NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 8TH, 2017

KARL SCHWEDE

0.1. **Fedder's Lemma on p^{-e}-linear maps.** Recall last time we showed that:

Lemma 0.1. *With notation as above,*

(i) *Suppose that $\phi(F^e_* J) \subseteq I$ for all $\phi \in \mathrm{Hom}_S(F^e_* S, S)$, then $J \subseteq I^{[p^e]}$.*

(ii)

$$\Big(F^e_*\big(I^{[p^e]} : J\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S) = \{\psi \in \mathrm{Hom}_S(F^e_* S, S) \mid \psi(F^e_* J) \subseteq I\}.$$

Theorem 0.2 (Fedder's Lemma)**.** *With notation as above*

$$\rho : \Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow \mathrm{Hom}_R(F^e_* R, R)$$

is surjective and $\ker \rho$ *is isomorphic to* $\Big(F^e_* I^{[p^e]}\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$*. In particular*

$$\mathrm{Hom}_R(F^e_* R, R) \cong \frac{\Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)}{\Big(F^e_* I^{[p^e]}\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)}.$$

Proof. First we prove that ρ is surjective. Choose $\alpha \in \mathrm{Hom}_R(F^e_* R, R)$. Consider the following diagram of S-modules

$$\begin{array}{ccc} F^e_* S & \overset{\overline{\alpha}}{\dashrightarrow} & S \\ \downarrow & & \downarrow \\ F^e_* R & \underset{\alpha}{\longrightarrow} & R. \end{array}$$

The dotted arrow $\overline{\alpha}$ exists since $F^e_* S$ is a projective S-module (although it is not unique). The commutativity of the diagram implies that $\overline{\alpha}(F^e_* I) \subseteq I$ (as $F^e_* I$ and I are kernels of the vertical projection maps) and therefore we see that $\overline{\alpha} \in \Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$ by **??**. This proves the surjectivity of ρ.

Next we identify the kernel of ρ. Suppose that $\psi \in \mathrm{Hom}_S(F^e_* S, S)$ satisfies $\psi(F^e_* I) \subseteq I$ and also that $\rho(\psi) = 0$. This second condition means that $\psi(F^e_* S) \subseteq I$. Applying **??** in the case that $J = S$ we see that $\psi \in \Big(F^e_* I^{[p^e]}\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$. The reverse inclusion also follows immediately from Lemma 0.1. The final isomorphism then of course follows from the first isomorphism theorem. $\square$

The real beauty of Fedder's Lemma is that it allows us to compute numerous things with ease!

0.2. **Computations with Fedder's Lemma.** Fedder's lemma gives us a very explicit way to compute the locus where a ring is not F-split.

Theorem 0.3. *Suppose that S is an F-finite regular ring and $R = S/I$. Let $J_e \subseteq S$ denote the image of the evaluation-at-1 map*

$$\mathrm{Image}\left(\left(F^e_*\big(I^{[p^e]} : I\big)\right) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow S\right)$$

for some integer $e > 0$. Then the set theoretic locus $V(J_e) \subseteq V(I) \subseteq \mathrm{Spec}\, S$ is the set of points of $V(I) \cong \mathrm{Spec}\, R$ where $\mathrm{Spec}\, R$ is not F-split.

Before proving this result, we notice that the result implies that $V(J_e)$ is independent of e. However, scheme theoretically, $V(J_e)$ is generally not independent of e.

Proof. Using Theorem 0.2, we see that the evaluation-at-1 map in the statement of the theorem is surjective at all points $\mathfrak{q} \in V(I) \subseteq \mathrm{Spec}\, S$ where $\mathrm{Hom}_R(F^e_* R, R) \longrightarrow R$ is also surjective. Of course, outside of $V(I)$, $(I^{[p^e]} : I)$ agrees with S and the surjectivity is obvious. The result follows since $R_{\mathfrak{q}} \longrightarrow F^e_* R_{\mathfrak{q}}$ splits if and only if $R_{\mathfrak{q}} \longrightarrow F_* R_{\mathfrak{q}}$ splits **??**. $\square$

Via the identification $\mathrm{Hom}_S(F^e_* S, S) \cong F^e_* S$ (sending Φ to 1, where Φ is the projection onto the $F^e_* \mathbf{x}^{\mathbf{p^e-1}}$-basis element), we get a map $F^e_* S \longrightarrow S$. It is not hard to see that this map is itself Φ. In particular:

Corollary 0.4. *The locus where $\mathrm{Spec}\, R$ is not split is closed and it is equal to $V\big(\Phi^e(F^e_*(I^{[p^e]} : I))\big)$.*

Remark 0.5. The ideal $\Phi^e(F^e_*(I^{[p^e]} : I))$ depends on the choice of e, although the locus it defines does not!

