killed by a transition map). Hence R^∞ is reduced even R is not. In particular, Frobenius always acts bijectively on R^∞.

Example 1.4. Note that R^∞ is rarely Noetherian even if R is (even though it is easy to check that $\operatorname{Spec} R^\infty \longrightarrow \operatorname{Spec} R$ is an isomorphism). If $R = R^\infty$ then R is called *perfect*. Indeed, let $R = \mathbb{F}_p[x]$ then $R^\infty = \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, \ldots]$.

Lemma 1.5. [BS15, Lemma 3.16, Lemma 5.10] *If $R \xleftarrow{g} S \xrightarrow{h} R'$ is are surjections of Noetherian rings of characteristic $p > 0$ with induced surjection $R^\infty \xleftarrow{g^\infty} S^\infty \xrightarrow{h^\infty} R'^\infty$ of perfect rings. Then $\operatorname{Tor}^i_{S^\infty}(R^\infty, R'^\infty) = 0$ for all $i \neq 0$ or in other words*

$$R^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty \simeq_{qis} R^\infty \otimes_{S^\infty} R'^\infty.$$

In particular, specializing to the case $R' = R$, the multiplication map $R^\infty \otimes^{\mathbf{L}}_{S^\infty} R^\infty \longrightarrow R^\infty$ is a quasi-isomorphism.

Proof. Let $I = \ker g = \langle f_1, \ldots, f_n \rangle$ so that $R = S/I$. It is easy to see that $\ker g^\infty = \ker(S^\infty \longrightarrow R^\infty) = \langle f_1^{1/p^e}, \ldots, f_n^{1/p^e} \rangle_{e \geq 0}$.

We now proceed by induction on n. Indeed, if we let $I_j = \langle f_1^{1/p^e}, \ldots, f_j^{1/p^e} \rangle_{e \geq 0} \subseteq S^\infty$ and $R_j^\infty = S^\infty / I_j$, then assuming the induction hypothesis

$$R_j^\infty \otimes^{\mathbf{L}}_{S^\infty} R' \simeq_{\text{qis}} (R_j^\infty \otimes^{\mathbf{L}}_{R^\infty_{j-1}} R^\infty_{j-1}) \otimes^{\mathbf{L}}_{S^\infty} R' = R_j^\infty \otimes^{\mathbf{L}}_{R^\infty_{j-1}} (R^\infty_{j-1} \otimes^{\mathbf{L}}_{S^\infty} R') \simeq_{\text{qis}} R_j^\infty \otimes_{R^\infty_{j-1}} (R^\infty_{j-1} \otimes_{S^\infty} R')$$

where the final quasi isomorphism is just assuming our induction hypothesis twice. Hence it suffices to prove the base case that $I = \langle f \rangle$ and $I^\infty = \langle f^{1/p^e} \rangle_{S^\infty}$.

Consider the directed system

$$\{S^\infty, \cdot f^{\frac{p-1}{p^n}}\} = S^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^2}}} \cdots \xrightarrow{\cdot f^{\frac{p-1}{p^{n-1}}}} S^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^n}}} S^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^{n+1}}}} \cdots$$

There is a map from this directed system to I^∞

$$\{S^\infty, \cdot f^{\frac{p-1}{p^n}}\} \longrightarrow I^\infty = \langle f^{1/p^e} \rangle.$$

sending s (from the nth spot) to $f^{1/p^n}a$. Note this really compatible with the maps of the directed system since $f^{1/p^{n+1}} f^{\frac{p-1}{p^{n+1}}} a = f^{1/p^n} a$. This obviously yields a surjective map

$$\mu : \varinjlim \{S^\infty, \cdot f^{\frac{p-1}{p^n}}\} \longrightarrow I^\infty.$$

Claim 1.6. *μ is an isomorphism.*

Proof of claim. We need to show that μ is injective, note this is trivial if S is a domain. For the general case suppose that $s \in S^\infty$ (living in the nth spot) is sent to zero. This means that $f^{1/p^n} s = 0 \in S^\infty$. But then since S^∞ is perfect and reduced, $f^{1/p^{n+1}} s^{1/p} = 0$ as well, and so $f^{1/p^{n+1}} s = 0$ which proves that s is killed by a transition map (which multiplies by even more). □

Likewise consider $IR' = \bigcup f^{1/p^e} R'^\infty$, the ideal generated by the image of f^{1/p^e} in R'^∞ and thus the direct system

$$\{R'^\infty, \cdot f^{\frac{p-1}{p^n}}\} = R'^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^2}}} \cdots \xrightarrow{\cdot f^{\frac{p-1}{p^{n-1}}}} R'^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^n}}} R'^\infty \xrightarrow{\cdot f^{\frac{p-1}{p^{n+1}}}} \cdots$$

and hence a map as before

$$\nu : \varinjlim \{R'^\infty, \cdot f^{\frac{p-1}{p^n}}\} \longrightarrow I^\infty R'^\infty$$

Claim 1.7. *μ is an isomorphism.*

Proof. The proof is the same as the previous claim. □

Now,

$$I^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty \cong \varinjlim \{S^\infty, \cdot f^{\frac{p-1}{p^n}}\} \otimes^{\mathbf{L}}_{S^\infty} R'^\infty \cong \varinjlim \{R'^\infty, \cdot f^{\frac{p-1}{p^n}}\} \cong I^\infty R'^\infty$$

and so it follows that for $i \neq 0$, $h^i\big(I^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty\big) = 0$. In particular, we have the map of distinguished triangles

$$\begin{array}{ccccccccc}
 & & I^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty & \longrightarrow & S^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty & \longrightarrow & R^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty & \xrightarrow{+1} & \\
 & & \sim\downarrow & & \sim\downarrow & & \downarrow & & \\
0 & \longrightarrow & I^\infty R'^\infty & \longrightarrow & R'^\infty & \longrightarrow & R^\infty \otimes_{S^\infty} R'^\infty & \longrightarrow & 0
\end{array}$$

The result follows. □

References

[BS15] B. Bhatt and P. Scholze: *Projectivity of the Witt vector affine Grassmannian*, arXiv:1507.06490.