NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 15TH, 2017

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1. A crash course in local cohomology

We'll be doing a series of crash courses over the next few days. We'll start with local cohomology and the Cohen-Macaulay condition, and then we'll move to Matlis and local duality (as well as a study of the dualizing complex). Today, local cohomology, tomorrow the world!

Let R be a Noetherian ring and I an ideal. There is a functor from R-modules to R-modules Γ_I which is defined by

$$\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n > 0\}.$$

Because I is finitely generated we also have

$$\Gamma_I(M) = \{m \in M \mid x^{n_x} m = 0 \text{ for all } x \in I \text{ and some } n_x > 0 \text{ depending on } x\}.$$

It is not difficult to see that $\Gamma_I(M)$ is left exact and so we make the following definition.

Definition 1.1. With R, I, M above, we denote by $H^i_I(M) = h^i \mathbf{R}\Gamma_I(M)$ the ith right derived functor of Γ_I. It is called the *ith local cohomology group.*

There is another important related functor. Let $U = \operatorname{Spec} R \setminus V(I)$. Choose a generating set $\langle f_0, \dots, f_t \rangle$ for I and for each m-tuple $f_{i_1}, \dots, f_{i_m}$ of these generators, we can form the localization $M_{f_{i_1} \cdots f_{i_m}}$. Most notably, we have

$$\alpha_0 : \bigoplus_j M_{f_j} \longrightarrow \bigoplus_{a<b} M_{f_a f_b}$$

where

$$d_0(m_1/f_1^{n_1}, \dots, m_t/f_t^{n_t}) = (\dots, m_a/f_a^{n_a} - m_b/f_b^{n_b}, \dots).$$

The kernel of d_0 is denoted by $\Gamma(U, M)$ and it is independent of the choice of generators of I.

Exercise 1.1. Prove that $\Gamma(U, M)$ really is independent of the choice of generators.

Hint: It suffices to consider the case where you add a single generator to the list.

Example 1.2. Suppose $M = R = k[x]$ and $I = \langle x \rangle$. Then $\Gamma(U, M) = k[x, x^{-1}]$ (there are no $M_{f_a f_b}$ terms).

Example 1.3. Suppose $M = R = k[x, y]$ and $I = \langle x, y \rangle$. Consider the kernel of

$$k[x, y, x^{-1}] \oplus k[x, y, y^{-1}] \xrightarrow{-} k[x, y, x^{-1}, y^{-1}]$$

and deduce that $\Gamma(U, M) = k[x, y]$.

On the other hand, if $M = \langle x, y \rangle$, then $\Gamma(U, M) = k[x, y]$ since $\langle x, y \rangle_x = k[x, y, x^{-1}]$ and $\langle x, y \rangle_y = k[x, y, y^{-1}]$.

Lemma 1.4. *Suppose that $Q \in U$, then $\Gamma(U, M)_Q = M_Q$.*

Proof. We localize the map α_0 at Q and notice that since $Q \in U$, at least one $f_j \notin Q$, and so $(M_{f_j})_Q = M_Q$. We may assume that $j = 1$ and so write

$$M_Q \oplus \bigoplus_{j>1} (M_Q)_{f_j} \longrightarrow \left(\bigoplus_j (M_Q)_{f_j} \right) \bigoplus \left(\bigoplus_{1<a<b} (M_Q)_{f_a f_b} \right).$$

In particular, it is easy to see that if an element is in the kernel, it is completely determined by its entry in $(M_Q)_{f_1} = M_Q$ and any such entry gives an element of the kernel. The lemma follows. $\square$

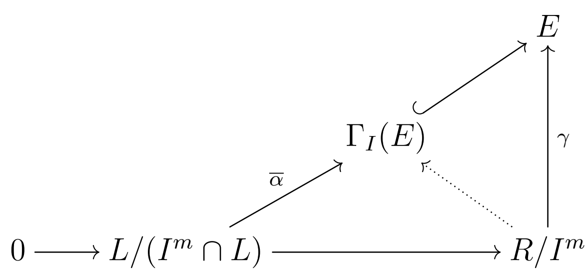
Alternate proof. Alternately, simply observe that $IR_Q = \langle 1 \rangle_{R_Q}$, and then the result immediately follows by change of the generating set. $\square$

It is also easy to see that $\Gamma(U, \bullet)$ is a left exact functor, and its higher derived functors are denoted by $H^i(U, \bullet)$.