Exercise 0.1. Show that $\Phi^e(F^e_*(I^{[p^e]} : I)) \supseteq \Phi^{e+1}(F^{e+1}_*(I^{[p^{e+1}]} : I))$.

Hint: Show that $\Phi^e(F^e_*(I^{[p^e]} : I)) \cdot R$ is the same as the image of the evaluation-at-1 map $\mathrm{Hom}_R(F^e_* R) \longrightarrow R$.

Question 0.6 (Open question). It is an open question whether the descending ideals from the previous exercise stabilize (are all equal for $e \gg 0$). This is known if R is a hypersurface or more generally Gorenstein or even more generally $\mathbb{Q}$-Gorenstein. The Gorenstein case is essentially a key step in a famous result of Hartshorne and Speiser [HS77].

Remark 0.7. Since Φ is additive, note that $\Phi(F^e_*\langle f_1, \ldots, \rangle f_m\rangle) = \Phi(F^e_*\langle f_1\rangle) + \Phi(F^e_*\langle f_2\rangle) + \cdots + \Phi(F^e_*\langle f_m\rangle)$. Hence from a computational perspective, it is sufficient to compute $\Phi(F^e_*\langle f\rangle)$.

Suppose now that k is perfect for simplicity, if one writes $F^e_* f$ in terms of the basis $F^e_* \mathbf{x}^\lambda$ as

$$F^e_* f = F^e_* \sum f_\lambda^{p^e} \mathbf{x}^\lambda = \sum f_\lambda F^e_* \mathbf{x}^\lambda$$

then we claim that $\Phi(F^e_*\langle f\rangle) = \langle \ldots, f_\lambda, \ldots\rangle$. The point is that $\Phi(F^e_* f)$ simply projects from the term $f_{(\mathbf{p^e-1})} F^e_* \mathbf{x}^{(\mathbf{p^e-1})}$, on the other hand $\mathbf{x}^\lambda f \in \langle f\rangle$ and $\Phi(F^e_* \mathbf{x}^{(\mathbf{p^e-1})-\lambda} f)$ projects from $f_\lambda F^e_* \mathbf{x}^\lambda$. Doing the various projections proves that

$$\Phi(F^e_*\langle f\rangle) = \langle \ldots, f_\lambda, \ldots\rangle$$

as claimed.

As another corollary of Fedder's Lemma, we state a frequently easy to check criterion for whether or not a ring is F-split at some point. Recall by **??**, to show that R is F-split, it is sufficient to show that there exists a single surjective $\phi : F^e_* R \longrightarrow R$.

Theorem 0.8 (Fedder's F-purity criterion)**.** *Suppose that S is an F-finite regular ring and $R = S/I$. Then R is F-split in a neighborhood of a prime ideal $\mathfrak{q} \in V(I) \subseteq \operatorname{Spec} S$ if and only if*

$$(I^{[p^e]} : I) \not\subseteq \mathfrak{q}^{[p^e]}.$$

Proof. Suppose that R is F-split in a neighborhood of a prime ideal $\mathfrak{q} \in V(I)$. It follows that the evaluation-at-1 map $\operatorname{Hom}_R(F^e_* R, R) \longrightarrow R$ surjects in a neighborhood of $\mathfrak{q}$. Let $\phi_R \in \operatorname{Hom}_R(F^e_* R, R)$ be such that $\phi(F^e_* a) \notin \mathfrak{q}/I$ for some $\overline{a} \in R$. It follows from Theorem 0.2 that there exists $\phi_S \in \left(F^e_*\big(I^{[p^e]} : I\big)\right) \cdot \operatorname{Hom}_S(F^e_* S, S)$ such that

$$\phi_S(F^e_* a) \notin \mathfrak{q}$$

where $a \in S$ maps to $\overline{a} \in R$. On the other hand, suppose for a contradiction now that $(I^{[p^e]} : I) \subseteq \mathfrak{q}^{[p^e]}$ and so $\phi_S \in \left(F^e_* \mathfrak{q}^{[p^e]}\right) \cdot \operatorname{Hom}_S(F^e_* S, S)$. But since $\mathfrak{q}^{[p^e]} = \mathfrak{q}^{[p^e]} : S$, we have that $\phi_S(F^e_* S) \subseteq \mathfrak{q}$ by Lemma 0.1. But this contradicts our choice of $\overline{a}$.

