NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 3RD, 2017

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1. A crash course in local cohomology continued

1.1. Vanishing and non-vanishing theorems continued.

Definition 1.1. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module. Then M has *depth* $\geq n$ if $H^i_{\mathfrak{m}}(M) = 0$ for $i < n$. M is called *Cohen-Macaulay* if $H^i_{\mathfrak{m}}(M) = 0$ for $i < \dim R$.

Example 1.2. A Noetherian regular local ring is Cohen-Macaulay. To see this we proceed by induction on dimension and note it is obvious in dimension zero (the case of a field). More generally let x be part of a regular system of parameters (a minimal generating set of the maximal ideal) and note we have $0 \longrightarrow R \xrightarrow{\cdot x} R \xrightarrow{R} /xR \longrightarrow 0$. As before, we have injections $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R)$ for $i < \dim R$ but since $H^i_{\mathfrak{m}}(R)$ is $\mathfrak{m}$-torsion, this is a contradiction.

The proof we just performed shows that in order to verify that R is Cohen-Macaulay, it suffices to show that there exists a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ such that x_{i+1} is not a zero divisor on $R/\langle x_1, \ldots, x_i \rangle$ (likewise for a finitely generated module). In fact, that is the usual definition of a Cohen-Macaulay ring (likewise module).

Lemma 1.3. *Suppose that M is an R-module but that R is not necessarily local. If $\mathfrak{m}$ is a maximal ideal then $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ where the second term is viewed as an R-module via restriction.*

Proof. It is easy to see that the functors $\Gamma_{\mathfrak{m}}(\bullet) \cdot R_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}R}(\bullet_{\mathfrak{m}})$ are equal and hence the same also holds for the associated local cohomology functors (since injective modules over a Noetherian ring stay injective after localization). But now the result follows from the following claim.

Claim 1.4. *If N is a $\mathfrak{m}$-torsion module, then $N \cong N_{\mathfrak{m}} = NR_{\mathfrak{m}}$ (as R-modules).*

Proof of claim. Consider the map $N \longrightarrow N_{\mathfrak{m}}$. The kernel is the set of elements $n \in N$ such that $un = 0$ for some $u \notin \mathfrak{m}$. Consider the submodule nR for such a n with fixed u. Since N is $\mathfrak{m}$-torsion, $\mathfrak{m}^l n = 0$ for some $l > 0$. Thus nR is compatibly a $R/\mathfrak{m}^l$-module. But $R/\mathfrak{m}^l$ is a local ring and $\overline{u}$ kills $n \in nR$, but $\overline{u}$ is a unit in $R/\mathfrak{m}^l$, a contradiction. $\square$

$\square$

Example 1.5. The ring $R = k[x, y, u, v]/\langle xu, xv, yu, vx \rangle = k[x, y, u, v]/\langle x, y \rangle \cap \langle u, v \rangle$ localized at the origin has depth 1. To see this, it suffices to show that $H^0_{\mathfrak{m}}(R) = 0$ and

$H^1_{\mathfrak{m}}(R) \neq 0$. The vanishing statement is obvious because no element is killed by all powers of $\mathfrak{m}$. For the second statement, note we have a short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & k[x,y,u,v]/\langle x,y\rangle \cap \langle u,v\rangle & \longrightarrow & k[x,y,u,v]/\langle x,y\rangle \oplus k[x,y,u,v]/\langle u,v\rangle & \xrightarrow{-} & k[x,y,u,v]/\langle x,y,u,v\rangle & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & k[x,y,u,v]/\langle x,y\rangle \cap \langle u,v\rangle & \longrightarrow & k[u,v] \oplus k[x,y] & \longrightarrow & k & \longrightarrow 0. \end{array}$$

Now apply $H^i_{\mathfrak{m}}(\bullet)$ and consider the long exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(k) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(k[u,v] \oplus k[x,y])$$

Now, $H^1_{\mathfrak{m}}(k[u,v] \oplus k[x,y]) = 0$ since this is just a direct sum of local cohomologies of regular local rings, and $H^0_{\mathfrak{m}}(k) = k$. The result follows. (Note we were not very careful about localization here, but it doesn't matter due to Lemma 1.3.

Now we move to a non-vanishing theorem which we state but do not prove.

Theorem 1.6. *Suppose that $(R, \mathfrak{m})$ is local and M is a nonzero finitely generated R-module of dimension n, then $H^n_{\mathfrak{m}}(M) \neq 0$.*

As an easy consequence, we obtain the following:

Corollary 1.7. *If Q is a prime ideal such that $M_Q \neq 0$ and $d = \dim R_Q$, then $H^d_Q(M) \neq 0$.*

Proof. $H^d_Q(M) \otimes_R R_Q = H^d_{QR_Q}(M_Q) \neq 0$. □

1.2. **F-splitting's implications for local cohomology.** Local cohomology $H^i_I(\bullet)$ is a functor and so if we consider the e-iterated Frobenius map $R \longrightarrow F^e_* R$, there is an induced map

$$H^i_I(R) \xrightarrow{F^e} H^i_I(F^e_* R) \cong F^e_* H^i_I(R).$$

called the Frobenius action on local cohomology.

Lemma 1.8. *If R is F-split, then Frobenius acts injectively on $H^i_I(R)$ for any ideal I and any $i \geq 0$.*

Proof. $H^i_I(\bullet)$ is a functor, apply it to $R \longrightarrow F^e_* R \xrightarrow{s} R$ where the composition is the identity. □

Thus we have the following definition which is a weakening of the F-splitting condition.

Definition 1.9. A Noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is called *F-injective* if $F : H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F_* R)$ injects for all $i \geq 0$.

Remark 1.10. Note we only looked at the Frobenius action on the local cohomology of the maximal ideal above, it doesn't necessarily imply injectivity of Frobenius on the local cohomology of other ideals. We will see later though, that under certain conditions (for example, R is Gorenstein and F-finite), R being F-injective implies that R is F-split.

Corollary 1.11. *Suppose that Frobenius acts injective only $H^i_I(R)$ for some $I \subseteq R$ and $i > 0$. Further suppose that $J \cdot H^i_I(R) = 0$, then also $\sqrt{J} \cdot H^i_I(R) = 0$. In particular, in an F-injective local ring $(R, \mathfrak{m}, k)$, if $H^i_{\mathfrak{m}}(R)$ has finite length then $H^i_{\mathfrak{m}}(R)$ is a k-vector space.*

Proof. Suppose that $x \in \sqrt{J}$ with $x^n \in I$ and hence that $x^{p^e} \in I$ for some $e > 0$. Choose now $z \in H^i_I(R)$ and suppose for a contradiction that $x \cdot z \neq 0$. □

References