Lemma 1.5. *With notation as above, suppose E is an injective module. Then $\Gamma_I(E)$ is also injective.*

Proof. Suppose we have $0 \longrightarrow L \xrightarrow{f} M$ exact as well as a map $\alpha : L \longrightarrow \Gamma_I(E)$. We need to show that there exists $\beta : M \longrightarrow \Gamma_I(E)$ such that $\alpha = \beta \circ f$. In fact, by [Sta16, Tag 0AVF][1], it suffices to consider the case when $M = R$ and L is an ideal (and so in particular, finitely generated since R is Noetherian). Consider the finitely generated submodule $\alpha(L) \subseteq \Gamma_I(E)$ and choose $n > 0$ so that $0 = I^n \alpha(L) = \alpha(I^n L)$. By Krull's theorem, there exists some m such that $I^m \cap L \subseteq I^n L$ and so $\alpha(I^m \cap L) = 0$ as well. In particular, α factors as $\alpha : L \longrightarrow L/(I^m \cap L) \xrightarrow{\overline{\alpha}} \Gamma_I(E)$. It thus suffices to show that $\overline{\alpha}$ extends to $\overline{\beta} : R/I^m \longrightarrow E$.

On the other hand, we certainly have



where the map labeled γ exists by the injectivity hypothesis on E. Applying $\Gamma_I(\bullet)$ to the entire diagram yields $\Gamma_I(\gamma) = \overline{\beta}$, the map we desired since $\Gamma_I(R/I^m) = R/I^m$. $\square$

Corollary 1.6. *If M is I-torsion (in other words $M = \Gamma_I(M)$), then M can be embedded in an injective I-torsion module $M \subseteq E$. In particular, $M \simeq_{qis} \mathbf{R}\Gamma_I(M)$ (and thus $H^i_I(M) = 0$ for $i > 0$).*

Proof. For the first statement, simply embed M in an I-torsion module $M \subseteq E$ and then apply the functor $\Gamma_I(\bullet)$ using Lemma 1.5. For the second statement, it follows from the

[1]The trick is to look at the largest submodule to which the map extends, and derive a contradiction if it's not M.

first that we can take an injective resolution of an I-torsion module by I-torsion injective modules. But then Γ_I acts as the identity on such a resolution. $\square$

There is a canonical map $M \longrightarrow \Gamma(U, M)$ (the diagonal), the kernel of which is easily seen to be exactly $\Gamma_I(M)$.

Theorem 1.7. [Har77, III, Exercise 2.3] *For any $I \subseteq R$, an ideal in a Noetherian ring, there is a long exact sequence*

$$\begin{array}{ccc} \cdots & \cdots & \cdots \\ H^i_I(M) \longrightarrow & \mathrm{Id}^i(M) \longrightarrow & H^i(U, M) \longrightarrow \\ H^{i+1}_I(M) \longrightarrow & \mathrm{Id}^{i+1}(M) \longrightarrow & H^{i+1}(U, M) \longrightarrow \\ \cdots & \cdots & \cdots \end{array}$$

Where $\mathrm{Id}^i(M)$ is the ith derived functor of the identity, in particular equal to zero for $i > 0$.

Exercise 1.2. Prove Theorem 1.7.

Hint: Show that if M is injective, then $0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow \Gamma(U, M) \longrightarrow 0$ is exact (we showed in class everything but the exactness on the right, to do that use the fact that if I is injective, then $I \longrightarrow I_f$ is surjective). Now take an injective resolution of M, use what you just proved, and chase.

Corollary 1.8. *With notation as above, $H^{i+1}_I(M) = H^i(U, M)$ for all $i \geq 1$ and*

$$0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow H^0(U, M) \longrightarrow H^1_I(M) \longrightarrow 0$$

is exact.