Conversely we suppose that $b \in (I^{[p^e]} : I) \setminus \mathfrak{q}^{[p^e]}$. Let $\Phi \in \operatorname{Hom}_S(F^e_* S, S)$ be the generating homomorphism as in **??** and let $\phi_S(F^e_* \_\_) = \Phi^e(F^e_*(b \cdot \_\_))$. Since $b \notin \mathfrak{q}^{[p^e]}$, we know that $\phi_S(F^e_* S) \notin \mathfrak{q}$ by Lemma 0.1. Hence there exists $a \in \phi_S(F^e_* S)$, $a \notin \mathfrak{q}$. Thus, $\overline{a} \in R_{\mathfrak{q}}$ is a unit. On the other hand, by our choice of ϕ_S, it induces $\phi_R : F^e_* R \longrightarrow R$ and so by localization, $\phi_{R_{\mathfrak{q}}} : F^e_* R_{\mathfrak{q}} \longrightarrow R_{\mathfrak{q}}$ and $\overline{a}$ is in the image. Thus $\phi_{R_{\mathfrak{q}}}$ surjects and so $R_{\mathfrak{q}}$ is F-split as desired. ☐

Exercise 0.2. Suppose that R is a regular Noetherian ring of characteristic $p > 0$ and that $\mathfrak{q}$ is a prime ideal. Prove that $\mathfrak{q}^{[p^e]}$ is $\mathfrak{q}$-primary.

Hint: Show that if $f \notin \mathfrak{q}$, then $0 \longrightarrow R/\mathfrak{q}^{[p^e]} \xrightarrow{\cdot f} R/\mathfrak{q}^{[p^e]}$ injects.

Corollary 0.9. *Suppose that $R = S/\langle f \rangle_S$. Then R is F-split at the origin if and only if $f^{p-1} \notin \mathfrak{m}^{[p]} = \langle x_1^p, \ldots, x_n^p \rangle$.*

Example 0.10. Consider the following examples of F-split rings. We assume S is as before and consider $R = S/\langle f \rangle$ where f is as specified in each case below.

(a) $f = z$. The ring R is regular so we already know it is F-split, but we can alternately observe that $z^{p-1} \notin \langle x^p, y^p, z^p \rangle$.
(b) If $f = xyz$, then R is F-split (at the origin) since $x^{p-1}y^{p-1}z^{p-1} \notin \langle x^p, y^p, z^p \rangle$.
(c) If $f = xy - z^2$ then R is F-split (at the origin) since

$$(xy - z^2)^{p-1} = x^{p-1}y^{p-1} + \text{other terms} \notin \langle x^p, y^p, z^p \rangle.$$

(d) If $p = 2$, then $R = S/\langle f \rangle$ is F-split (at the origin) if and only if $f \notin \langle x^2, y^2, z^2 \rangle$ (note $p - 1 = 2 - 1 = 1$). So for example $f = x^7 + y^4 + z^3 + xyz$ yields an F-split ring.
(e) Consider $f = x^3 + y^3 + z^3$ and suppose $1 \equiv p \pmod{3}$. Note that the degree of every monomial of f^{p-1} is equal to $3(p-1)$. Thus the only way that $f^{p-1} \notin \langle x^p, y^p, z^p \rangle$ is if $x^{p-1}y^{p-1}z^{p-1}$ has non-zero coefficient in f^{p-1}. Since each monomial x^3, y^3 and z^3 to be raised to the same power we must have $3|(p-1)$ which implies that

$1 \equiv p \pmod 3$ as we already assumed. Now we need the multinomial coefficent of $x^{p-1}y^{p-1}z^{p-1}$ to not be divisible by p. But this coefficient is

$$\binom{p-1}{\frac{p-1}{3}, \frac{p-1}{3}, \frac{p-1}{3}} \equiv \frac{(p-1)!}{\left(\frac{p-1}{3}\right)!\left(\frac{p-1}{3}\right)!\left(\frac{p-1}{3}\right)!}.$$

which clearly is not divisible by p.

Now we consider several non-F-split rings.

(a′) $f = z^2$. The ring R is not reduced, so it can't be F-split, but also $z^{2(p-1)} \in \langle x^p, y^p, z^p\rangle$.

(b′) $f = x^2y - z^2$ with $p = 2$. Note that $f \in \langle x^2, y^2, z^2\rangle$. R actually is F-split if $p \neq 2$.

(c′) $f = x^4 + y^4 + z^4$. This is not F-split since every monomial in the expansion of $(x^4 + y^4 + z^4)^{p-1}$ has degree equal to $4 \cdot (p-1)$. In particular, each such monomial is divisible by x^p, y^p or z^p by the pigeon-hole-principal.

(d′) $f = x^3 + y^3 + z^3$ and $1 \not\equiv p \pmod 3$. In this case, there is no $x^{p-1}y^{p-1}z^{p-1}$ term in the expansion of $(x^3 + y^3 + z^3)^{p-1}$ by the argument in (e) above. Thus since each monomial in said expansion has degree $3(p-1)$, we see that $f^{p-1} \in \langle x^p, y^p, z^p\rangle$ which implies that R is not F-split.

References

[HS77] R. Hartshorne and R. Speiser: *Local cohomological dimension in characteristic p*, Ann. of Math. (2) **105** (1977), no. 1, 45–79. MR0441962 (56 #353)