1.1. **Vanishing and non-vanishing theorems.**

Proposition 1.9. *If R is Noetherian, $I \subseteq R$ is an ideal and M is an R-module then $H^i_I(M) = 0$ for $i > \dim M =: d$,*[2] *in particular for $i > \dim R$.*

Proof. It is easy to see that $H^i_I(M)_{\mathfrak{m}} = H^i_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$ and so it suffices to work in the local case (in the Noetherian case, a localization of an injective module is still injective).

Because the local cohomology functor commutes with direct limits, it is harmless to assume that M is finitely generated. First consider the short exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow M/\Gamma_I(M) \longrightarrow 0.$$

Note that $\Gamma_I(M)$ is I-torsion and so $H^i_I(\Gamma_I(M)) = 0$ for $i > 0$ by Corollary 1.6. Hence $H^i_I(M) \cong H^i_I(M/\Gamma_I(M))$ for all $i > 0$ and so it suffices to prove the proposition in the case that $M = \Gamma_I(M)$ is I-torsion free.

We now proceed by induction on the dimension of M. In the case that M is dimension zero, we are already done since then M is I-torsion and hence zero.

Claim 1.10. *There exists choose $x \in I \subseteq \mathfrak{m}$ such that x is a regular element for M (in particular, $M \xrightarrow{\cdot x} M$ is injective.*

[2]The dimension of a module is the maximal length of a chain of primes $\mathfrak{q}_0 \subsetneq \mathfrak{q}_n$ such that $M_{\mathfrak{q}_i} \neq 0$.

Proof of claim. Since M is already assumed to be finitely generated, it has finitely many associated primes $P_i = \mathrm{Ann}_R m_i$. On the other hand, since M is I-torsion free, $I \cdot m_i \neq 0$, thus $I \not\subseteq P_i$. By *Prime Avoidance*[3], we have that $I \not\subseteq \bigcup P_i$. So choose $x \in I \setminus \left(\bigcup P_i\right)$. We claim that x is a regular element M. If $M \xrightarrow{\cdot x} M$ is not injective with kernel K, then K has an associated prime containing x, and thus so does M. This is a contradiction which proves the claim. □

Using the claim, consider now the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0$$

and note that $\dim(M/xM) \leq \dim M - 1 = d - 1$ by [AM69] (in that source, this is only proved for rings but since M is finitely generated, one can reduce to the case that $\operatorname{supp} M = \operatorname{supp} R$ and so this is easy since M/xM is a R/xR-module). By our induction hypothesis, we have

$$\ldots \longrightarrow H_I^{i-1}(M/xM) \longrightarrow H_I^i(M) \xrightarrow{\cdot x} H_I^i(M) \longrightarrow \ldots$$

and so $H_I^i(M) \xrightarrow{\cdot x} H_I^i(M)$ injects for $i < \dim M$.

Claim 1.11. *The induced map on local cohomology is still multiplication by x.*

Proof of claim. The functor $H_I^i(M)$ is R-linear (since it is a derived functor of an R-linear functor). Note that being R-linear that $\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(H_I^i(M), H_I^i(N))$ is an R-module homomorphism. In the case that $N = M$, this means that not only the identity is sent to the identity but also that the identity multiplied by r is sent to the identity multiplied by r, which is exactly what we want. □

Now that we have proved the claim, this implies that $H_I^i(M) \xrightarrow{\cdot x^m} H_I^i(M)$ injects for each integer m. But every element of $H_I^i(M)$ is I-torsion, and so killed by some x^m, which proves that $H_I^i(M) = 0$ for $i < \dim M$. □

References

[AM69] M. F. Atiyah and I. G. Macdonald: *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802 (39 #4129)

[Har77] R. Hartshorne: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[Sta16] T. Stacks Project Authors: *stacks project*, 2016.

[3]This says that if an ideal I is not contained in any of a set of prime ideals P_i, then $I \not\subseteq_i P_i$.